Elect. Comm. in Probab. **11** (2006), 230–241

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON THE EXISTENCE OF RECURRENT EXTENSIONS OF SELF-SIMILAR MARKOV PROCESSES

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Submitted 29 March 2006, *accepted in final form* 4 October 2006

AMS 2000 Subject classification: Primary 60G18, secondary 60G51, 60J45, 60J55.
Keywords: self-similar, semi-stable, Lamperti transformation, recurrent extension, Cramér condition, excursion

Abstract
Let $X = (X_t)_{t\geq 0}$ be a self-similar Markov process with values in $[0, \infty[$, such that the state 0 is a trap. We present a necessary and sufficient condition for the existence of a self-similar recurrent extension of X that leaves 0 continuously. This condition is expressed in terms of the Lévy process associated with X by the Lamperti transformation.

1 Introduction

In his pioneering study [15] of the structure of self-similar Markov processes with state space $[0, \infty[$, Lamperti posed the problem of determining those self-similar Markov processes that agree with a given self-similar Markov process up to the time the latter process first hits 0. Our goal in this paper is to present a necessary and sufficient condition for the existence of such a "recurrent extension" that, in addition, leaves 0 continuously. Our work was inspired by that of Vuolle-Apiala [22] and Rivero [18].

To state our results precisely, we introduce some notation and recall some of the basic theory of self-similar Markov processes. A Borel right process $X = ((X_t)_{t\geq 0}, (\mathbf{P}^x)_{x\geq 0})$ with values in $[0, \infty)$ is *self-similar* provided there exists $H > 0$ such that, for each $c > 0$ and $x \geq 0$, the law of the rescaled process $(c^{-H} X_{ct})_{t\geq 0}$, is $\mathbf{P}^{x/c^H}$ when X has law $\mathbf{P}^x$. The number H is the *order* of X, and when there is a need to emphasize H, we shall describe X as being H-self-similar. By the discussion in section 2 of [15], we can (and do) assume that X is a Hunt process; thus in addition to being a right-continuous strong Markov process, the sample paths of X are quasi-left-continuous.

One of several zero-one laws developed by Lamperti states that if

$$T_0 := \inf\{t > 0 : X_t = 0\}, \tag{1.1}$$

then either $\mathbf{P}^x[T_0 < \infty] = 0$ for all $x > 0$ or $\mathbf{P}^x[T_0 < \infty] = 1$ for all $x > 0$. Our interest is in the latter situation, *which we assume to be the case throughout the rest of the paper.*

For definiteness, we take X to be realized as the coordinate process $X_t(\omega) = \omega(t)$ on the sample space Ω of all right-continuous left-limited paths from $[0,\infty[$ to itself. *We assume that* 0 *is a trap for* X, so that each of the laws $\mathbf{P}^x$ governing X is carried by $\{\omega \in \Omega : \omega(t) = 0, \forall t \geq T_0(\omega)\}$. The natural filtration on Ω is $(\mathcal{F}_t)$, and $\mathcal{F}_\infty := \vee_{t\geq 0}\mathcal{F}_t$. We write $P_t f(x) = P_t(x,f) := \mathbf{P}^x[f(X_t)]$ for the transition semigroup of X, and $U^q := \int_0^\infty e^{-qt} P_t\, dt$, $q > 0$, for the associated resolvent operators.
Define, for $c > 0$, $\Phi_c : \Omega \to \Omega$ by $\Phi_c\omega(t) := c^{-H}\omega(ct)$. The H-self-similarity of X means that

$$\Phi_c\mathbf{P}^x(B) := \mathbf{P}^x[\Phi_c^{-1}B] = \mathbf{P}^{x/c^H}[B], \qquad \forall B \in \mathcal{F}_\infty, x \geq 0, c > 0. \tag{1.2}$$

Observe that

$$T_0\circ\Phi_c = c^{-1}T_0, \tag{1.3}$$

identically on Ω.

Definition 1. A Borel right Markov process $\overline{X} = (\overline{X}_t, \overline{\mathbf{P}}^x)$ with state space $[0,\infty[$ is a *recurrent extension* of X provided (i) the stopped process $((X_{t\wedge\overline{T}_0})_{t\geq 0}, \overline{P}^x)$ has the same law as $(X_t, \mathbf{P}^x)$, for each $x \geq 0$, and (ii) 0 is *not* a trap for $\overline{X}$.

Typically, $\overline{X}$ will be realized as the coordinate process on Ω, in which case $\overline{X}_t(\omega) = \omega(t)$. What distinguishes $\overline{X}$ from X is the collection of laws $(\overline{\mathbf{P}}^x)_{x\geq 0}$. For emphasis or clarity we may at times write $\overline{T}_0$ instead of T_0, etc.
In view of (1.3), if a recurrent extension $\overline{X}$ is self-similar, then its order must be the same as that of X.
Let $\overline{X}$ be a recurrent extension of X, with resolvent $\overline{U}^q$, $q > 0$. Writing $\overline{T}_0$ for the hitting time of 0 by $\overline{X}$, we have $\psi_q(x) := \overline{\mathbf{P}}^x[\exp(-q\overline{T}_0)] = \mathbf{P}^x[\exp(-qT_0)]$, and by the strong Markov property of $\overline{X}$ at time $\overline{T}_0$,

$$\overline{U}^q f(x) = U^q f(x) + \psi^q(x)\overline{U}^q f(0), \qquad \forall x \geq 0, q > 0. \tag{1.4}$$

Excursion theory [13, 16, 3, 10] leads to an expression for $\overline{U}^q f(0)$ in terms of a certain entrance law for X. Let $\overline{M}$ denote the closure of the zero set $\{t \geq 0 : \overline{X}_t = 0\}$, and let $\overline{G}$ denote the set of strictly positive left endpoints of the maximal intervals in the complement of $\overline{M}$. The excursions of $\overline{X}$ from 0 are indexed by the elements of $\overline{G}$: The excursion e^s associated with $s \in \overline{G}$ is the Ω-valued path defined by

$$e^s_t := \begin{cases} \overline{X}_{s+t}, & 0 \leq t < T_0\circ\theta_s, \\ 0, & t \geq T_0\circ\theta_s, \end{cases}$$

where θ_s is the shift operator on Ω. Let $\overline{L} = (\overline{L}_t)_{t\geq 0}$ denote $\overline{X}$-local time at 0, normalized so that $\overline{\mathbf{P}}^0 \int_0^\infty e^{-t}\, d\overline{L}_t = 1$. Then there is a σ-finite measure $\mathbf{n}$ on $(\Omega, \mathcal{F}_\infty)$ such that, for predictable $Z \geq 0$ and $\mathcal{F}_\infty$-measurable $F \geq 0$,

$$\overline{\mathbf{P}}^0\left[\sum_{s\in\overline{G}} Z_s\, F(e^s)\right] = \overline{\mathbf{P}}^0\left[\int_0^\infty Z_t\, d\overline{L}_t\right] \cdot \mathbf{n}[F]. \tag{1.5}$$

Formula (1.5) determines $\mathbf{n}$ uniquely, and under $\mathbf{n}$ the coordinate process $(X_t)_{t>0}$ is a strong Markov process with transition semigroup (P_t) and one-dimensional distributions

$$n_t(B) := \mathbf{n}[X_t \in B, t < T_0], \qquad t > 0, B \in \mathcal{B}_{]0,\infty[}. \tag{1.6}$$

One can show that there exists $\ell \geq 0$ such that $\int_0^t 1_{\{0\}}(\overline{X}_s)\, ds = \ell \cdot \overline{L}_t$ for all $t \geq 0$, $\overline{\mathbf{P}}^0$-a.s.; it follows from this and (1.5) (applied to $Z_s = e^{-qs}$ and $F = \int_0^{T_0} e^{-qt} f(X_t)\, dt$, first with a general f and then with $f = 1$) that

$$\overline{U}^q f(0) = \frac{\ell f(0) + n^q(f)}{q\ell + qn^q(1)}, \qquad f \in bp\mathcal{B}_{[0,\infty[}, \tag{1.7}$$

where $n^q := \int_0^\infty e^{-qt} n_t\, dt$. (Here $bp\mathcal{B}_{[0,\infty[}$ denotes the class of bounded non-negative Borel functions on $[0, \infty[$). The denominator in (1.7) is equal to

$$q\ell + \mathbf{n}\left[\int_0^{T_0} qe^{-qt}\, dt\right] = q\ell + \mathbf{n}[1 - e^{-qT_0}],$$

which is finite for all $q > 0$, as follows from (1.5) with $F = 1 - \exp(-qT_0)$ and $Z_s = e^{-qs}$. The family $(n_t)_{t>0}$ is an *entrance law*: $n_t P_s = n_{t+s}$ for all $t, s > 0$. The measure $\mathbf{n}$ is uniquely determined by (n_t) and (P_t).
If the recurrent extension $\overline{X}$ is self-similar, then (i) $\ell = 0$ [22, p. 551] and (ii) either $\mathbf{n}[X_0 = 0] = 0$ or $\mathbf{n}[X_0 > 0] = 0$ [22, Thm. 1.2].

Definition 2. A recurrent extension $\overline{X}$ is said to *leave* 0 *continuously* provided $\mathbf{n}[X_0 > 0] = 0$.

It follows easily from (1.5) that $\overline{X}$ leaves 0 continuously if and only if

$$\overline{\mathbf{P}}^0[\overline{X}_s = 0 \text{ for all } s \in \overline{G}] = 1.$$

In this paper we focus on recurrent extensions that leave 0 continuously, referring the interested reader to [22] and [18] for discussions of extensions for which $\mathbf{n}[X_0 = 0] = 0$.
We shall produce an H-self-similar recurrent extension of X by constructing a suitable excursion measure $\mathbf{n}$ as above. Motivated by the preceding discussion, we make the following definitions. Recall that $(P_t)_{t \geq 0}$ is the transition semigroup for X.

Definition 3. (a) A measure $\mathbf{n}$ on $(\Omega, \mathcal{F}_\infty)$ is an *excursion measure* provided (i) the measure n_t defined on $\mathcal{B}_{]0,\infty[}$ by formula (1.6) is σ-finite for each $t > 0$, (ii) $(X_t)_{t>0}$ under $\mathbf{n}$ is Markovian with transition operators (P_t) and one-dimensional distributions (n_t), and (iii) $\mathbf{n}$ puts no mass on the zero path $[0] : t \mapsto 0$.
(b) We say that an excursion measure $\mathbf{n}$ is *self-similar* if

$$\Phi_c \mathbf{n} = c^{-\gamma} \mathbf{n}, \qquad \forall c > 0, \tag{1.8}$$

for some $\gamma > 0$. Equivalently,

$$n_{ct}(c^H B) = c^{-\gamma} n_t(B), \qquad \forall c > 0, t > 0, B \in \mathcal{B}_{]0,\infty[}, \tag{1.9}$$

(c) An excursion measure $\mathbf{n}$ is *admissible* provided $\mathbf{n}[1 - \exp(-T_0)] < \infty$.

As noted above, the excursion measure associated with a recurrent extension of X is necessarily admissible.
Itô [13] showed that the excursion measure $\mathbf{n}$ determined by a recurrent extension $\overline{X}$ (as in (1.5)) satisfies a set of six necessary conditions. These conditions are not quite sufficient

for the existence of a recurrent extension, but Salisbury [20, 21] discovered that if two of the conditions are strengthened, then the resulting set of conditions is sufficient (and still necessary). These results hold for processes with very general state spaces. The special case of $[0,\infty[$-valued Feller processes was treated by Blumenthal in [2]. Vuolle-Apiala [22] verified that the conditions imposed by Blumenthal are satisfied in the setting of self-similar Markov processes on $[0,\infty[$. Thus, if $\mathbf{n}$ is an admissible excursion measure, then X admits recurrent extensions $\overline{X}^{\ell}$ (one for each $\ell \geq 0$) such that (1.7) holds. The process $\overline{X}^{0}$ has the additional property that $\overline{U}^{q}(x, 1_{\{0\}}) = 0$ for all $x \geq 0$, a condition that is necessary for self-similar recurrent extensions, as was noted above. By [22, (1.5)][(see also [18, Lem. 2]), this extension is self-similar if and only if $\mathbf{n}$ is self-similar.
Our hypotheses will be stated in terms of the Lévy process associated with X by the *Lamperti transformation.* For this consider the continuous additive functional A defined by

$$A_t := \int_0^t X_s^{-1/H}\, ds, \qquad t \geq 0, \tag{1.10}$$

and its right continuous inverse τ defined by

$$\tau(u) := \inf\{t > 0 : A_t > u\}, \qquad u \geq 0, \tag{1.11}$$

in which we follow the usual convention that $\inf \emptyset = +\infty$. According to [15, Thm. 4.1], the $[-\infty,\infty[$-valued process

$$Z_u := \log X_{\tau(u)}, \qquad u \geq 0, \tag{1.12}$$

is a Lévy process (*i.e.*, a process with stationary independent increments). Moreover, by [15, Lem. 3.2], we have either

$$\mathbf{P}^x[X_{T_0-} > 0, T_0 < \infty] = 1, \qquad \forall x > 0, \tag{1.13}$$

or

$$\mathbf{P}^x[X_{T_0-} = 0, T_0 < \infty] = 1, \qquad \forall x > 0. \tag{1.14}$$

If (1.13) holds then the random variable $\zeta := A_{T_0-}$ is exponentially distributed (with rate $\delta > 0$, say) and there is a real-valued Lévy process $\overline{Z}$ independent of ζ such that Z is $\overline{Z}$ killed at time ζ:

$$Z_u = \begin{cases} \overline{Z}_u, & 0 \leq u < \zeta; \\ -\infty, & u \geq \zeta. \end{cases} \tag{1.15}$$

If (1.14) holds then $\zeta = A_{T_0-} = +\infty$, $\mathbf{P}^x$-a.s. for all $x > 0$, and Z is a real-valued Lévy process that drifts to $-\infty$. Let $\mathbf{Q}^z$ denote the law of Z under the initial condition $Z_0 = z$. The process $Z = (Z_t, \mathbf{Q}^z)$ is referred to as the *Lévy process underlying* X. We shall write (Q_t) for the transition semigroup of Z.
Because A and τ are strictly increasing on their respective domains of finiteness, the Lamperti transformation can be inverted as follows. Let $(Z_t, \mathbf{Q}^z)$ be a $[-\infty,\infty[$-valued Lévy process that drifts to $-\infty$ in case $\zeta := \inf\{t : Z_t = -\infty\} = +\infty$, a.s. (The state $-\infty$ is a trap, and serves as the cemetery state for Z.) Then

$$\tau(u) := \int_0^u \exp(Z_s/H)\, ds \tag{1.16}$$

is finite for all $u \geq 0$; see [1, Thm. 1]. Define the inverse of τ:

$$A(t) = A_t := \inf\{u : \tau(u) > t\}, \qquad t \geq 0. \tag{1.17}$$

For each $x > 0$, the process defined by

$$\exp(Z_{A(t)}), \qquad t \geq 0, \tag{1.18}$$

has, under $\mathbf{Q}^{\log x}$, the same law as X under $\mathbf{P}^x$.
Here is the main result of the paper. A recent result of V. Rivero [19, Lem. 2] (obtained after this manuscript was submitted for publication) shows that a hypothesis included in the original statement of our theorem (namely, the finiteness of the expectation $\mathbf{Q}^0[I^{\kappa H-1}]$ appearing in (2.15) below) is in fact implied by condition (1.19) when $0 < \kappa H < 1$. We present this improved version of our result, noting that Rivero's paper [19] also contains a proof of Theorem 1 (below) by methods that are very different from those used here.

Theorem 1. *(a) The H-self-similar Markov process X admits a self-similar recurrent extension that leaves 0 continuously if and only if there exists $\kappa \in]0, 1/H[$ such that Cramér's condition*

$$\mathbf{Q}^0[\exp(\kappa Z_t)] = 1, \qquad \textit{for some (or all) } t > 0. \tag{1.19}$$

holds.
(b) There is at most one self-similar recurrent extension that leaves 0 continuously.

Remark 1. (a) To see why the condition $0 < \kappa < 1/H$ is natural, observe that if there is to be a recurrent extension $\overline{X}$ of X, then the inverse local time of $\overline{X}$ at 0 is a stable subordinator of index $\gamma := \kappa H$.

(b) Let $C_0]0, \infty[$ denote the class of continuous real-valued functions f on $]0, \infty[$ such that $\lim_{x\to 0} f(x) = \lim_{x\to+\infty} f(x) = 0$. Vuolle-Apiala [22] introduced the following condition (VA): There exists $k > 0$ such that the limit

$$\lim_{x\to 0+} \frac{\mathbf{P}^x[1 - \exp(-T_0)]}{x^k} \tag{VA.1}$$

exists in $]0, \infty[$, and the limit

$$\lim_{x\to 0+} \frac{U^q f(x)}{x^k} \tag{VA.2}$$

exists for all $q > 0$ and all $f \in C_0]0, \infty[$, and is strictly positive for at least one non-negative $f \in C_0]0, \infty[$. Vuolle-Apiala showed that (VA) implies the existence (and uniqueness) of a self-similar recurrent extension of X that leaves 0 continuously.

(c) The meaning of the parameter k in (VA) was clarified by Rivero [18], who introduced the following condition (R), expressed in terms of the underlying Lévy process Z:

$$\text{The law of } Z_1 \text{ is not supported by a lattice } r\mathbf{Z}; \tag{R.1}$$

there exists $\theta > 0$ such that

$$\mathbf{Q}^0[\exp(\theta Z_t)] = 1, \text{ for some (or all) } t > 0 \text{ and} \tag{R.2}$$

$$\mathbf{Q}^0[Z_1^+ \exp(\theta Z_1)] < \infty. \tag{$R.3$}$$

Suppose that (R) holds. Rivero showed that if $\theta H \geq 1$, then condition (VA) must fail, while if $0 < \theta H < 1$ then (VA) holds for $k = \theta$. Conversely, Rivero showed that if (VA) holds then $0 < kH < 1$ and $(R.2)$ and $(R.3)$ hold with $\theta = k$. Thus, modulo the technical condition $(R.1)$, (VA) and (R) are equivalent, for $k = \theta \in]0, 1/H[$. Moreover, under (VA), $k = \theta = \kappa$ (as in (1.19)).
(d) The essentially equivalent conditions (VA) and (R) imply Cramér's condition (1.19) (with $0 < \kappa H < 1$). It is not clear how broad the gap is between these conditions.

2 Proof of Theorem 1

(a) Let us first suppose that X admits a recurrent extension leaving 0 continuously. By the discussion in section 1 there is an admissible excursion measure $\mathbf{n}$ such that $\mathbf{n}[X_0 > 0] = 0$ and such that the scaling property

$$\Phi_c \mathbf{n} = c^{-\gamma} \mathbf{n}, \qquad \forall c > 0, \tag{2.1}$$

holds for some $\gamma \in]0, 1[$. The "mean occupation measure" m defined on the Borel subsets of $]0, \infty[$ by

$$m(B) := \mathbf{n} \int_0^{T_0} 1_B(X_s)\, ds, \qquad B \in \mathcal{B}_{]0,\infty[}, \tag{2.2}$$

is an X-excessive measure; see [10, (4.5)]. As a consequence of (2.1), m takes the form

$$m(dx) = C x^{-1+(1-\gamma)/H}\, dx, \tag{2.3}$$

for some constant $0 < C < \infty$; see [18, Lem. 3]. [Briefly, (2.1) implies that m scales as follows: $m(b^H B) = b^{1-\gamma} m(B)$; this in turn implies (2.3), the admissibility of m guaranteeing the finiteness of C.] By the theory of time changes for Markov processes, as found for example in [9], the measure $\nu(dx) := x^{-1-\gamma/H}\, dx$ is therefore excessive for the time-changed process $t \mapsto X_{\tau(t)}$. Define $\kappa := \gamma/H$. Then $\xi(dz) := e^{-\kappa z}\, dz$ (the image of ν under the mapping $x \mapsto \log x$) is excessive for the Lévy process Z.
Let $(Y_t, \mathbb{Q})$ denote the Kuznetsov measure associated with Z and ξ. In more detail, let W be the space of paths $w : \mathbf{R} \to [-\infty, +\infty[$ that are $\mathbf{R}$-valued and right-continuous with left limits on an open interval $]\alpha(w), \beta(w)[$ and that hold the value $-\infty$ outside that interval. (Thus $-\infty$ serves as cemetery state for Y.) The coordinate maps $Y_t(w) := w(t)$, $t \in \mathbf{R}$, generate $\mathcal{G} := \sigma\{Y_t : t \in \mathbf{R}\}$, and $\mathbb{Q}$ is the unique measure on $(W, \mathcal{G})$ not charging the dead path $[-\infty] : t \mapsto -\infty$, and such that

$$\begin{aligned} &\mathbb{Q}[Y_{t_1} \in dx_1, Y_{t_2} \in dx_2, \ldots, Y_{t_n} \in dx_n] \\ &\qquad = \xi(dx_1) Q_{t_2 - t_1}(x_1, dx_2) \cdots Q_{t_n - t_{n-1}}(x_{n-1}, dx_n), \end{aligned} \tag{2.4}$$

for all real $t_1 < t_2 < \cdots < t_n$ and $x_1, x_2, \ldots, x_n$. Thus $Y = (Y_t)$ under $\mathbb{Q}$ is a stationary Markov process with random times of birth and death (namely, α and β), with one-dimensional distributions (while alive) all equal to ξ, and with transition probabilities those of the Lévy process Z. For the construction and various properties of such processes the reader is referred to [17, 6, 11].

We now use a time-reversal argument to show that ξ is in fact invariant for Z. To this end define a semigroup $(\widehat{Q}_t^\kappa)$ by the formula

$$\widehat{Q}_t^\kappa f(x) := \mathbf{Q}^0[f(x - Z_t)e^{\kappa Z_t}], \tag{2.5}$$

and observe that $(\widehat{Q}_t^\kappa)$ is the semigroup dual to (Q_t) with respect to ξ. That is,

$$\int_{\mathbf{R}} f \cdot Q_t g \, d\xi = \int_{\mathbf{R}} g \cdot Q_t^\kappa f \, d\xi, \qquad t > 0, f, g \in p\mathcal{B}_{\mathbf{R}};$$

cf. the computation in (2.15) below. Thus

$$\begin{aligned} &\mathbb{Q}[Y_{t_1} \in dx_1, Y_{t_2} \in dx_2, \ldots, Y_{t_{n-1}} \in dx_{n-1}, Y_{t_n} \in dx_n] \\ &\qquad = \xi(dx_n)\widehat{Q}^\kappa_{t_n - t_{n-1}}(x_n, dx_{n-1}) \cdots \widehat{Q}^\kappa_{t_2 - t_1}(x_2, dx_1), \end{aligned} \tag{2.6}$$

The moment generating function $\mathbf{Q}^0[\exp(\lambda Z_t)]$, $\lambda \in \mathbf{R}$, is necessarily of the form

$$\mathbf{Q}^0[\exp(\lambda Z_t)] = \exp(t\psi(\lambda)), \tag{2.7}$$

where $\psi : \mathbf{R} \to]-\infty, +\infty]$. By Hölder's inequality, $\log \psi$ is convex (strictly convex on the interior of the interval where it is finite). Now either Z drifts to $-\infty$ (in which case $\psi(0) = 0$ and $\mathbf{Q}^0[Z_1] = \psi'(0+) \in [-\infty, 0[$) or jumps to $-\infty$ (in which case $\psi(0) < 0$). Clearly $1 \geq \widehat{Q}_t^\kappa 1(x) = \exp(\psi(\kappa)t)$, so $\psi(\kappa) \leq 0$. If $\psi(\kappa) < 0$ then $\mathbf{Q}^0_\kappa[\widehat{Z}^\kappa_t = -\infty] = 1 - \exp(\psi(\kappa)t) > 0$. If $\psi(\kappa) = 0$ then $\widehat{\mathbf{Q}}^0_\kappa[\widehat{Z}^\kappa_t] = -\mathbf{Q}^0[Z_t e^{\kappa Z_t}] = -\psi'(\kappa-) \in [-\infty, 0[$ by convexity. Therefore $\widehat{Z}^\kappa$ either jumps to $-\infty$ at a finite time (if $\psi(\kappa) < 0$) or drifts to $-\infty$ (if $\psi(\kappa) = 0$). In either case, the exponential integral

$$\widehat{I} := \int_0^\infty \exp(\widehat{Z}^\kappa_s / H) \, ds$$

is finite, $\widehat{\mathbf{Q}}^z_\kappa$-a.s., for all $z \in \mathbf{R}$. Thus if we define

$$\rho(u) := \int_{-\infty}^u \exp(Y_s / H) \, ds, \qquad u \in \mathbf{R}, \tag{2.8}$$

then

$$\mathbb{Q}[\rho(u) = \infty, \alpha < u < \beta] = \int_{\mathbf{R}} \widehat{\mathbf{Q}}^z_\kappa[\widehat{I} = \infty] \, \xi(dz) = 0, \tag{2.9}$$

by [17, (4.7)]. In particular,

$$\lim_{u \downarrow \alpha} \rho(u) = 0, \qquad \mathbb{Q}\text{-a.s.}$$

It follows that the inverse process K defined by

$$K(t) := \inf\{u : \rho(u) > t\}, \qquad t \geq 0,$$

is strictly increasing and continuous on $[0, \rho(\infty)[$. Recalling from section 1 the discussion of the (inverse) Lamperti transformation, we see that the measure m is proportional to the image under the mapping $z \mapsto \exp(z)$ of the Revuz measure of the CAF τ relative to the Z-excessive measure ξ. By a result of Kaspi (see (2.3) and (2.8) in [14]), the formula

$$\eta_t(f) := \mathbb{Q}[f(\exp(Z_{K(t)})), 0 < K(t) \leq 1], \qquad f \in p\mathcal{B}_{]0,\infty[}, t > 0, \tag{2.10}$$

defines an entrance law for (P_t) and

$$m = \int_0^\infty \eta_t \, dt. \tag{2.11}$$

But by (2.2) and Tonelli's theorem we also have $m = \int_0^\infty n_t \, dt$, so by the uniqueness of such a representation [11, (5.25)], $\eta_t = n_t$ for all $t > 0$. Let Ω^+ be the space of right-continuous left-limited paths from $]0,\infty[$ to $[0,\infty[$, and let $\mathbf{n}^+$ be the image of $\mathbf{n}$ under the mapping $\Omega \ni \omega \mapsto \omega|_{]0,\infty[} \in \Omega^+$. Let $\mathcal{F}_\infty^+$ be the σ-field on Ω^+ generated by the coordinate maps X_t^+, $t > 0$. Theorem (2.1) of [14] now tells us that if $\Pi : W \to \Omega^+$ denotes the (inverse Lamperti) transformation $w \mapsto (t \mapsto \exp(Y_{K(t,w)}(w)),\, t > 0)$ then

$$\mathbf{n}^+[A, t < T_0] = \mathbb{Q}[\Pi^{-1}(A), 0 < K(t) \le 1], \qquad A \in \mathcal{F}_\infty^+. \tag{2.12}$$

In particular

$$\begin{aligned} \mathbb{Q}\left[\limsup_{u \downarrow \alpha} Y_u > -\infty, 0 < K(t) \le 1\right] &= \mathbf{n}^+\left[\limsup_{s \downarrow 0} X_s^+ > 0, t < T_0^+\right] \\ &= \mathbf{n}[X_0 > 0, t < T_0] = 0, \end{aligned}$$

for all $t > 0$, from which it follows that $\lim_{u \downarrow \alpha} Y_u = -\infty$, $\mathbb{Q}$-a.s. By time reversal (as in (2.9)), this means that the dual process $\widehat{Z}^\kappa$ does not jump to $-\infty$ ($\widehat{\mathbf{Q}}^z$-a.s. for ξ-a.e. $z \in \mathbf{R}$). That is, $\psi(\kappa) = 0$; equivalently,

$$\mathbf{Q}^0[e^{\kappa Z_t}] = 1, \qquad t > 0,$$

which is (1.19).
Conversely, suppose that (1.19) holds for some $\kappa > 0$. Then $\xi(dz) = e^{-\kappa z}\, dz$ is an *invariant* measure for Z:

$$\begin{aligned} \xi Q_t(f) &= \int_{\mathbf{R}} e^{-\kappa z} \mathbf{Q}^z[f(Z_t)]\, dz = \int_{\mathbf{R}} e^{-\kappa z} \mathbf{Q}^0[f(z + Z_t)]\, dz \\ &= \mathbf{Q}^0\left[\int_{\mathbf{R}} e^{-\kappa z} f(z + Z_t)\, dz\right] = \mathbf{Q}^0\left[\int_{\mathbf{R}} e^{\kappa(Z_t - y)} f(y)\, dz\right] \\ &= \int_{\mathbf{R}} e^{-\kappa y} f(y) \mathbf{Q}^0[e^{\kappa Z_t}]\, dy = \int_{\mathbf{R}} e^{-\kappa y} f(y), dy = \xi(f). \end{aligned} \tag{2.13}$$

Let $(Y, \mathbb{Q})$ be the Kuznetsov process for Z and ξ as before. Because ξ is invariant, $\mathbb{Q}[\alpha > -\infty] = 0$; see [11, (6.7)]. Making use of (2.5) and the discussion following (2.6) we see that

$$\widehat{\mathbf{Q}}^0_\kappa[\widehat{Z}^\kappa_t] = -t\psi'(\kappa-) \in [-\infty, 0[,$$

so that $\widehat{Z}^\kappa$ drifts to $-\infty$. As before, (2.6) holds; consequently the time-reversal argument used before to show the finiteness of the integral in (2.8) now permits us to deduce that $\mathbb{Q}$ is carried by $\{\alpha = -\infty, \lim_{t \downarrow \alpha} Y_t = -\infty\}$.
Define $m(dx) = x^{-1-\kappa+1/H}\, dx$. The Revuz measure of the continuous additive functional $\tau(u) := \int_0^u \exp(Z_v/H)\, dv$, relative to ξ, is the measure $\mu(dz) := e^{z/H}\, \xi(dz)$ on $\mathbf{R}$. The image of μ under the mapping $z \mapsto \exp(z)$ is the measure m; another application of Kaspi's theorem shows that m is purely excessive for X, so there is a uniquely determined entrance law $(\eta_t)_{t>0}$ such that $m = \int_0^\infty \eta_t \, dt$. As before let Ω^+ be the space of right-continuous left-limited paths

from $]0,\infty[$ to $[0,\infty[$. Let $\mathbf{n}^+$ be the measure on Ω^+ under which the coordinate process $(X_t^+)_{t>0}$ is Markovian with one-dimensional distributions $(\eta_t)_{t>0}$ and transition semigroup (P_t); see [12] for the construction of such measures. Then by [14, Thm. 2.1], writing $\Pi : W \to \Omega^+$ for the map $w \mapsto (t \mapsto \exp(Y_{K(t,w)}(w)),\, t>0)$, we have

$$\mathbf{n}^+[B, t<T_0] = \mathbb{Q}[\Pi^{-1}(B), 0<K(t)\le 1], \qquad B \in \mathcal{F}^+_\infty. \tag{2.14}$$

In particular, the choice $B = B_0 := \{\omega \in \Omega^+ : \lim_{t\downarrow 0}\omega(t) = 0\}$ in (2.14) shows that that $\mathbf{n}^+$ is carried by B_0 because $\mathbb{Q}$ is carried by $\{w \in W : \lim_{t\downarrow -\infty} w(t) = -\infty, \alpha(w) = -\infty\}$. Identifying Ω with B_0 and $\mathcal{F}_\infty$ with $\mathcal{F}^+_\infty \cap B_0$, we see that $\mathbf{n}^+$ induces an excursion measure $\mathbf{n}$ on $(\Omega, \mathcal{F}_\infty)$ as in Definition 3(a). Now write $\Psi_x : w \mapsto (w(t)+x)_{t\in\mathbf{R}}$ for the spatial translation operator on W; a check of finite dimensional distributions shows that

$$\Psi_x \mathbb{Q} = e^{-\kappa x}\mathbb{Q}.$$

This, in combination with (2.14), implies that $\mathbf{n}$ is self-similar, with similarity index $\gamma = \kappa H$. From the proof of Lemma 3 in [18], we have

$$\mathbf{n}[1-\exp(-T_0)] = H \cdot \Gamma(1-\kappa H)\cdot \mathbf{Q}^0[I^{\kappa H-1}], \tag{2.15}$$

where $I := \int_0^\infty \exp(Z_v/H)\,dv$. Lemma 2 of [19] tells us that $\mathbf{Q}^0[I^{\kappa H-1}] < \infty$ because $\kappa H \in]0,1[$, thereby guaranteeing the admissibility of $\mathbf{n}$.

(b) Turning to the uniqueness assertion, let $\overline{X}^1$ and $\overline{X}^2$ be self-similar recurrent extensions of X that leave 0 continuously. In view of (1.4) and (1.7), the resolvent (and so the distribution) of $\overline{X}^j$ is uniquely determined by the associated entrance law (n_t^j), $j=1,2$. Because of a uniqueness theorem [11, (5.25)] cited earlier, each entrance law is in turn uniquely determined by its integral $m^j = \int_0^\infty n_t^j\,dt$. But as noted at the beginning of this section, m^j has the form $m^j(dx) = C_j x^{-1+(1-\gamma)/H}\,dx$, where $\gamma = \kappa H$. (Both $\overline{X}^1$ and $\overline{X}^2$ satisfy (1.19), by the part of Theorem 1 already proved.) It follows that $n_t^1 = (C_1/C_2)n_t^2$ for all $t>0$, so $\overline{X}^1$ and $\overline{X}^2$ have the same resolvent. □

Remark 2. The argument just given for the "if" portion of part (a) of Theorem 1 proves a bit more than is asserted in the statement of the theorem. Namely, if $\kappa > 0$ is such that (1.19) holds, then there is a self-similar excursion measure $\mathbf{n}$ (with index $\gamma := \kappa H$) such that $\mathbf{n}[X_0 > 0] = 0$. Notice that (1.3) implies the existence of a constant $\widetilde{C} \in]0,\infty]$ such that

$$\mathbf{n}[T_0 > t] = \widetilde{C}t^{-\gamma}, \qquad t>0. \tag{2.16}$$

If, as in the proof above, we normalize $\mathbf{n}$ so that $\mathbf{n}\int_0^\infty 1_B(X_t)\,dt = \int_B x^{-1+(1-\gamma)/H}\,dx$, then it is clear from (2.15) and [19, Lem. 2] that the constant $\widetilde{C}$ if finite if $\kappa H < 1$. It is not clear whether $\widetilde{C}$ is finite when $\kappa H \ge 1$.

3 An Application

Suppose that our self-similar Markov process X satisfies (1.13). In this section we shall apply Theorem 1 to obtain the following improvement of Theorem 6(i) of Chaumont and Rivero [4].

Theorem 2. *Suppose there exists $\kappa > 0$ such that*

$$\mathbf{Q}^0[\exp(-\kappa Z_t)] = 1, \qquad \textit{for some (or all) } t > 0. \tag{3.1}$$

Then $h : x \mapsto x^{-\kappa}$ is a purely excessive function for X, and the h-transform process X^h with laws

$$\mathbf{P}_h^x[F, t < T_0] = x^\kappa \cdot \mathbf{P}^x[F X_t^{-\kappa}], \qquad t > 0, F \in b\mathcal{F}_t, x > 0, \tag{3.2}$$

is "X conditioned to converge to 0":

$$\mathbf{P}_h^x[X_{T_0-} = 0, T_0 < \infty] = 1, \qquad \forall x > 0. \tag{3.3}$$

Proof. Let us start with the Lévy process $\widehat{Z} := -Z$, the dual of Z with respect to Lebesgue measure on $\mathbf{R}$, and apply the inverse Lamperti transformation to obtain a self-similar Markov process $\widehat{X} = ((\widehat{X}_t)_{t\geq 0}, (\widehat{\mathbf{P}}^x)_{x\geq 0})$ on $[0, \infty[$ with 0 as a trap. The process $\widehat{X}$ is in weak duality with X with respect to the measure $\eta(dx) := x^{-1+1/H}\, dx$; see [23]. By hypothesis, $\widehat{X}$ satisfies the condition (1.19) of part (a) of Theorem 1. In particular, $\widehat{\mathbf{P}}^x[\widehat{T}_0 < \infty] = 1$ for all $x > 0$. (Otherwise, the Lévy process $\widehat{Z}$ has infinite lifetime and $\limsup_{t\to+\infty} \widehat{Z}_t = +\infty$, a.s., hence the moment generating function $\widehat{\psi}$ of $\widehat{Z}$ (defined by analogy with (2.7)) vanishes at 0 as well as at κ. Since $\widehat{\psi}$ in convex on $[0, \kappa]$, this yields $\widehat{\psi}'(0+) \in [-\infty, 0[$, implying that $\widehat{Z}$ drifts to $-\infty$, a contradiction.) By the proof of Theorem 1 (see Remark 2), applied to $\widehat{X}$, there is an $\widehat{X}$-excursion measure $\widehat{\mathbf{n}}$ under which the coordinate process on $(\Omega, \mathcal{F}_\infty)$ is a strong Markov process with transition semigroup $(\widehat{P}_t)$ (that of $\widehat{X}$) and entrance law $(\widehat{n}_t)_{t>0}$, say. Moreover,

$$m(B) := \int_0^\infty \widehat{n}_t(B)\, dt = \int_B x^{-1-\kappa+1/H}\, dx, \qquad \forall B \in \mathcal{B}_{]0,\infty[}. \tag{3.4}$$

By Nagasawa's theorem, as found for example in the section 4 of [7] (see also [5] and the appendix of [8]), the process $\widetilde{X}$ defined by

$$\widetilde{X}_t := \begin{cases} X_{(T_0 - t)-}, & \text{if } 0 \leq t < T_0, \\ 0, & \text{otherwise,} \end{cases} \tag{3.5}$$

is Markovian under $\widehat{\mathbf{n}}$, with transition semigroup

$$P_t^h f(x) := x^\kappa \mathbf{P}^x[f(X_t) X_t^{-\kappa}; t < T_0] \tag{3.6}$$

i.e., the h-transform of (P_t) corresponding to $h(x) = x^{-\kappa}$. In other words, the image $\widetilde{\mathbf{n}}$ of $\widehat{\mathbf{n}}$ under the mapping $\omega \mapsto \widetilde{X}(\omega)$ is an X^h-excursion measure. Because $\widetilde{\mathbf{n}}[T_0 = \infty] = 0$, we have, for $\epsilon > 0$,

$$0 = \widetilde{\mathbf{n}}[\epsilon < T_0 = \infty] = \widetilde{\mathbf{n}}\left[\mathbf{P}_h^{X_\epsilon}[T_0 = \infty]; T_0 > \epsilon\right],$$

so $\mathbf{P}_h^x[T_0 = \infty] = 0$, first for Lebesgue a.e. $x > 0$, then for all $x > 0$ because this probability does not depend on x—apply (1.2) and (1.3) to X^h. Similarly,

$$\widetilde{\mathbf{n}}[X_0 \neq 0] = \widehat{\mathbf{n}}\left[\limsup_{t\uparrow \widetilde{T}_0} \widetilde{X}_t > 0\right] = \widehat{\mathbf{n}}[X_0 \neq 0] = 0,$$

so if $\epsilon > 0$,

$$0 = \widetilde{\mathbf{n}}\left[\{T_0 > \epsilon\} \cap \theta_\epsilon^{-1}\{\limsup_{t \uparrow T_0} X_t > 0\}\right] = \widetilde{\mathbf{n}}\left[\mathbf{P}_h^{X_\epsilon}\left[\limsup_{t \uparrow T_0} X_t > 0\right]; T_0 > \epsilon\right].$$

It follows that

$$\mathbf{P}_h^x\left[\limsup_{t \uparrow T_0} X_t > 0\right] = 0,$$

first for m-a.e. $x > 0$, and then for all $x > 0$ by the reasoning used above. □

Acknowledgment. I thank Victor Rivero for an early look at [19], which allowed the improvement discussed before the statement of Theorem 1.

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