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SURVIVAL PROBABILITIES FOR BRANCHING BROWNIAN MOTION WITH ABSORPTION

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Abstract
We study a branching Brownian motion (BBM) with absorption, in which particles move as Brownian motions with drift $-\rho$, undergo dyadic branching at rate $\beta > 0$, and are killed on hitting the origin. In the case $\rho > \sqrt{2\beta}$ the extinction time for this process, ζ, is known to be finite almost surely. The main result of this article is a large-time asymptotic formula for the survival probability $P^x(\zeta > t)$ in the case $\rho > \sqrt{2\beta}$, where P^x is the law of the BBM with absorption started from a single particle at the position $x > 0$.
We also introduce an additive martingale, V, for the BBM with absorption, and then ascertain the convergence properties of V. Finally, we use V in a 'spine' change of measure and interpret this in terms of 'conditioning the BBM to survive forever' when $\rho > \sqrt{2\beta}$, in the sense that it is the large t-limit of the conditional probabilities $P^x(A|\zeta > t + s)$, for $A \in \mathcal{F}_s$.

1 Introduction and summary of results

The object of study in this article is a (dyadic) branching Brownian motion with absorption. This is a branching process in which all particles diffuse spatially as Brownian motions with drift $-\rho$, where $\rho \in \mathbb{R}$, and are killed (removed from the process) on hitting the origin. All living particles undergo binary fission at exponential rate $\beta > 0$, with offspring moving off independently from their birth positions and repeating stochastically the behaviour of their parent, and so on. Let the configuration of particles at time t be $\{Y_u(t) : u \in N_t^{-\rho}\}$, where $N_t^{-\rho}$ is the set of surviving particles. We shall refer to this process as a $(-\rho, \beta; \mathbb{R}^+)$-BBM,

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with probabilities $\{P^x : x \in \mathbb{R}\}$, where P^x is the law of the process initiated from a single particle at $x > 0$.

We define $\zeta := \inf\{t > 0 : N_t^{-\rho} = \varnothing\}$ to be the extinction time of the $(-\rho, \beta; \mathbb{R}^+)$-BBM, so that $\{\zeta = \infty\}$ is understood to be the event that the process survives forever. The right-most particle in this process is defined to be $R_t := \sup\{Y_u(t) : u \in N_t^{-\rho}\}$ on $\{\zeta > t\}$, and 0 otherwise. We also introduce the notation $N_t^{-\rho}(a,b) := \sum_{u \in N_t^{-\rho}} \mathbf{1}_{\{Y_u(t) \in (a,b)\}}$ for the number of surviving particles with spatial positions in the interval (a,b) at time t, where $a < b$ and $a, b \in [0, \infty]$. Trivially, we note that $\{R_t > 0\} = \{\zeta > t\} = \{N_t^{-\rho}(0,\infty) > 0\}$.

This model was studied in Harris *et al.* [14], where it was shown using martingale arguments that for all $x > 0$, $P^x(\zeta < \infty) = 1$ when $\rho \geq \sqrt{2\beta}$, and further that $P^x(\zeta < \infty) \in (0,1)$ when $\rho < \sqrt{2\beta}$. The main result of this article gives an asymptotic expression for the (decaying) survival probability $P^x(R_t > 0)$ when $\rho > \sqrt{2\beta}$.

Theorem 1. *For $\rho > \sqrt{2\beta}$ and $x > 0$,*

$$\lim_{t\to\infty} P^x(R_t > 0) \frac{\sqrt{2\pi t^3}}{x} e^{-\rho x + (\frac{1}{2}\rho^2 - \beta)t} = K, \tag{1}$$

for some constant $K > 0$ that is independent of x. This is equivalent to

$$P^x(R_t > 0) \underset{t\to\infty}{\sim} \frac{1}{2}\rho^2 K \times E^x(N_t^{-\rho}(0,\infty)). \tag{2}$$

This new result will be proved in Section 2 (although some of the details will be delayed until Section 3) and represents the bulk of the original work of this paper. Theorem 1 was motivated by the related results of Chauvin and Rouault [6, Thm. 2] (for standard BBM) and Harris *et al.* [14, Thm. 6] (for the $(-\rho, \beta; \mathbb{R}^+)$-BBM), which give asymptotic expressions for the probabilities $P^x(R_t > \lambda t + \theta)$, where $\theta \in \mathbb{R}$ and $\lambda > 0$ is chosen sufficiently large that $P^x(R_t > \lambda t + \theta) \to 0$ as $t \to \infty$.

Theorem 1 is closely related to the standard BBM study of Chauvin and Rouault [6], where there is no killing at the origin and no drift on the particles' motion. Let $\mathbf{P}^x$ (with associated expectation operator $\mathbf{E}^x$) be the law of standard BBM with dyadic branching at rate β and configuration of particles alive at time t given by $\{Y_u(t) : u \in \mathcal{N}_t\}$, where $\mathcal{N}_t$ is the set of particles alive at time t. Then clearly

$$\begin{aligned} P^x(N_t^{-\rho}(0,\infty) > 0) &= \mathbf{P}^x\Big(\exists u \in \mathcal{N}_t : Y_u(s) > \rho s, \forall s \in [0,t]\Big) \\ &\leq \mathbf{P}^x\Big(\exists u \in \mathcal{N}_t : Y_u(t) > \rho t\Big) = \mathbf{P}^x(\mathcal{N}_t(\rho t, \infty) > 0). \end{aligned}$$

Here $\mathcal{N}_t(a,b) := \sum_{u \in \mathcal{N}_t} \mathbf{1}_{\{Y_u(t) \in (a,b)\}}$, in keeping with our notation for the killed BBM. Freidlin [9, 10] treated similar problems in the context of the KPP equation, and when $\rho > \sqrt{2\beta}$ the exponential decay rate of $\mathbf{P}^x(\mathcal{N}_t(\rho t, \infty) > 0)$, namely $\beta - \frac{1}{2}\rho^2$, can be deduced from Freidlin's work. Chauvin and Rouault [6, Thm. 2] is a finer result which shows that

$$\mathbf{P}^x(\mathcal{N}_t(\rho t, \infty) > 0) \sim \rho K' \times \mathbf{E}^x(\mathcal{N}_t(\rho t, \infty)) \sim \frac{K'}{\sqrt{2\pi t}} e^{\rho x - (\frac{1}{2}\rho^2 - \beta)t}$$

as $t \to \infty$, where $\rho > \sqrt{2\beta}$ and $K' > 0$ is some constant (which is independent of x). Comparing this with Theorem 1, we see that $P^x(N_t^{-\rho}(0,\infty) > 0)$ and $\mathbf{P}^x(\mathcal{N}_t(\rho t, \infty) > 0)$ have the same exponential decay rate, but the polynomial corrections decay like $x t^{-\frac{3}{2}}$ and $t^{-\frac{1}{2}}$, respectively.

The gap between these asymptotics comes from the extra effort particles need to make to avoid the origin, and we can intuitively explain the difference as follows.
Given a large number of *independent* particles, n, each with a very small probability, p, of a certain success (where np is small), then the probability at least one succeeds is $1-(1-p)^n \approx np$, the expected number of successes. Although particles in the BBM are *dependent* on one another through their common ancestors, we might still expect the probability that at least one particle succeeds in some difficult event to agree (asymptotically) with the expected number of particles that succeed. To calculate expected numbers of particles, we can make use of the 'Many-to-One' lemma (see Hardy and Harris [11], for example), which implies that for measurable f

$$\mathbf{E}^x \sum_{u \in N_t} f(Y_u(s) : s \leq t) = e^{\beta t} \mathbb{E}_0^x f(Y_s : s \leq t), \tag{3}$$

where Y is a standard Brownian motion under the measure $\mathbb{P}_0^x$. It follows that $\mathbf{E}^x(\mathcal{N}_t(\rho t, \infty)) = e^{\beta t}\mathbb{P}_0^x(Y(t) > \rho t)$ and $E^x_{-\rho}(N_t(0,\infty)) = e^{\beta t}\mathbb{P}^x_{-\rho}(Y(s) > 0, \forall s \in [0,t])$, giving rise to the required asymptotic in each case. Since $E(N) = E(N|N > 0)\mathbb{P}(N > 0)$ for $N \geq 0$, our result also says that the expected number of particles alive conditioned on at least one being alive tends to $2/\rho^2$. In fact, we would expect to have a Yaglom type result with

$$\mathbb{P}^x_{-\rho}\Big(N_t(0,\infty) = j \Big| N_t(0,\infty) > 0\Big) \to \pi_j$$

as $t \to \infty$, for some probability distribution $(\pi_j)_{j\geq 1}$ with finite mean equal to $2/\rho^2$. We do not offer a proof of this result here since, given the asymptotic result of Theorem 1, the method of Chauvin and Rouault [6, Thm. 3] used in the analogous standard BBM result should readily adapt.
Theorem 1 also complements the work on survival probabilities for the $(-\rho, \beta; \mathbb{R}^+)$-BBM of Kesten [16], who considered the 'critical' case $\rho = \sqrt{2\beta}$. (Kesten [16] actually allowed random numbers of offspring, but our dyadic branching results would generalise to this set-up without too much difficulty.) In this more delicate case, Kesten showed that when $\rho = \sqrt{2\beta}$, there exists a constant $C \in (0,\infty)$, depending only on β, such that for $x > 0$,

$$xe^{\rho x - C(\log t)^2} \leq P^x(R_t > 0)\exp\left(\frac{3\rho^2\pi^2}{2}\right)^{\frac{1}{3}} t^{\frac{1}{3}} \leq (1+x)e^{\rho x - C(\log t)^2}.$$

However the Many-to-One lemma implies that $E^x N_t^{-\rho}(0,\infty)$ is of order $t^{-\frac{3}{2}}$ when $\rho = \sqrt{2\beta}$, which is a slower decay in t than $P^x(R_t > 0)$. No precise asymptotic for $P^x(R_t > 0)$ is known when $\rho = \sqrt{2\beta}$, although we hope to treat this case in future work.
We now define an additive martingale for the $(-\rho, \beta; \mathbb{R}^+)$-BBM.

Lemma 2. *For all $\beta, \rho > 0$ the process*

$$V(t) := \sum_{u \in N_t^{-\rho}} Y_u(t) e^{\rho Y_u(t) + (\frac{1}{2}\rho^2 - \beta)t}$$

defines an additive martingale for the $(-\rho, \beta; \mathbb{R}^+)$-BBM.

Proving this is an easy application of the Many-to-One lemma and the branching Markov property. We can now use the martingale V to change measure on the probability space of the $(-\rho, \beta; \mathbb{R}^+)$-BBM, and use a spine construction to describe the behaviour of the BBM under the new measure. A spine change of measure is one in which the law of the single initial particle (the spine) is altered, and all sub-trees branching off from the spine behave as if under the original law P. We define a new measure $\mathbb{Q}^x$ on the same probability space as P^x via

$$\left.\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}P^x}\right|_{\mathcal{F}_s} = \frac{1}{x}e^{-\rho x}\sum_{u\in N_s^{-\rho}} Y_u(s)e^{\rho Y_u(s)+(\frac{1}{2}\rho^2-\beta)s} = \frac{V(s)}{V(0)}.$$

For full details of the construction of such changes of measure we refer the reader to the spine theory given in Hardy and Harris [11], but also see Chauvin and Rouault [6], and Kyprianou [18]. We note that $V(t)$ may also be obtained on taking a suitable conditional expectation of a single-particle martingale for the spine only, in the manner of Hardy and Harris [11], and then in calculations following their methods, one can prove the following.

Lemma 3. *Under $\mathbb{Q}^x$, the $(-\rho,\beta;\mathbb{R}^+)$-BBM can be reconstructed in law as:*

- *starting from position x, the initial ancestor diffuses as a Bessel-3 process;*
- *at rate 2β the initial ancestor undergoes fission producing two particles;*
- *one of these particles is selected uniformly at random;*
- *this chosen particle (the spine) repeats stochastically the behaviour of their parent;*
- *the other particle initiates from its birth position an independent copy of a $(-\rho,\beta;\mathbb{R}^+)$-BBM with law $P^{\cdot}$.*

The next result describes the convergence properties of V, and is proved in Section 4.

Theorem 4. *If $0<\rho<\sqrt{2\beta}$, then V is uniformly integrable and the events $\{V(\infty)>0\}$ and $\{\zeta=\infty\}$ agree up to a P^x-null set. If $\rho\geq\sqrt{2\beta}$ then, P^x-almost surely, $V(\infty)=0$.*

Moreover it can be shown that changing measure with V corresponds to conditioning the sub-critical process to 'survive forever', in the following sense.

Theorem 5. *Let $\rho>\sqrt{2\beta}$. For $s>0$ fixed and $A\in\mathcal{F}_s$, as $t\to\infty$,*

$$P^x(A|R_{s+t}>0)\to\mathbb{Q}^x(A).$$

Again, given the asymptotic expression for $P^x(R_t>0)$ from Theorem 1, the proof of Theorem 5 is nearly unchanged from the proof of the analogous result in Chauvin and Rouault [6, Thm. 4] for standard BBM, and so omitted.
Obtaining a spine construction by 'conditioning' on a null event, in the manner of Theorem 5, has been seen before for BBM in Chauvin and Rouault [6], and a related conditioning for the $(-\rho,\beta;\mathbb{R}^+)$-BBM appears in Harris *et al.* [14]; in both these cases, the conditioning event is $R_t>\lambda t+\theta$, for $\theta\in\mathbb{R}$ and suitably large $\lambda>0$. Similar ideas for superprocesses arise in 'immortal particle' constructions – see, for example, Evans [8]. More generally, in recent years spine constructions have been seen to be a powerful technique in the analysis of branching processes, and have yielded conceptually simple, elegant proofs of some important results. In particular, the reader is referred to Lyons *et al.* [20], Lyons [19], Kurtz *et al.* [17], Biggins and Kyprianou [2], Athreya [1], Kyprianou [18], Hardy and Harris [13], and references therein, for applications of the spine approach.

2 Survival probabilities

In our proof of Theorem 1 we use similar ideas to those seen in Chauvin and Rouault [6], involving links with nonlinear partial differential equations and the Brownian bridge. However, there are significant novelties here – in particular Proposition 6, and the study of the Brownian bridge conditioned to avoid the origin, which comes in Section 3.

Proof of Theorem 1. Define $v(t,x) := P^x(\zeta < t)$ and let $s \in [0,t]$. Then it follows from the branching Markov property that

$$v(t,x) = E^x\Big(P^x(\zeta < t|\mathcal{F}_s)\Big) = E^x\bigg(\prod_{u \in N_s^{-\rho}} v(t-s, Y_u(s))\bigg),$$

whence $\prod_{u\in N_s^{-\rho}} v(t-s, Y_u(s))$ is a product martingale on $[0,t]$. If we now set $u(t,x) := P^x(R_t > 0) = 1 - v(t,x)$ we have that $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^+)$ and satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial u}{\partial x} + \beta u(1-u)$$

with initial condition $u(0,x) = \mathbf{1}_{\{x>0\}}$ for $x > 0$, and boundary condition $u(t,0) \equiv 0$ for $t > 0$ (cf. Champneys *et al.* [5], or Chauvin and Rouault [6]).
Let Y be a Brownian motion with drift $-\rho$ started at $x > 0$ under $\mathbb{P}^x_{-\rho}$ (with associated expectation $\mathbb{E}^x_{-\rho}$) and let $\tau_0 := \inf\{s > 0 : Y_s = 0\}$. Then Itô's formula reveals that

$$M_t(s) := u(t-(s\wedge\tau_0), Y(s\wedge\tau_0))\exp\left(\beta\int_0^{s\wedge\tau_0}(1-u(t-\phi, Y(\phi))\,\mathrm{d}\phi\right)$$

is a $\mathbb{P}^x_{-\rho}$-local martingale on $[0,t]$. Furthermore, since $0 \le M_t(s) \le e^{\beta t}$ for all $s \in [0,t]$, $M_t(s)$ is a uniformly integrable martingale on $[0,t]$ and hence

$$u(t,x) = \mathbb{E}^x_{-\rho}\Big(\mathbf{1}_{\{Y_t>0\}}e^{\beta\int_0^t(1-u(t-\phi,Y_\phi))\,\mathrm{d}\phi}; \tau_0 > t\Big). \tag{4}$$

For later use, we note that applying Chernov's bound gives

$$\mathbb{P}^x_{-\rho}(Y_t > 0) \le \inf_{s\ge0}\mathbb{E}^x_{-\rho}(e^{sY_t}) = \inf_{s\ge0} e^{s(x-\rho t)+\frac{1}{2}s^2t} = e^{-\frac{x^2}{2t}+x\rho-\frac{1}{2}\rho^2t}$$

and so we obtain a simple (but useful) upper bound for u:

$$u(t,x) \le \mathbb{E}^x_{-\rho}(\mathbf{1}_{\{Y_t>0\}}e^{\beta t}) \le e^{\rho x-(\frac{1}{2}\rho^2-\beta)t}, \qquad \text{for all } t,x > 0. \tag{5}$$

We will now re-write our expression for u at (4) in terms of the Brownian bridge. Under $\mathbb{P}$, let $\{\mathbb{B}^t_{x,z}(\phi) : \phi \in [0,t]\}$ be a Brownian bridge from x to z over time t, with associated expectation $\mathbb{E}$. For further details about the Brownian bridge, and its properties we use below, we refer the reader to Borodin and Salminen [4, Ch. IV, 20–26]. Roughly speaking, a Brownian motion started at x and conditioned to be at the point z at time t is a Brownian bridge from x to z.

Then, by changing measure (twice) to deal with the drift, we find

$$\begin{aligned} u(t,x) &= \mathbb{E}^x_{-\rho}\Big(e^{\beta\int_0^t(1-u(t-\phi,Y_\phi))\,\mathrm{d}\phi};\tau_0>t\Big)\\ &= e^{\beta t}\mathbb{E}^x_0\Big(e^{-\rho Y_t-\frac12\rho^2 t}e^{-\beta\int_0^t u(t-\phi,Y_\phi)\,\mathrm{d}\phi};\tau_0>t\Big)\\ &= e^{\beta t}\int_0^\infty \mathbb{P}^x_0(Y_t\in\mathrm{d}z)\mathbb{E}\Big(e^{-\rho z-\frac12\rho^2 t}e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z}(\phi))\,\mathrm{d}\phi};\tau^t_0>t\Big)\\ &= e^{\beta t}\int_0^\infty \mathbb{P}^x_{-\rho}(Y_t\in\mathrm{d}z)\mathbb{E}\Big(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z}(\phi))\,\mathrm{d}\phi};\tau^t_0>t\Big), \end{aligned}$$

where $\tau_0^t=\tau_0^t(x,z):=\inf\{\phi\in[0,t]:\mathbb{B}^t_{x,z}(\phi)=0\}$. Hence

$$u(t,x)\frac{\sqrt{2\pi t^3}}{xe^{\rho x}}e^{(\frac12\rho^2-\beta)t}=\int_0^\infty \frac{t}{x}e^{-\rho z}e^{-\frac{1}{2t}(x-z)^2}\mathbb{E}\Big(e^{-\beta\int_0^t u(t-\phi,\mathbb{B}^t_{x,z}(\phi))\,\mathrm{d}\phi};\tau^t_0>t\Big)\,\mathrm{d}z. \tag{6}$$

Using the distributional equivalence

$$\int_0^t u(t-\phi,\mathbb{B}^t_{x,z}(\phi))\,\mathrm{d}\phi \overset{d}{=} \int_0^t u(\phi,\mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi,$$

obtained by a time-reversal, and the explicit form for the probability that the Brownian bridge avoids the origin, $\mathbb{P}(\tau_0^t>t)=1-\exp\big(-\frac{2xz}{t}\big)$, we may re-write the right-hand side of (6) as

$$\int_0^\infty \frac{t}{x}(1-e^{-\frac{2xz}{t}})e^{-\rho z}e^{-\frac{1}{2t}(x-z)^2}\mathbb{E}\Big(e^{-\beta\int_0^t u(\phi,\mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi}\Big|\tau^t_0>t\Big)\,\mathrm{d}z. \tag{7}$$

Now as $t\to\infty$, $0\le\frac{t}{x}(1-e^{-\frac{2xz}{t}})\uparrow 2z$, and so it is sufficient to show that, as $t\to\infty$,

$$\mathbb{E}\Big(e^{-\beta\int_0^t u(\phi,\mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi}\Big|\tau^t_0>t\Big)\to g(z) \tag{8}$$

for some function $g:[0,\infty)\to(0,1]$, since then by dominated convergence we have that the expression given at equation (7) tends to $\int_0^\infty 2ze^{-\rho z}g(z)\,\mathrm{d}z$ as $t\to\infty$.
The intuition for the convergence of the conditional expectation to a function of z is clear when we consider how the Brownian bridge behaves for large t. We can represent the Brownian bridge $\mathbb{B}^t_{z,x}(\phi)$ as

$$\mathbb{B}^t_{z,x}(\phi)=Y(\phi)-\frac{\phi}{t}Y(t)+\frac{\phi}{t}(x-z)+z,$$

where Y is a standard Brownian motion started at the origin. Since $Y_t/t\to0$ almost surely as $t\to\infty$, for $0<\phi\ll t$ the bridge is approximately $\mathbb{B}^t_{z,x}(\phi)\approx z+Y(\phi)$, which is to say the conditioning event $\{Y(t)\in\mathrm{d}x\}$ does not significantly affect the Brownian motion for small ϕ. Now a Brownian motion conditioned to avoid the origin is a Bessel-3 process and so, if we additionally condition the bridge to avoid the origin, we might expect that $\mathbb{B}^t_{z,x}(\phi)\approx X(\phi)$ for $\phi\ll t$, where X is a Bessel-3 process started at z. (In fact, we will see later that a Brownian bridge conditioned to avoid the origin is equal in law to a Bessel-3 bridge with the same start and end points. See, for example, Revuz and Yor [21, pp463–470] for the Bessel bridge definition and properties.) To make this idea watertight we will show that $\{\mathbb{B}^t_{z,x}(\phi)|\tau^t_0>t\}_{\phi\ge0}$ converges in distribution to $\{X(\phi)\}_{\phi\ge0}$ as $t\to\infty$. For all $t>0$ we define $\mathbb{B}^t_{z,x}(\phi)\equiv x$ for $\phi>t$ to ensure that the conditioned processes have paths in $D_{[0,\infty)}[0,\infty)$ – the set of càdlàg paths in $[0,\infty)$.

The exponential decay of $u(t,x)$ with respect to t – recall equation (5) – allows us essentially to neglect the contribution from the tail of the integral to the conditional expectation in equation (8). As the dependence of the conditioned bridge $\{\mathbb{B}^t_{z,x}(\phi)|\tau^t_0 > t\}_{\phi\geq 0}$ on x is only apparent as ϕ approaches t, this means that in the limit as $t \to \infty$ the conditional expectation in (8) is independent of x. In the next section we turn this intuitive explanation into a rigorous proof of the following proposition.

Proposition 6. *For each $z > 0$,*

$$\mathbb{E}\Big(e^{-\beta\int_0^t u(\phi,\mathbb{B}^t_{z,x}(\phi))\,\mathrm{d}\phi}\Big|\tau^t_0 > t\Big) \to \mathbb{E}^z_B\Big(e^{-\beta\int_0^\infty u(\phi,X(\phi))\,\mathrm{d}\phi}\Big) =: g(z) \in (0,1]$$

as $t \to \infty$, where $\{X(\phi) : \phi \geq 0\}$ is a Bessel-3 process under $\mathbb{P}^z_B$.

This completes the proof of equation (1). Equation (2) will follow shortly from the Many-to-One lemma and some one-particle calculations. For $t > 0$ and $x, y \in \mathbb{R}$, let $p_t(x,y)$ be the standard (driftless) Brownian transition density (with respect to Lebesgue measure). By the reflection principle, if $xy > 0$ then

$$q_t(x,y) := p_t(x,y) - p_t(x,-y)$$

is the transition density for Brownian motion killed at the origin. Then

$$\begin{aligned} e^{-\beta t}E^x_{-\rho}(N_t(0,\infty)) &= \mathbb{P}^x_{-\rho}(\tau_0 > t) \\ &= \mathbb{P}^x_0(e^{\rho(x-Y_t)-\frac{1}{2}\rho^2 t}; \tau_0 > t) = e^{\rho x-\frac{1}{2}\rho^2 t}\int_0^\infty e^{-\rho y}q_t(x,y)\,\mathrm{d}y \\ &\sim e^{\rho x-\frac{1}{2}\rho^2 t}\int_0^\infty e^{-\rho y}\frac{2xy}{t\sqrt{2\pi t}}\,\mathrm{d}y = \frac{2}{\rho^2}\times\frac{x}{\sqrt{2\pi t^3}}e^{\rho x-\frac{1}{2}\rho^2 t} \end{aligned}$$

as $t \to \infty$, as required. □

3 Proof of Proposition 6

To simplify notation, we will use the family of measures indexed by t, $\mathbb{P}^{z,x}_t$, with associated expectation $\mathbb{E}^{z,x}_t$, for the laws of the Brownian bridges from z to x over time t conditioned to avoid the origin (but recall that we have extended the bridge definitions to include times $\phi > t$). For the remainder of this section, it will be convenient to use the notation $X = \{X\}_{\phi\geq 0}$ for a process with paths in $C_{[0,\infty)}[0,\infty)$, where X is the conditioned Brownian bridge under $\mathbb{P}^{z,x}_t$, whilst X is a Bessel-3 process started at z under $\mathbb{P}^z_B$.

The main reference for the theory on weak convergence necessary in this section is Ethier and Kurtz [7]. For a metric space (E,r) denote by $D_E[0,\infty)$ the set of càdlàg paths in E; that is, right continuous functions $\nu : [0,\infty) \mapsto E$ with left limits. A metric d can be defined on $D_E[0,\infty)$ that generates the Skorohod topology and makes $(D_E[0,\infty), d)$ a complete and separable metric space whenever (E,r) is complete and separable. See Ethier and Kurtz [7, Ch. 3.5]. Even though all the processes we deal with are continuous, for convenience we make use of this general theory on $D_{[0,\infty)}[0,\infty)$.

We will make important use of the following weak convergence result, the proof of which is deferred until later.

Lemma 7. *As $t \to \infty$, $\mathbb{P}^{z,x}_t \Rightarrow \mathbb{P}^z_B$.*

Proof of Proposition 6. Fix $T > 0$ and let $t > T$. Define a functional of the process X for all $0 \le a \le b$ by

$$I(a,b) := \int_a^b u(\phi, X(\phi))\,\mathrm{d}\phi.$$

Now let $\varepsilon > 0$ and define

$$A(t) := \mathbb{E}_t^{z,x}\left(e^{-\beta I(0,T)-\beta I(T,t)}; \sup_{T\le\phi} \frac{X(\phi)}{\phi} > \varepsilon\right)$$
$$B(t) := \mathbb{E}_t^{z,x}\left(e^{-\beta I(0,T)-\beta I(T,t)}; \sup_{T\le\phi} \frac{X(\phi)}{\phi} \le \varepsilon\right),$$

so that $\mathbb{E}_t^{z,x}(e^{-\beta I(0,t)}) = A(t) + B(t)$.
Now by Lemma 7 and Ethier and Kurtz [7, Ch. 3, Thm. 3.1],

$$A(t) \le \mathbb{P}_t^{z,x}\left(\sup_{T\le\phi} \frac{X(\phi)}{\phi} > \varepsilon\right) \to \mathbb{P}_B^{z}\left(\sup_{T\le\phi} \frac{X(\phi)}{\phi} > \varepsilon\right) \qquad \text{as } t\to\infty.$$

Since $X(\phi)/\phi \to 0$ $\mathbb{P}_B^z$-almost surely as $\phi\to\infty$, the final probability in the line above can be made arbitrarily small (for any $\varepsilon > 0$) by letting $T\to\infty$.
To deal with the term $B(t)$, we bound it above and below with expressions that are equal (in the limit) to the required expectation as we first let $t\to\infty$, and then let $T\to\infty$. For the upper bound we have

$$B(t) \le \mathbb{E}_t^{z,x}(e^{-\beta I(0,T)}) \to \mathbb{E}_B^{z}(e^{-\beta I(0,T)})$$

as $t\to\infty$. This follows from Lemma 7 because $e^{-\beta\int_0^T u(\phi,\cdot)\,\mathrm{d}\phi}$ is a continuous bounded function of the sample paths. Letting $T\to\infty$, and using bounded convergence, we obtain

$$\limsup_{t\to\infty} B(t) \le \mathbb{E}_B^{z}(e^{-\beta I(0,\infty)}). \tag{9}$$

For the lower bound, recall that $\rho > \sqrt{2\beta}$, and so we can take $\varepsilon > 0$ sufficiently small that $\delta := -\rho\varepsilon + \left(\frac{1}{2}\rho^2 - \beta\right) > 0$. Then if $X(\phi) \le \varepsilon\phi$, using the upper bound for $u(t,x)$ at equation (5) we have $u(\phi, X(\phi)) \le e^{-\delta\phi}$ and hence

$$\begin{aligned}
B(t) &\ge \mathbb{E}_t^{z,x}\left(e^{-\beta I(0,T)-\beta\int_T^t e^{-\delta\phi}\,\mathrm{d}\phi}; \sup_{T\le\phi}\frac{X(\phi)}{\phi} \le \varepsilon\right)\\
&= \exp\Big(-\frac{\beta}{\delta}(e^{-\delta T} - e^{-\delta t})\Big)\mathbb{E}_t^{z,x}\left(e^{-\beta I(0,T)}; \sup_{T\le\phi}\frac{X(\phi)}{\phi} \le \varepsilon\right)\\
&\to \exp\Big(-\frac{\beta}{\delta}e^{-\delta T}\Big)\mathbb{E}_B^{z}\left(e^{-\beta I(0,T)}; \sup_{T\le\phi}\frac{X(\phi)}{\phi} \le \varepsilon\right)
\end{aligned}$$

as $t\to\infty$. Note that, although $e^{-\beta\int_0^T u(\phi,v(\phi))\,\mathrm{d}\phi}\mathbf{1}_{\{\sup_{T\le\phi}\frac{v(\phi)}{\phi}\le\varepsilon\}}$ is not continuous as a function of $v(\cdot)\in D_{[0,\infty)}[0,\infty)$, the set of discontinuities,

$$\left\{v(\cdot)\in D_{[0,\infty)}[0,\infty) : \sup_{T\le\phi}\frac{v(\phi)}{\phi} = \varepsilon\right\},$$

has $\mathbb{P}_B^z$-measure zero, and so the expectation does converge (see Billingsley [3, Thm. 2.7]). On letting $T\to\infty$, using bounded convergence we have

$$\liminf_{t\to\infty} B(t) \ge \mathbb{E}_B^{z}(e^{-\beta I(0,\infty)}), \tag{10}$$

and it follows from (9) and (10) that $\lim_{t\to\infty} B(t) = \mathbb{E}^z_B(e^{-\beta I(0,\infty)})$. Recalling that $A(t) \to 0$ as $t \to \infty$, we have shown that

$$\mathbb{E}^{z,x}_t\Big(e^{-\beta\int_0^t u(\phi,X(\phi))\,\mathrm{d}\phi}\Big) \to \mathbb{E}^z_B\Big(e^{-\beta\int_0^\infty u(\phi,X(\phi))\,\mathrm{d}\phi}\Big)$$

as $t \to \infty$.
It remains to show that the limit in the line above is strictly positive. Since the limit is bounded in $[0,1]$, it is sufficient to prove a $\mathbb{P}^z_B$-almost sure domination of $\int_0^\infty u(\phi, X(\phi))\,\mathrm{d}\phi$ by some finite quantity. By the law of large numbers, there exists a random $T_0 < \infty$ such that $\mathbb{P}^z_B$-almost surely, for all $\phi > T_0$, $X(\phi)/\phi < \varepsilon$. Then $\mathbb{P}^z_B$-almost surely $u(\phi, X(\phi)) \le e^{-\delta\phi}$ for all $\phi > T_0$, and as $u(\phi, X(\phi)) \le 1$ for $0 \le \phi \le T_0$ we have an almost sure domination. □

The remainder of this section is devoted to the proof of Lemma 7.

Proof of Lemma 7. We will first show that a Brownian bridge that is conditioned to avoid the origin has the same law as a Bessel bridge; and we will then note that the Bessel bridges have a density with respect to the Bessel process, from which weak convergence will easily follow. We shall use some standard h-transform theory – for further details and rigorous justification for any slight abuse of notation, see Revuz and Yor [21, Ch. XI] or Rogers and Williams [22, 23]. See also Borodin and Salminen [4, IV.20–IV.26] for some related calculations.
Let X be a Brownian motion under $\mathbb{P}^z$ and define the σ-algebras $\mathcal{F}_s := \sigma(X_u : 0 \le u \le s)$. Since a Brownian bridge from z to x over time t can be thought of as a standard Brownian motion Y started at z and conditioned to be at position x at time t, the law $\mathbb{P}^{z,x}_t$ of the conditioned bridge is related to $\mathbb{P}^z$ through the change of measure

$$\frac{\mathrm{d}\mathbb{P}^{z,x}_t}{\mathrm{d}\mathbb{P}^z}\bigg|_{\mathcal{F}_s} = \frac{\mathbb{P}^z(X(t)\in \mathrm{d}x; \tau_0 > t|\mathcal{F}_s)}{\mathbb{P}^z(X(t)\in \mathrm{d}x; \tau_0 > t)} = \mathbf{1}_{\{\tau_0>s\}}\frac{q_{t-s}(X(s),x)}{q_t(z,x)}. \tag{11}$$

(Recall that $q_t(z, x)$ is the transition density of a Brownian motion killed at the origin.)
It is also true that a Brownian motion conditioned to avoid the origin is a Bessel-3 process, and, since X is Bessel-3 process under $\mathbb{P}^z_B$, we have

$$\frac{\mathrm{d}\mathbb{P}^z_B}{\mathrm{d}\mathbb{P}^z}\bigg|_{\mathcal{F}_s} = \mathbf{1}_{\{\tau_0>s\}}\frac{X(s)}{z}.$$

In particular, this gives the transition density of the Bessel-3 process, $xq_s(z, x)/z$. Further, if X is a Bessel bridge under the measure $\mathbb{P}^{z,x}_{B,t}$ then we have

$$\frac{\mathrm{d}\mathbb{P}^{z,x}_{B,t}}{\mathrm{d}\mathbb{P}^z_B}\bigg|_{\mathcal{F}_s} = \frac{\mathbb{P}^z_B(X(t)\in \mathrm{d}x|\mathcal{F}_s)}{\mathbb{P}^z_B(X(t)\in \mathrm{d}x)} = \frac{xq_{t-s}(X(s),x)/X(s)}{xq_t(z,x)/z} = \frac{z}{X(s)}\frac{q_{t-s}(X(s),x)}{q_t(z,x)}.$$

Hence, we can write

$$\frac{\mathrm{d}\mathbb{P}^{z,x}_{B,t}}{\mathrm{d}\mathbb{P}^z}\bigg|_{\mathcal{F}_s} = \frac{\mathrm{d}\mathbb{P}^{z,x}_{B,t}}{\mathrm{d}\mathbb{P}^z_B}\frac{\mathrm{d}\mathbb{P}^z_B}{\mathrm{d}\mathbb{P}^z}\bigg|_{\mathcal{F}_s} = \mathbf{1}_{\{\tau_0>s\}}\frac{q_{t-s}(X(s),x)}{q_t(z,x)}. \tag{12}$$

Comparing the two changes of measure at (11) and (12), we immediately see that the law $\mathbb{P}^{z,x}_t$ of the Brownian bridge conditioned to avoid the origin and the law $\mathbb{P}^{z,x}_{B,t}$ of the Bessel bridge are identical. From now on we shall use only the $\mathbb{P}^{z,x}_t$ notation, and we can very conveniently interpret the conditioned Brownian bridges as Bessel bridges.

A calculation with the explicit expressions for the transition densities shows that

$$\frac{z}{y}\frac{q_{t-s}(y,x)}{q_t(z,x)} = \frac{z}{y}\sqrt{\frac{t}{t-s}}\frac{\exp\big(-\frac{(y-x)^2}{2(t-s)}\big)\big(1-e^{\frac{-2yx}{t-s}}\big)}{\exp\big(-\frac{(z-x)^2}{2t}\big)\big(1-e^{\frac{-2zx}{t}}\big)} \to 1$$

as $t \to \infty$. This yields

$$\frac{\mathrm{d}\mathbb{P}_t^{z,x}}{\mathrm{d}\mathbb{P}_B^z}\bigg|_{\mathcal{F}_s} = \frac{z}{X(s)}\frac{q_{t-s}(X(s),x)}{q_t(z,x)} =: M_t(s) \to 1$$

as $t \to \infty$. Furthermore, since the martingale property gives $\mathbb{E}_B^z(M_t(s)) = 1$ for all $t > s$, we find that $M_t(s) \to 1$ in $\mathcal{L}^1(\mathbb{P}_B^z)$. (See Revuz and Yor [21, Ch. XI Ex. (3.10)] or Kallenberg [15, Lem. 1.32].) Now for any $A \in \mathcal{F}_s$, $\mathbf{1}_A M_t(s) \le M_t(s)$, and so dominated convergence (as stated in Kallenberg [15, Thm. 1.21]) gives

$$\mathbb{P}_t^{z,x}(A) = \mathbb{P}_B^z(\mathbf{1}_A M_t(s)) \to \mathbb{P}_B^z(A) \qquad \text{as } t \to \infty.$$

It is now immediate that we have weak convergence on *any* finite time interval $[0,s]$, and this in turn implies that we have weak convergence on the whole of $[0,\infty)$. (For example, see Billingsley [3, Thm. 16.7].) $\square$

4 Convergence properties of V

The uniform-integrability assertion of Theorem 4 follows from the following, slightly stronger, convergence result.

Proposition 8. *For $x > 0$ and any $p \in (1,2]$,*

(i) the martingale V is $\mathcal{L}^p(P^x)$-convergent if $p\rho^2/2 < \beta$;

(ii) almost surely under P^x, $V(\infty) = 0$ when $\rho \ge \sqrt{2\beta}$.

We do not give a proof of this result here: it is a simple adaptation of the spine approach to $\mathcal{L}^p$ convergence described in Hardy and Harris [12].

Proof of Theorem 4. If $\rho \ge \sqrt{2\beta}$ then, by Proposition 8, $V(\infty) = 0$ almost surely.
If $\rho < \sqrt{2\beta}$ then it follows from Proposition 8 that there exists a $p > 1$ such that V converges in $\mathcal{L}^p(P^x)$, and so, by Doob's $\mathcal{L}^p$ inequality, V is uniformly integrable. It remains to show that $V(\infty) > 0$ almost surely on $\{\zeta = \infty\}$.
To prove that $P^x(V(\infty) = 0; \zeta = \infty) = 0$ when $\rho < \sqrt{2\beta}$, we note that

$$\begin{aligned} P^x(V(\infty) = 0) &= P^x(V(\infty) = 0; \zeta = \infty) + P^x(V(\infty) = 0; \zeta < \infty) \\ &= P^x(V(\infty) = 0; \zeta = \infty) + P^x(\zeta < \infty) \end{aligned}$$

and hence it suffices to show that $P^x(V(\infty) = 0) = P^x(\zeta < \infty)$.
To show this, we make use of the following result on existence and uniqueness for solutions of the 'one-sided FKPP equation', proved in Harris *et al.* [14].

Theorem 9. *The system*

$$\begin{aligned}\frac{1}{2}f'' - \rho f' + \beta(f^2 - f) &= 0 \text{ on } (0,\infty)\\ f(0+) &= 1 \\ f(\infty) &= 0\end{aligned} \tag{13}$$

has a unique solution in $\{f \in C^2(0,\infty) : 0 \le f(x) \le 1, \forall x \in (0,\infty)\}$ *if and only if* $-\infty < \rho < \sqrt{2\beta}$, *in which case the unique solution is* $f(x) = P^x(\zeta < \infty)$.
If $\rho \ge \sqrt{2\beta}$ *there is no solution to the system* (13).

We now show that $p(x) := P^x(V(\infty) = 0)$ is a solution to (13) when $0 < \rho < \sqrt{2\beta}$. We have

$$p(x) = E^x\Big(P^x(V(\infty) = 0|\mathcal{F}_t)\Big) = E^x\bigg(\prod_{u\in N_t^{-\rho}} p(Y_u(t))\bigg), \tag{14}$$

whence $p(x)$ satisfies the ODE in the system (13). Since extinction in a finite time guarantees that $V(\infty) = 0$, we also have $\lim_{x\downarrow 0} p(x) = 1$. Considering the process path-wise, we see that increasing x increases the value of V under the law P^x. Recall here that $\rho > 0$, so $xe^{\rho x}$ is increasing in x. Hence $p(x)$ is monotone decreasing in x and $p(x) \downarrow p(\infty)$ as $x \to \infty$.
Now consider the $(-\rho, \beta; \mathbb{R}^+)$-BBM as a subprocess of the BBM we would see if the killing barrier at the origin were removed; retaining our naming convention we will call this process the $(-\rho, \beta; \mathbb{R})$-BBM. Let $\mathcal{N}_t^{-\rho}$ be the set of particles alive at time t in the $(-\rho, \beta; \mathbb{R})$-BBM. Take any fixed infinite BBM tree started at x. For any fixed time $t > 0$, we have $N_t^{-\rho} \uparrow \mathcal{N}_t^{-\rho}$ as $x \to \infty$. Looking at the process path-wise again, for all $u \in N_t^{-\rho}$ we have $Y_u(t) \uparrow \infty$ as $x \to \infty$. Taking the limit $x \to \infty$ in (14) we then have $p(\infty) = E^0\big(\prod_{u\in\mathcal{N}_t^{-\rho}} p(\infty)\big)$, whence $p(\infty) \in \{0, 1\}$ and now uniform integrability of V forces $p(\infty) = 0$. Uniqueness of the one-sided FKPP solution (Theorem 9) now finishes the argument. □

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