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LARGE DEVIATION PRINCIPLES FOR MARKOV PROCESSES VIA Φ-SOBOLEV INEQUALITIES

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Abstract
Via Φ-Sobolev inequalities, we give some sharp integrability conditions on F for the large deviation principle of the empirical mean $\frac{1}{T}\int_0^T F(X_s)ds$ for large time T, where F is unbounded with values in some separable Banach space. Several examples are provided.

1 Introduction

Let $(\Omega, (\mathcal{F}_t)_{t\in R^+}, (X_t)_{t\in R^+}, (\mathbb{P}_x)_{x\in E})$ be a conservative *càdlàg* Markov process with values in a Polish space E, with semigroup of transition probability $P_t(x, dy)$. We assume that μ is a probability measure on E (equipped with the Borel σ-field $\mathcal{B}$),which is invariant with respect to (w.r.t.) (P_t). We assume throughout this paper that P_s is ergodic w.r.t. μ for some $s > 0$, i.e., if a bounded measurable function f satisfies $P_s f = f$, $\mu - a.e.$, then f is $\mu - a.e.$ constant. For any initial measure ν on E, write $\mathbb{P}_\nu := \int_E \mathbb{P}_x \nu(dx)$.
Given a measurable function $F : E \to \mathbb{B}$, μ-integrable where $(\mathbb{B}, \|\cdot\|)$ is some separable Banach space, we are interested in the probability of large deviation of the empirical mean $L_T(F) := \dfrac{1}{T}\displaystyle\int_0^T F(X_s)ds$ from its space mean $\mu(F) := \int_E F d\mu$, i.e.

$$\mathbb{P}_\nu\left(\left\|\frac{1}{T}\int_0^T F(X_s)ds - \mu(F)\right\| > r\right).$$

In [26, Corollary 5.4], the following uniform integrability criterion is proved: if

$$\{f^2;\ f \in \mathbb{D}_2(\mathcal{L}),\ \langle -\mathcal{L}f, f\rangle_\mu + \langle f, f\rangle_\mu \le 1\} \tag{1.1}$$

is μ-uniformly integrable, then the law of the occupation measure $L_t := \frac{1}{t}\int_0^t \delta_{X_s} ds$ under $\mathbb{P}_\nu$ satisfies the large deviation principle (LDP in short) on the space of probability measures $M_1(E)$ on E with respect to the τ-topology (i.e., the weak topology $\sigma(M_1(E), b\mathcal{B})$, here $b\mathcal{B}$ is the space of real bounded and $\mathcal{B}$-measurable functions on E), for every initial measure $\nu \ll \mu$ with $d\nu/d\mu \in L^{1+\delta}(\mu)$. Here $\mathcal{L}$ with domain $\mathbb{D}_p(\mathcal{L})$ is the generator of the transition semigroup (P_t) in $L^p(\mu) := L^p(E, \mathcal{B}, \mu)$, $1 \le p \le \infty$. The symmetrized Dirichlet form is given by

$$\mathcal{E}(f, g) = \frac{1}{2}(\langle -\mathcal{L}f, g\rangle_\mu + \langle -\mathcal{L}g, f\rangle_\mu), \forall f, g \in \mathbb{D}_2(\mathcal{L}). \tag{1.2}$$

Furthermore the uniform integrability condition above becomes necessary for that LDP in the symmetric case ([26, Corollary 5.5]).
When F is bounded, the mapping $\nu \to \nu(F)$ is continuous from $M_1(E)$ (equipped with the τ-topology) to $\mathbb{B}$ ([7]). By the contraction principle, $\mathbb{P}_\nu(L_t(F) \in \cdot)$ satisfies the LDP for all bounded measurable $F : E \to \mathbb{B}$ under the uniform integrability condition (1.1). Our main purpose is to extend the last result for unbounded observable F. It turns out that this question is intimately related with the Φ-Sobolev inequality studied in Gong-Wang [13]:

$$\int_E \Phi(f^2) d\mu \le C_1 \langle -\mathcal{L}f, f\rangle_\mu + C_2,\ \forall f \in \mathbb{D}_2(\mathcal{L}),\ \int_E f^2 d\mu = 1 \tag{1.3}$$

for some even function $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, convex on $\mathbb{R}^+$ such that $\Phi(0) = 0$ and

$$\lim_{r\to+\infty} \frac{\Phi(r)}{r} = +\infty. \tag{1.4}$$

The Φ-Sobolev inequality (1.3), giving a quantitative measurement of the uniform integrability of the set (1.1), is stronger than the more classical Φ-Sobolev inequality below

$$\|f^2\|_\Phi \le C_1 \langle -\mathcal{L}f, f\rangle_\mu + \tilde{C}_2 \langle f, f\rangle_\mu,\ \forall f \in \mathbb{D}_2(\mathcal{L}) \tag{1.5}$$

where $\tilde{C}_2 = \max\{C_2, 1\}$, $\|\cdot\|_\Phi$ denotes the Orlicz norm of the Orlicz space $L^\Phi(\mu)$ associated with Φ. This last inequality is studied by Roberto-Zegarlinski [19], who proved in further that a tight version of (1.5) implies (1.3) (this was due to Bobkov-Götze [2] when $\Phi(x) = x\log x$). See also Chafai [4] for related Φ-entropy Sobolev inequality.
Obviously the Φ-Sobolev inequalities (1.3) and (1.5) imply (1.1), and the inverse is true for some such Φ as shown by Gong-Wang [13]. Two typical examples are

(i) $\Phi(x) = |x|\log|x|$: the Φ-Sobolev inequality (1.3) becomes the defective logarithmic Sobolev inequality.

(ii) $\Phi(x) = |x|^p$ for some $p \in (1, +\infty)$, (1.5) is exactly the classical Sobolev inequality.

Our objective is to derive the LDP of $L_t(F)$ under sharp integrability condition on F via the Φ-Sobolev inequality (1.3) or (1.5). This becomes important and very practical because of recent great progress on functional inequalities, see the textbooks by M.F. Chen [5], M. Ledoux [16], F.Y. Wang [21], and the recent works by Barthe, Cattiaux and Roberto [1] and Cattiaux-Guillin [3] and the references therein.
Our question is motivated directly by the well-known Donsker-Varadhan's theorem: for a sequence $(\xi_k)_{k\ge1}$ of independent and identically distributed (i.i.d.) random variables valued in $\mathbb{B}$, if

$$\mathbb{E}e^{\lambda\|\xi_1\|} < +\infty, \ \forall \lambda > 0, \tag{1.6}$$

then $\frac{1}{n}\sum_{k=1}^n \xi_k$ satisfies the LDP on $\mathbb{B}$. Furthermore if $\mathbb{B}$ is of infinite dimension, the condition (1.6) can not be weakened to

$$\mathbb{E}e^{\delta\|\xi_1\|} < +\infty, \quad \text{for some } \delta > 0. \tag{1.7}$$

This last condition is the necessary and sufficient condition in the finite dimension case $\mathbb{B} = \mathbb{R}^d$ (well known), and for general separable Banach space $\mathbb{B}$ see F.Q. Gao [10] for a necessary and sufficient condition.

It is striking that for continuous time Markov processes, the exponential integrability conditions (1.6) and (1.7) can be sometimes largely weakened as shown in this paper.

We mention also another important application of the Φ-Sobolev inequalities (1.3) and (1.5) to the concentration inequalities for

$$\mathbb{P}_\nu\left(L_t(V) - \mu(V) \geq r\right)$$

where V is real valued. Indeed the first named author [25] proved that for every measurable function $V : E \to \mathbb{R}$,

$$\mathbb{P}_\nu\left(L_t(V) - \mu(V) \geq r\right) \leq \left\|\frac{d\nu}{d\mu}\right\|_2 e^{-tI_V(\mu(V)+r)}, \ \forall r > 0, t > 0 \tag{1.8}$$

where

$$I_V(r) := \inf\{\langle -\mathcal{L}f, f\rangle_\mu \mid f \in \mathbb{D}_2(\mathcal{L}) \cap L^2(|V|d\mu), \ \int_E f^2 V d\mu = r\}, \tag{1.9}$$

(this is sharp in the large deviation scale in the symmetric case) and derived a Cramer-Chernoff type concentration inequality under the log-Sobolev inequality. Lezaud [17] established a Cramer-Chernoff exponential inequality, which is sharp in the scale of moderate deviations, for real bounded V from the Poincaré inequality

$$\lambda_1\langle f, f\rangle_\mu \leq \langle -\mathcal{L}f, f\rangle_\mu, \ \forall f \in \mathbb{D}_2(\mathcal{L}), \ \mu(f) = 0.$$

Recently P. Cattiaux and A. Guillin [3] study thoroughly the concentration inequalities under various functional inequalities.

For literatures on large deviations of Markov processes the reader is referred to [9, 7, 8, 26] etc. For preceding works on applications of Φ-Sobolev inequality in large deviations, see F.Q. Gao [11]. See also the first named author [27] for LDP of $L_t(F)$ with unbounded F under the Lyapunov function condition.

This note is organized as follows. The main results are stated in the next section. Several examples are provided in §3, based on recent progresses on functional inequalities. Finally we prove the main results in the last sections, by means of approximation techniques in large deviations and convex analysis : the key being the inequality (1.8) and the theory of Orlicz spaces.

2 Main results

2.1 Orlicz space

A Young function Φ is a convex even function $\Phi : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$, left continuous on $\mathbb{R}^+$ such that:

$$i)\ \Phi(0) = 0; \quad ii)\ \lim_{x\to\infty} \Phi(x) = +\infty. \quad iii)\ \sup\{x > 0; \Phi(x) < +\infty\} > 0.$$

Any Young function Φ admits an integral representation, i.e., $\Phi(x) = \int_0^x \varphi(t)dt\ x \in [0, \infty)$ where $\varphi : [0, \infty) \to [0, \infty]$ is nondecreasing, right continuous.
Let ψ be the generalized inverse of φ, that is

$$\psi(s) = \inf\{t|\ \varphi(t) > s\},\ \forall s \in [0, +\infty)$$

($\inf \emptyset := +\infty$), then the conjugate function of Φ is the Young function $\Psi = \Phi^*$ defined by its Legendre transformation:

$$\Psi(y) = \sup_{x\in\mathbb{R}} (xy - \Phi(x)) = \int_0^{|y|} \psi(t)dt, \quad y \in \mathbb{R}. \tag{2.1}$$

It follows from the definition that $\Psi(0) = 0, \Psi(-y) = \Psi(y)$, and, what is important, $\Psi(\cdot)$ is a convex increasing function satisfying $\lim_{y\to\infty} \Psi(y) = +\infty$.
From (2.1) it is evident that the couple (Φ, Ψ) satisfies Young's inequality:

$$|xy| \le \Phi(x) + \Psi(y), \quad x, y \in \mathbb{R}, \tag{2.2}$$

and the equality holds if $y = \varphi(x)$ or $x = \psi(y)$(see [18, theorem 3, p10]).
Let $(E, \mathcal{B}, \mu)$ be a probability space. The Orlicz space $L^\Phi(\mu)$ associated with the Young function Φ is the space of all real measurable functions $u : E \to \mathbb{R} \cup \{+\infty\}$ such that $\Phi(\alpha|u|)$ is μ-integrable for some $\alpha > 0$. Thus

$$L^\Phi(\mu) = \{u : E \to \mathbb{R} \cup \{+\infty\}| \int_E \Phi(\alpha|u|)d\mu < +\infty,\ for\ some\ \alpha > 0\}.$$

Define the gauge norm

$$N_\Phi(u) := \inf\left\{K > 0| \int_E \Phi\left(\frac{|u|}{K}\right) d\mu \le 1\right\} \tag{2.3}$$

Then [18, Theorem 3, p54] tells us that $(L^\Phi(\mu), N_\Phi(\cdot))$ is a Banach space when μ-equivalent functions are identified. Moreover $N_\Phi(u) \le 1$ iff $\int_E \Phi(u)d\mu \le 1$.
Define the Orlicz norm $\|\cdot\|_\Phi$ as:

$$\|f\|_\Phi = \sup\left\{\int_E |fg|d\mu;\ \int_E \Psi(|g|)d\mu \le 1\right\} \tag{2.4}$$

Then [18, Proposition 4, p61] gives a useful relation between the Orlicz and gauge norms, i.e.

$$N_\Phi(f) \le \|f\|_\Phi \le 2N_\Phi(f), \quad \forall f \in L^\Phi(\mu).$$

2.2 Main results

Now recall the modified Donsker-Varadhan's level-2 entropy functional $J_\mu : M_1(E) \to [0, +\infty]$, introduced in [26]. It is given by (see [26, Proposition B.10(B.26)])

$$J_\mu(\nu) := \begin{cases} \sup_{1 \le u \in \mathbb{D}_\infty(\mathcal{L})} \int_E -\frac{\mathcal{L}u}{u} d\nu, & \text{if } \nu \ll \mu; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.5}$$

Here we say that $u \in \mathbb{D}_\infty(\mathcal{L})$ and $v = \mathcal{L}u$ if $u, v \in L^\infty(\mu)$ and $P_t u - u = \int_0^t P_s v ds, \mu - a.s.$ for every $t \ge 0$.
The main result of this note is

Theorem 2.1. *Assume either*

$$\int_E \Phi(f^2) d\mu \le C_1 \langle -\mathcal{L}f, f \rangle_\mu + C_2, \quad \forall f \in \mathbb{D}_2(\mathcal{L}), \int_E f^2 d\mu = 1 \tag{2.6}$$

where $\Phi : \mathbb{R} \to \mathbb{R} \bigcup \{+\infty\}$ *is even, convex on* $\mathbb{R}^+$ *and satisfies (1.4); or*

$$\|f^2\|_\Phi \le C_1 \langle -\mathcal{L}f, f \rangle_\mu + C_2 \mu(f^2), \quad \forall f \in \mathbb{D}_2(\mathcal{L}), \tag{2.7}$$

where Φ *is a Young function satisfying (1.4).*
Let $\Psi = \Phi^*$ *be its conjugate function given in the first equality of (2.1) and* $(\mathbb{B}, \|\cdot\|)$ *a separable Banach space. Then for every measurable* $\mathbb{B}$*-valued function* $F : E \to \mathbb{B}$ *verifying*

$$\int_E \Psi(\lambda \|F\|) d\mu < +\infty, \ \forall \lambda > 0 \tag{2.8}$$

$\mathbb{P}_\nu(L_t(F) \in \cdot)$ *satisfies as* t *goes to infinity the large deviation principle (LDP in short) on* $\mathbb{B}$ *with speed* t *and with the rate function given*

$$J_\mu^F(z) = \inf \left\{ J_\mu(\beta); \ \beta \in M_1(E), \int_E \|F\| d\beta < +\infty, \ \int_E F d\beta = z \right\}, \ z \in \mathbb{B} \tag{2.9}$$

uniformly over initial measures ν *in*

$$\mathcal{A}_L := \{\nu \in M_1(E); \ \nu \ll \mu; \left\| \frac{d\nu}{d\mu} \right\|_2 \le L\}$$

for every $L \ge 1$*. More precisely, the following three properties hold:*

(i) $J_\mu^F : \mathbb{B} \to [0, +\infty]$ *is inf-compact, i.e.,* $[J_\mu^F \le l]$ *is compact in* $\mathbb{B}$ *for every* $l \in \mathbb{R}^+$*;*

(ii) (lower bound) *for all* $L \ge 1$ *and each open subset* O *of* $\mathbb{B}$*,*

$$\liminf_{t \to \infty} \frac{1}{t} \log \inf_{\nu \in \mathcal{A}_L} \mathbb{P}_\nu(L_t(F) \in O) \ge - \inf_O J_\mu^F;$$

(iii) (upper bound) *for all* $L \ge 1$ *and each closed subset* C *of* $\mathbb{B}$*,*

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{\nu \in \mathcal{A}_L} \mathbb{P}_\nu(L_t(F) \in C) \le - \inf_C J_\mu^F.$$

By means of the super-Poincaré inequality due to Wang [23], we can deduce from Theorem 2.1,

Theorem 2.2. *Let Φ be some finite Young function satisfying (1.4) and the Δ_2-growth condition*

$$\sup_{x\gg 1} \Phi(2x)/\Phi(x) < \infty. \tag{2.10}$$

Assume the following super Φ-Sobolev inequality: for every $\varepsilon > 0$, there is some constant $C(\varepsilon) \geq 1$ such that

$$\int_E \Phi(f^2) d\mu \leq \varepsilon \langle -\mathcal{L} f, f \rangle_\mu + C(\varepsilon), \quad \forall f \in \mathbb{D}_2(\mathcal{L}), \int_E f^2 d\mu = 1. \tag{2.11}$$

Let $\Psi, \mathbb{B}$ be as in Theorem 2.1. Then for every measurable $\mathbb{B}$-valued function $F : E \to \mathbb{B}$ satisfying

$$\int_E \Psi(\delta \|F\|) d\mu < +\infty, \text{ for some } \delta > 0, \tag{2.12}$$

the LDP for $L_t(F)$ in Theorem 2.1 holds true.

2.3 Several corollaries

Corollary 2.3. *Assume the defective log-Sobolev inequality:*

$$\int_E f^2 \log f^2 d\mu - \langle f^2 \rangle_\mu \log \langle f^2 \rangle_\mu \leq C_1 \langle -\mathcal{L} f, f \rangle_\mu + C_2 \int_E f^2 d\mu, \quad f \in \mathbb{D}_2(\mathcal{L}) \tag{2.13}$$

for some constants $C_1, C_2 > 0$. If $F : E \to \mathbb{B}$ verifies

$$\int_E e^{\lambda \|F\|} d\mu < +\infty, \ \forall \lambda > 0, \tag{2.14}$$

then the LDP for $L_t(F)$ in Theorem 2.1 holds true.

Proof. By (2.13), the Φ-Sobolev inequality (2.6) holds with

$$\Phi(t) = (|t| + 1) \log(|t| + 1) - |t|, \ \forall t \in \mathbb{R}$$

for $\Phi(t) \leq C(|t| \log |t| + 1)$ over $\mathbb{R}$ for some $C > 1$. By easy calculus the conjugate function of Φ is given by $\Psi(s) = e^{|s|} - |s| - 1$ for all $s \in \mathbb{R}$. Thus the corollary follows by Theorem 2.1. □

Corollary 2.4. *Assume the super log-Sobolev inequality: $\forall \varepsilon > 0, \ \exists C(\varepsilon) > 0$ such that*

$$\int_E f^2 \log f^2 d\mu - \langle f^2 \rangle_\mu \log \langle f^2 \rangle_\mu \leq \varepsilon \langle -\mathcal{L} f, f \rangle_\mu + C(\varepsilon) \int_E f^2 d\mu, \quad f \in \mathbb{D}_2(\mathcal{L}). \tag{2.15}$$

If $F : E \to \mathbb{B}$ satisfies the exponential integrability condition of type (1.7), i.e.,

$$\int e^{\delta \|F\|} d\mu < +\infty \ \text{ for some } \ \delta > 0,$$

then the LDP for $L_t(F)$ in Theorem 2.1 holds true.

Proof. By the proof of Corollary 2.3, the super Φ-Sobolev inequality (2.11) holds with $\Phi(t) = (|t|+1)\log(|t|+1) - |t|$. The corollary follows by Theorem 2.2. □

Remarks 2.5. When (P_t) is symmetric on $L^2(\mu)$, then the defective log-Sobolev inequality (2.13) is equivalent to the hypercontractivity of $e^{-C_2 t}P_t$, due to Gross [14], and super log-Sobolev inequality (2.15) is equivalent to the supercontractivity of (P_t) below: for all $t > 0, 1 < p < q < +\infty$, $\|P_t\|_{L^p(\mu)\to L^q(\mu)} < +\infty$, see E. Davies [6] and Kavian, Kerkyacharian and Roynette [15].

Remarks 2.6. Under the defective log-Sobolev inequality, we re-find the sharp integrability condition (1.6) to the classical LDP of Donsker-Varadhan's theorem in [9] in the i.i.d. case, recalled in the introduction. But our condition (2.15) in the super log-Sobolev inequality case is much weaker, but it coincides with the necessary and sufficient condition (1.7) to the LDP of empirical mean in the finite dimension case $\mathbb{B} = \mathbb{R}^d$.

A surprise in the continuous time case is : the strong exponential integrability condition (2.14) can be largely weakened in the case of Sobolev inequality:

Corollary 2.7. *Assume the Sobolev inequality: for some $p \in (1, +\infty]$ and constants $C_1, C_2 > 0$*

$$\|f^2\|_p \le C_1\langle -\mathcal{L}f, f\rangle_\mu + C_2 \tag{2.16}$$

for any $f \in \mathbb{D}(\mathcal{L})$ with $\int_E f^2 d\mu = 1$. If $F : E \to \mathbb{B}$ verifies $\|F\| \in L^q(\mu)$ where $1/p + 1/q = 1$, then the LDP for $L_t(F)$ in Theorem 2.1 holds true.

Proof. It follows directly by Theorem 2.1 with $\Phi(t) = |t|^p$ when $p < +\infty$ and with $\Phi(t) = +\infty 1_{(1,+\infty)}(|t|)$ when $p = +\infty$. □

3 Several Examples

In this section, basing on recent progresses on functional inequalities we present several Markov processes which satisfy the Φ-Sobolev inequalities in our assumptions.

Example 3.1. As a well known fact (see Saloff-Coste [20]), the Brownian Motion (B_t) on a compact connected Riemannian manifold M of dimension n with the invariant measure μ given by the normalized Riemannian measure $\frac{dx}{V(M)}$(where $V(M)$ is the volume of M), the Dirichlet form $\int |\nabla f|^2 d\mu$ satisfies the Φ-Sobolev inequality (2.7) with

$$\Phi(t) = \begin{cases} +\infty I_{(1,\infty)}(|t|), & \text{if} \quad n = 1, \\ \exp(C|t|) - 1, & \text{if} \quad n = 2, \\ |t|^{\frac{2n}{n-2}}, & \text{if} \quad n \ge 3. \end{cases}$$

Then the corresponding integrability condition for the LDP of $L_t(F)$ in Theorem 2.1 (and Corollary 2.7) becomes

$$\|F\| \in \begin{cases} L^1(\mu), & \text{if} \quad n = 1, \\ L^1 \log L^1, & \text{if} \quad n = 2, \\ L^{\frac{2n}{n+2}}(\mu), & \text{if} \quad n \ge 3. \end{cases}$$

Those still hold for diffusions generated by $\Delta + b\cdot\nabla$ with smooth vector field b on connected compact manifolds.

Example 3.2. Consider the measure $\mu_\beta(dx) = \dfrac{\exp(-|x|^\beta)}{Z_\beta}$ (where Z_β is the normalized constant), and $\beta > 1$. For the diffusion process corresponding to the Dirichlet form $\langle -\mathcal{L}f, f\rangle_\mu = \frac{1}{2}\int |\nabla f|^2 d\mu$, it satisfies Φ-Sobolev inequality (2.7) with

$$\Phi_\alpha(x) = x\log^\alpha(1+x),\ \ \alpha = 2(1-1/\beta)$$

according to Barthe, Cattiaux and Roberto [1, section 7]. Our integrability condition for F in Theorem 2.1 becomes

$$\int \exp\left(\lambda\|F\|^{\beta/(2\beta-2)}\right) d\mu < +\infty,\ \ \forall\lambda > 0. \tag{3.1}$$

Example 3.3. *(Φ-Sobolev inequalities for one-dimensional diffusion process, see [1, 2, 5, 19])* Let $\mathcal{L} = a(x)d^2/dx + b(x)d/dx$ be an elliptic operator on an interval $(-\infty,+\infty)$(*i.e.* $E = (-\infty,+\infty)$), where $a > 0$ and b are locally bounded Borel measurable function and $1/a$ is also locally bounded. Set $C(x) = \int_0^x b/a dx$. Let $d\mu := e^C/adx$, assume $Z := \int_0^{+\infty} e^C/adx < \infty$ (i.e. the invariant measure is finite).
A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called a N-function if it is non-negative, continuous, convex, even and satisfies the following conditions:

$$\Phi(x) = 0 \Leftrightarrow x = 0, \quad \lim_{x\to 0}\Phi(x)/x = 0, \quad \lim_{x\to\infty}\Phi(x)/x = \infty.$$

Assume that Φ is a N-function and satisfies the $\triangle_2$-condition (2.10). Let $\theta \in (-\infty,+\infty)$ be a reference point. Define $h^{1\theta}(x) = \int_\theta^x e^{-C}dx$, $h^{2\theta}(x) = \int_x^\theta e^{-C}dx$ and let

$$B_\Phi^{1\theta} = \sup_{x\in(\theta,+\infty)} h^{1\theta}(x)\|I_{(x,+\infty)}\|_\Phi, \quad B_\Phi^{2\theta} = \sup_{x\in(-\infty,\theta)} h^{2\theta}(x)\|I_{(-\infty,x)}\|_\Phi \tag{3.2}$$

the modified Muckenhoupt's constants corresponding to the intervals $(\theta,+\infty)$ and $(-\infty,\theta)$.
According to [1] or [5, Chapiter 6] or [19] (which is a combination of Muckenhoupt's generalized Hardy inequality together with the theory of Orlicz spaces), *the Φ-Sobolev inequality (2.7) holds iff* $B_\Phi^{1\theta}\vee B_\Phi^{2\theta} < \infty$.

Example 3.4. Let $E = M$ be a noncompact, connected, complete Riemannian manifold. Consider $\mathcal{L} = \Delta + Z$ with some C^1-vector field Z such that the $\mathcal{L}$-diffusion process has an invariant probability measure μ. Let $\mathcal{E}(f,g) = \langle -Lf, g\rangle_\mu$ with $\mathcal{D}(\mathcal{E})$ being the closure of $C_0^\infty(M)$ under the H^1-norm $\sqrt{\mu(f^2) + \langle -\mathcal{L}f, f\rangle_\mu}$. We assume that there exists $K \geq 0$ such that

$$Ric(X,X) - \langle\nabla_X Z, X\rangle \geq -K|X|^2, \quad X\in TM. \tag{3.3}$$

Let $\rho(x)$ denote the Riemannian distance between x and a fixed point $o\in M$.

1) If $\mu(\exp[\delta\rho^2]) < \infty$ for some $\delta > K/2$, then μ satisfies the log-Sobolev inequality (Wang's criterion, [23]). Thus Corollary 2.3 is applicable.

2) If $\mu(\exp[\lambda\rho^2]) < \infty$ for any $\lambda > 0$, then by [22, Theorem 5.4, p283], μ satisfies the super log-Sobolev inequality. That's a nice example for application of our Corollary 2.4.

3) ([22, Corollary 5.3, p285]) For $\alpha \in (1,2)$, the Φ-Sobolev inequality (2.6) holds with $\Phi(r) = r[\log(1+r)]^\alpha \iff \mu(exp[c\rho^{2/(2-\alpha)}]) < \infty$ for some $c > 0$. In such case, the integrability condition for F in Theorem 2.1 becomes

$$\int e^{\lambda\|F\|^{1/\alpha}} d\mu < +\infty, \ \forall \lambda > 0.$$

This gives again an interesting example for which our integrability condition on F for the LDP is much weaker than the i.i.d. case.

4 Proofs of the mains results

4.1 Proof of Theorem 2.1

We need the following powerful result from convex analysis ([12, Theorem 8,p.110]).

Lemma 4.1. *Let $\Lambda : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function where X is a locally convex topological space. If Λ is bounded above on a neighborhood of zero, then Λ is continuous in the interior of* $[\Lambda < +\infty]$.

Proof of Theorem 2.1. **Step 1.** By [26, Corollary 5.4], $\mathbb{P}_\nu(L_t \in \cdot)$ satisfies as t goes to infinity the LDP on $M_1(E)$ equipped with the τ-topology, with speed t and with the rate function J_μ, uniformly over $\nu \in \mathcal{A}_L$. Now for every measurable $F : E \to \mathbb{B}$ which is bounded, since the mapping $\beta \to \int_E F(x)d\beta(x) =: \beta(F)$ is continuous from $(M_1(E), \tau)$ to $\mathbb{B}$ (by [7, Lemma 3.3.8]), hence the result in the theorem holds true for F which is bounded by the contraction principle.
Step 2. Now for unbounded F, let $F_R(x) = F(x)I_{[\|F\| \le R]}$. By Step 1 and the approximation lemma in the theory of large deviations [8, Theorem 4.2.16], for the LDP in the theorem it is enough to establish

$$\lim_{R\to\infty} \lim_{T\to\infty} \frac{1}{T} \log \sup_{\nu\in\mathcal{A}_L} \mathbb{P}_\nu \left(\frac{1}{T}\int_0^T \|(F - F_R)(X_t)\|dt > \delta \right) = -\infty, \ \forall\ \delta > 0. \tag{4.1}$$

The following two steps are devoted to the proof of the key (4.1).
Step 3. For every $V : E \to \mathbb{R}$ consider the Feynman-Kac semigroup

$$P_t^V f(x) := \mathbb{E}^x f(X_t) \exp \int_0^t V(X_s)ds.$$

Set

$$\|P_t^V\|_2 := \sup\ \{\|P_t^V f\|_{L^2(\mu)} :\ f \ge 0, \langle f^2\rangle_\mu \le 1\}$$

and consider the Cramer functional $\Lambda : L^\Psi(\mu) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\Lambda(V) = \lim_{t\to\infty} \frac{1}{t} \log \|P_t^V\|_2, \ \forall V \in L^\Psi(\mu) \tag{4.2}$$

(the last limit exists for $\log \|P_t^V\|_2$ is sub-additive in t). The aim of this step is to show that Λ is continuous at 0 in the norm topology of $L^\Psi(\mu)$. By [25, Theorem 1] we have

$$\Lambda(V) \le \sup \left\{ \int_E V f^2 d\mu - \langle -\mathcal{L}f, f\rangle_\mu | \int_E f^2 d\mu = 1, f \in \mathbb{D}_2(\mathcal{L}) \right\}.$$

Under the Φ-Sobolev inequality (2.6), we have

$$\begin{aligned}\Lambda(V) &\le \sup\left\{\int_E Vf^2 d\mu - \frac{1}{C_1}\int_E \Phi(f^2)d\mu \Big| \int_E f^2 d\mu = 1, f\in \mathbb{D}_2(\mathcal{L})\right\} + \frac{C_2}{C_1}\\ &= \frac{1}{C_1}\sup\{\int_E C_1 Vf^2 d\mu - \int_E \Phi(f^2)d\mu | \int_E f^2 d\mu = 1, f\in \mathbb{D}_2(\mathcal{L})\} + \frac{C_2}{C_1}.\end{aligned}$$

Since $C_1Vf^2 - \Phi(f^2) \le \Psi(C_1V)$ by (2.2), we get

$$\Lambda(V) \le \frac{1}{C_1}\int_E \Psi(C_1V)d\mu + \frac{C_2}{C_1}. \tag{4.3}$$

On the other hand if we assume the second assumption (2.7), we have

$$\begin{aligned}\Lambda(V) &\le \sup\left\{\int_E Vf^2 d\mu - \frac{1}{C_1}\|f^2\|_\Phi \Big| \int_E f^2 d\mu = 1, f\in \mathbb{D}_2(\mathcal{L})\right\} + \frac{C_2}{C_1}\\ &= \frac{1}{C_1}\sup\{\int_E C_1 Vf^2 d\mu - \|f^2\|_\Phi | \int_E f^2 d\mu = 1, f\in \mathbb{D}_2(\mathcal{L})\} + \frac{C_2}{C_1}\end{aligned}$$

Now if $\int \Psi(C_1V)d\mu \le 1$ (which is equivalent to $N_\Psi(V) \le \frac{1}{C\,1}$), by the definition of $\|\cdot\|_\Phi$ we have $\int_E C_1Vf^2d\mu \le \|f^2\|_\Phi$, and the inequality above yields to

$$\Lambda(V) \le \frac{C_2}{C_1}.$$

Consequently $\Lambda(\cdot)$ is upper-bounded in the neighborhood $\{V|N_\Psi(V) \le \frac{1}{C\,1}\}$ of zero in $L^\Psi(\mu)$, in both the two cases. Thus it is continuous on the interior of $[\Lambda < +\infty]$ by Lemma 4.1.
Step 4. By our assumption on F, $V_R := \|F - F_R\| \in L^\Psi(\mu)$. By Chebyshev inequality we have for any $\lambda > 0$,

$$\mathbb{P}_\nu\left(\frac{1}{T}\int_0^T \|(F-F_R)(X_t)\|dt > \delta\right) \le e^{-\lambda\delta T}\mathbb{E}^\nu \exp\left(\lambda\int_0^T V_R(X_t)dt\right).$$

As $\nu \in \mathcal{A}_L$, the r.h.s. above is bounded from above by $L\|P_T^{\lambda V_R}\|_2$. Therefore we get:

$$\text{the l.h.s. of (4.1)} \le \lim_{R\to\infty}(-\lambda\delta + \Lambda(\lambda V_R)),\ \forall\ \lambda > 0.$$

Then (4.1) will be proved if we show $\lim_{R\to\infty}\Lambda(\lambda V_R) = 0, \forall\ \lambda > 0$. By Step 3, we have only to show that $V_R \to 0$ in $L^\Psi(\mu)$.
To that end, notice that $\mathbb{E}_\mu\Psi(\lambda V_R)$ converges to 0 as $R \to \infty$ for every $\lambda > 0$ by the dominated convergence theorem and our integrability assumption of $\|F\|$. So for every $\lambda \in \mathbb{N}$ we can find a constant $R_\lambda > 0$ such that $\mathbb{E}_\mu\Psi(\lambda V_R) \le 1$ for any $R \ge R_\lambda$. That is $N_\Psi(V_R) \le \frac{1}{\lambda}$ for all $R \ge R_\lambda$ by the definition of the gauge norm, the desired claim. □

4.2 Proof of Theorem 2.2

Our main idea is to deduce from the super Φ-Sobolev inequality (2.11) some $\tilde{\Phi}$-Sobolev inequality (2.6), for some convex function $\tilde{\Phi} \gg \Phi$, i.e.,

$$\lim_{t\to\infty} \frac{\tilde{\Phi}(t)}{\Phi(t)} = \infty. \tag{4.4}$$

Then $\tilde{\Psi} = (\tilde{\Phi})^*$ verifies $\lim_{t\to\infty} \frac{\tilde{\Psi}(t)}{\Psi(t)} = 0$. So our condition (2.12) implies

$$\int \tilde{\Psi}(\lambda\|F\|)d\mu < +\infty, \forall\lambda > 0,$$

which allow us to conclude Theorem 2.2 by Theorem 2.1.
For the construction of the new function $\tilde{\Phi}$, we need the super-Poincaré inequality of Wang [23]:

$$\mu(f^2) \le r\mathcal{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}) \tag{4.5}$$

where $\beta : (0,+\infty) \to \mathbb{R}^+$ is some non-increasing function; and

Lemma 4.2. *Let $(E,\mathcal{F},\mu)$ be a measure space and $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ a Dirichlet form on $L^2(\mu)$.*

(a) ([23, Theorem 3.3.1])) If the Φ-Sobolev inequality (2.6) is true, then the super-Poincaré inequality (4.5) holds with

$$\beta(r) = \left(1 + \frac{rC_2}{2C_1}\right)^2 F^{-1}\left(C_2 + \frac{2C_1}{r}\right). \tag{4.6}$$

where $F(t) = \Phi(t)/t$, $F^{-1}(r) = \inf\{s \ge 0|F(s) > r\}$ ($\inf\emptyset := +\infty$ as usual).

(b) ([22, Theorem 1.2]) Conversely if the super Poincaré inequality (4.5) holds, then for any nondecreasing F verifying

$$F(r) \le \frac{1}{\beta^{-1}(r/2)}, \ \beta^{-1}(r) := \inf\{s > 0|\beta(s) \le r\},$$

there are constants $C_1, C_2 \ge 0$ so that

$$\mu(f^2F(f^2)) \le C_1\mathcal{E}(f,f) + C_2, \quad f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1 \tag{4.7}$$

Remarks 4.3. The explicit version (4.6) is important for us. Though it is not written explicitly in [23], but it is given by the proof therein.

Proof of Theorem 2.2. As said before it is enough to construct some $\tilde{\Phi} \gg \Phi$ in the sense of (4.4) so that the Φ-Sobolev inequality (2.6) holds for $\tilde{\Phi}$.
For any $\varepsilon \in (0,1)$, thanks to our condition (2.11) and Lemma 4.2(a), the super Poincaré inequality (4.5) holds with

$$\beta_\varepsilon(r) = (1 + \frac{rC(\varepsilon)}{2\varepsilon})^2 F^{-1}(C(\varepsilon) + \frac{2\varepsilon}{r}),$$

then by monotonicity it holds with $\tilde{\beta}_\varepsilon(r) = \inf_{s<r}\beta_\varepsilon(s)$, where $F(x) = \frac{\Phi(x)}{x}$. As that is true for any $\varepsilon > 0$, the super Poincaré inequality (4.5) continues to hold with

$$\beta(r) := \inf_{\varepsilon>0} \tilde{\beta}_\varepsilon(r) = \inf_{s<r} \inf_{\varepsilon>0} \beta_\varepsilon(s), \ r > 0.$$

β is non-increasing and left-continuous. By Lemma 4.2(b), the inequality (4.7) holds for the non-decreasing function $\tilde{F}(t) = 1/\beta^{-1}(t/2)$. Let us prove that

$$\lim_{t\to\infty} \frac{F(t)}{\tilde{F}(t)} = 0. \tag{4.8}$$

Indeed for any fixed $\varepsilon \in (0,1)$, if $r > 0$ satisfies

$$\frac{rC(\varepsilon)}{2\varepsilon} \le 1, \tag{4.9}$$

we obtain

$$\beta(r) \le \beta_\varepsilon(r) = 4F^{-1}(C(\varepsilon) + \frac{2\varepsilon}{r}). \tag{4.10}$$

Let $r = \beta^{-1}(t/2)$, then $\beta(r) \ge t/2$, $\tilde{F}(t) = \dfrac{1}{r}$ and r satisfies (4.9) as $t \to \infty$. Now (4.10) turns to

$$F(\frac{t}{8}) \le C(\varepsilon) + \frac{2\varepsilon}{r}. \tag{4.11}$$

Then we have $\limsup_{t\to\infty} \frac{F(t/8)}{\tilde{F}(t)} \le 2\varepsilon$. Thus (4.8) follows since $F(t) = \dfrac{\Phi(t)}{t}$ satisfies also $\triangle_2$-condition (2.10) as Φ.

Finally let $\tilde{\Phi}(t)$ be the greatest left continuous convex function $\le t\tilde{F}(t)$ for all $t \ge 0$, it satisfies $\tilde{\Phi}(t) \gg \Phi(t)$ in the sense of (4.4) and the inequality (4.7) for $\tilde{F}$ implies the $\tilde{\Phi}$-Sobolev inequality (2.6). Thus Theorem 2.2 follows by Theorem 2.1, as said at the beginning. □

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