

DISTRIBUTION OF A RANDOM FUNCTIONAL OF A FERGUSON-DIRICHLET PROCESS OVER THE UNIT SPHERE

THOMAS J. JIANG

Department of Mathematical Sciences, National Chengchi University, Taipei 11605, Taiwan

email: jiangt@nccu.edu.tw

KUN-LIN KUO

Institute of Statistical Science, Academia Sinica, Taipei 11529, Taiwan

email: KunLin.Kuo@gmail.com

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Abstract

Jiang, Dickey, and Kuo [12] gave the multivariate c -characteristic function and showed that it has properties similar to those of the multivariate Fourier transformation. We first give the multivariate c -characteristic function of a random functional of a Ferguson-Dirichlet process over the unit sphere. We then find out its probability density function using properties of the multivariate c -characteristic function. This new result would generalize that given by [11].

1 Introduction

Ferguson [5] introduced the Ferguson-Dirichlet process and studied its applications to nonparametric Bayesian inference. He also showed that when the prior distribution is a Ferguson-Dirichlet process with parameter μ , then the posterior distribution, given the sample s_1, s_2, \dots, s_n , is also a Ferguson-Dirichlet process having parameter $\mu + \sum_{j=1}^n \delta_{s_j}$, where δ_{s_j} denotes point mass at s_j . The most natural use of random functionals of a Ferguson-Dirichlet process is to make Bayesian inferences concerning the parameters of a statistical population. Hence, the expression for the probability density function of any random functional of a Ferguson-Dirichlet process can be employed both for prior and posterior Bayesian analyses. Further applications related to the random functional can be seen in [3] and other references. For example, random means and random variances of a Ferguson-Dirichlet process can be used for smooth Bayesian nonparametric density estimation (see [15]) and for quality control problems (see [4] for further discussions), respectively.

Research on the distribution of a random functional of a Ferguson-Dirichlet process has been ongoing for decades. A partial list of papers in this area are [2, 3, 8, 9, 11, 12, 14, 16, 17]. In particular, [11] gave the distribution of a random functional of a Ferguson-Dirichlet process over the unit circle. In this paper, we shall use the multivariate c -characteristic function, a tool given

by [12], to generalize the result to the case over the unit sphere in three-dimension.

In Section 2, we first review the definition of the multivariate c -characteristic function and some of its properties. We then compute a multivariate c -characteristic function of an interesting distribution. The multivariate c -characteristic function of the random mean of a Ferguson-Dirichlet process over the unit sphere is given in Section 3. Using the uniqueness property of the multivariate c -characteristic function, we then determine the distribution of the random mean of a Ferguson-Dirichlet process over the unit sphere. Conclusions are given in Section 4.

2 Multivariate c -characteristic function

Jiang [10] first gave a univariate c -characteristic function. Jiang, Dickey, and Kuo [12] generalized it to a multivariate c -characteristic function, which can be very useful when a distribution is difficult to deal with by traditional characteristic function. See [12] for detailed results. First, we state the definition of the multivariate c -characteristic function.

Definition 1. If $\mathbf{u} = (u_1, \dots, u_L)'$ is a random vector on a subset S of $A = [-a_1, a_1] \times \dots \times [-a_L, a_L]$, its multivariate c -characteristic function is defined as

$$g(\mathbf{t}; \mathbf{u}, c) = E[(1 - i\mathbf{t} \cdot \mathbf{u})^{-c}], \quad |\mathbf{t}| < a^{-1},$$

where $c > 0$, $a = \sqrt{\sum_{i=1}^L a_i^2}$, $\mathbf{t}' = (t_1, \dots, t_L)$, $|\mathbf{t}| = \sqrt{\sum_{i=1}^L t_i^2}$, and $\mathbf{t} \cdot \mathbf{u}$ is the inner product of \mathbf{t} and \mathbf{u} .

The above assumptions that c is positive and \mathbf{u} has a bounded support are needed in [12, Lemma 2.2], which shows that, for any positive c , there is a one-to-one correspondence between $g(\mathbf{t}; \mathbf{u}, c)$ and the distribution of \mathbf{u} .

Next, we give the multivariate c -characteristic function of an interesting distribution in the next lemma.

Lemma 2. Let $\mathbf{u} = (u_1, u_2, u_3)'$ be a distribution on the inside of a unit ball, i.e., $\{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 < 1\}$, with the probability density function

$$f(\mathbf{u}) = \frac{-e}{4\pi^2 r} (1+r)^{-(1+r)/2} (1-r)^{-(1-r)/2} \left(-\pi \sin \frac{\pi r}{2} + \ln \frac{1-r}{1+r} \cos \frac{\pi r}{2} \right),$$

where $r = |\mathbf{u}|$. Then the multivariate 1-characteristic function of \mathbf{u} is

$$g(\mathbf{t}; \mathbf{u}, 1) = \exp \left(\sum_{n=1}^{\infty} \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)} \right). \quad (1)$$

Proof. Let $C = \{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 < 1\}$. Eq. (1) is equivalent to the following identity

$$\int_C (1 - i\mathbf{t} \cdot \mathbf{u})^{-1} f(\mathbf{u}) d\mathbf{u} = \exp \left(\sum_{n=1}^{\infty} \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)} \right).$$

To prove the above identity, we establish the following four equations first. From [7, p. 105], we have

$$\int_0^{2\pi} (a \cos \alpha + b \sin \alpha)^n d\alpha = \begin{cases} \frac{(1/2, n/2) 2(a^2 + b^2)^{n/2} \pi}{(n/2)!}, & n \text{ is even,} \\ 0, & n \text{ is odd,} \end{cases} \quad (2)$$

where a and b are real numbers and $(a, k) = a(a+1)\cdots(a+k-1)$. We also can obtain the following equation from [6, Eq. 3.621.5],

$$\int_0^\pi \sin^{a-1} x \cos^{b-1} x dx = \begin{cases} \frac{B(a/2, b/2)}{2}, & \text{Re } a > 0, b > 0 \text{ is odd,} \\ 0, & \text{Re } a > 0, b > 0 \text{ is even.} \end{cases} \quad (3)$$

Using integration by parts, we have the following identity,

$$\begin{aligned} & \int_0^1 r^{2n+1} (1+r)^{-(1+r)/2} (1-r)^{-(1-r)/2} \left(-\pi \sin \frac{\pi r}{2} \right) dr \\ &= - \int_0^1 r^{2n+1} (1+r)^{-(1+r)/2} (1-r)^{-(1-r)/2} \ln \frac{1-r}{1+r} \cos \frac{\pi r}{2} dr \\ & \quad - \int_0^1 2(2n+1)r^{2n} (1+r)^{-(1+r)/2} (1-r)^{-(1-r)/2} \cos \frac{\pi r}{2}. \end{aligned} \quad (4)$$

Using [13, Lemma 8 and Example 2], we can obtain the following equality:

$$\exp \left(- \int_{-1}^1 \ln(1-itx) \frac{1}{2} dx \right) = \int_{-1}^1 (1-itx)^{-1} \frac{e}{\pi} (x+1)^{-(x+1)/2} (1-x)^{-(1-x)/2} \cos \frac{\pi x}{2} dx.$$

Since

$$\exp \left(- \int_{-1}^1 \ln(1-itx) \frac{1}{2} dx \right) = \exp \left(\sum_{n=1}^{\infty} \frac{(-t^2)^n}{2n(2n+1)} \right)$$

and

$$\begin{aligned} & \int_{-1}^1 (1-itx)^{-1} \frac{e}{\pi} (x+1)^{-(x+1)/2} (1-x)^{-(1-x)/2} \cos \frac{\pi x}{2} dx \\ &= \sum_{n=0}^{\infty} \int_{-1}^1 \frac{e i^n t^n}{\pi} x^n (x+1)^{-(x+1)/2} (1-x)^{-(1-x)/2} \cos \frac{\pi x}{2} dx, \end{aligned}$$

and by the fact that the function $(x+1)^{-(x+1)/2} (1-x)^{-(1-x)/2} \cos \frac{\pi x}{2}$ is symmetric at $x=0$, we have

$$\exp \left(\sum_{n=1}^{\infty} \frac{(-t^2)^n}{2n(2n+1)} \right) = \frac{2e}{\pi} \sum_{n=0}^{\infty} (-t^2)^n \int_0^1 x^{2n} (x+1)^{-(x+1)/2} (1-x)^{-(1-x)/2} \cos \frac{\pi x}{2} dx. \quad (5)$$

Setting

$$g(r) = \frac{-er}{4\pi^2} (1+r)^{-(1+r)/2} (1-r)^{-(1-r)/2} \left(-\pi \sin \frac{\pi r}{2} + \ln \frac{1-r}{1+r} \cos \frac{\pi r}{2} \right),$$

and using the spherical coordinate transformation, we have

$$\begin{aligned}
& \int_C (1 - it \cdot \mathbf{u})^{-1} f(\mathbf{u}) d\mathbf{u} \\
&= \int_0^1 \int_0^{2\pi} \int_0^\pi (1 - it_1 r \cos \theta \sin \phi - it_2 r \sin \theta \sin \phi - it_3 r \cos \phi)^{-1} \sin \phi g(r) d\phi d\theta dr \\
&= \int_0^1 \sum_{n=0}^{\infty} (ir)^n g(r) \int_0^{2\pi} \int_0^\pi (t_1 \cos \theta \sin \phi + t_2 \sin \theta \sin \phi + t_3 \cos \phi)^n \sin \phi d\phi d\theta dr \\
&= \int_0^1 \sum_{n=0}^{\infty} (ir)^n g(r) \int_0^{2\pi} \int_0^\pi \sum_{k=0}^n \binom{n}{k} (t_1 \cos \theta + t_2 \sin \theta)^k t_3^{n-k} \sin^{k+1} \phi \cos^{n-k} \phi d\phi d\theta dr \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{4\pi(-t_1^2 - t_2^2 - t_3^2)^n r^{2n}}{2n+1} g(r) dr \tag{6}
\end{aligned}$$

$$= \frac{2e}{\pi} \sum_{n=0}^{\infty} (-t_1^2 - t_2^2 - t_3^2)^n \int_0^1 r^{2n} (1+r)^{-(1+r)/2} (1-r)^{-(1-r)/2} \cos \frac{\pi r}{2} dr \tag{7}$$

$$= \exp \left(\sum_{n=1}^{\infty} \frac{(-t_1^2 - t_2^2 - t_3^2)^n}{2n(2n+1)} \right). \tag{8}$$

Identity (6) can be obtained by Eqs. (2) and (3). Identities (7) and (8) follow from Eq. (4) and Eq. (5), respectively. \square

3 Distribution of a random functional of a Ferguson-Dirichlet process over the unit sphere

Ferguson [5] first defined the Ferguson-Dirichlet process. Let μ be a finite non-null measure on (Y, A) , where Y is a Borel set in Euclidean space \mathbb{R}^n and A is the σ -field of Borel subsets of Y , and let U be a stochastic process indexed by elements of A . We say that U is a Ferguson-Dirichlet process with parameter μ , if for every finite measurable partition $\{B_1, \dots, B_m\}$ of Y , the random vector $(U(B_1), \dots, U(B_m))$ has a Dirichlet distribution with parameter $(\mu(B_1), \dots, \mu(B_m))$, where $\mu(B_j) > 0$ for all $j = 1, \dots, m$. A random vector $\mathbf{v} = (v_1, \dots, v_m)'$ is said to have a Dirichlet distribution with parameter $\mathbf{b} = (b_1, \dots, b_m)'$ where each $b_j > 0$, if \mathbf{v} has the probability density function

$$f(\mathbf{v}; \mathbf{b}) = \frac{\Gamma(b_1 + \dots + b_m)}{\prod_{j=1}^m \Gamma(b_j)} \prod_{j=1}^m v_j^{b_j-1},$$

for all \mathbf{v} in the probability simplex $\{\mathbf{v} \mid \text{each } v_j \geq 0, v_1 + \dots + v_m = 1\}$.

First, we give a trivariate c -characteristic function expression of any trivariate random functional of a Ferguson-Dirichlet process over a Borel set Y in Euclidean space in the next lemma.

Lemma 3. *Let $\mathbf{w} = \int_Y \mathbf{h}(\mathbf{x}) dU(\mathbf{x})$ be a random functional where $\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), h_2(\mathbf{x}), h_3(\mathbf{x}))'$ is a bounded measurable function defined on a Borel set Y in Euclidean space \mathbb{R}^n , and U is a Ferguson-Dirichlet process with parameter μ on (Y, A) . Then the trivariate c -characteristic function of \mathbf{w} can*

be expressed as

$$g(\mathbf{t}; \mathbf{w}, c) = \exp\left(-\int_Y \ln(1 - \mathbf{it} \cdot \mathbf{h}(\mathbf{x})) d\mu(\mathbf{x})\right), \text{ where } c = \mu(Y).$$

Proof. For any $k \geq 2$, let $\{B_{k1}, B_{k2}, \dots, B_{kk}\}$ be a partition of Y , $\mathbf{b}_{kj} \in B_{kj}$, $v_k = \max\{\text{volume}(B_{kj}) \mid 1 \leq j \leq k\}$, and $\lim_{k \rightarrow \infty} v_k = 0$. Then $(U(B_{k1}), \dots, U(B_{kk}))$ follows a Dirichlet distribution with parameter $(\mu(B_{k1}), \dots, \mu(B_{kk}))$. In addition, $\sum_{j=1}^k U(B_{kj}) = 1$ for all $k \geq 2$. Define $\mathbf{g}_k(\mathbf{x}) = \sum_{j=1}^k \mathbf{h}(\mathbf{b}_{kj}) \delta_{B_{kj}}(\mathbf{x})$ and $\mathbf{w}_k = \int_Y \mathbf{g}_k(\mathbf{x}) dU(\mathbf{x})$, where $\delta_{B_{kj}}(\mathbf{x})$ is 1, for $\mathbf{x} \in B_{kj}$; and is 0, otherwise. Then $\lim_{k \rightarrow \infty} \mathbf{g}_k(\mathbf{x}) = \mathbf{h}(\mathbf{x})$ for all $\mathbf{x} \in Y$, and $\mathbf{w}_k = \sum_{j=1}^k \mathbf{g}_k(\mathbf{b}_{kj}) U(B_{kj})$. The trivariate c -characteristic function of \mathbf{w}_k can be expressed as

$$\begin{aligned} g(\mathbf{t}; \mathbf{w}_k, c) &= E(1 - \mathbf{it} \cdot \mathbf{w}_k)^{-c} \\ &= E\left(1 - i \sum_{j=1}^k [\mathbf{t} \cdot \mathbf{g}_k(\mathbf{b}_{kj})] U(B_{kj})\right)^{-c} \\ &= E\left(\sum_{j=1}^k U(B_{kj}) [1 - \mathbf{it} \cdot \mathbf{g}_k(\mathbf{b}_{kj})]\right)^{-c} \\ &= \mathcal{R}_{-c}(\mu(B_{k1}), \dots, \mu(B_{kk}); 1 - \mathbf{it} \cdot \mathbf{g}_k(\mathbf{b}_{k1}), \dots, 1 - \mathbf{it} \cdot \mathbf{g}_k(\mathbf{b}_{kk})) \\ &= \prod_{j=1}^k (1 - \mathbf{it} \cdot \mathbf{g}_k(\mathbf{b}_{kj}))^{-\mu(B_{kj})}, \end{aligned}$$

where \mathcal{R} is a Carlson's multiple hypergeometric function ([1]), and the last equality can be obtained by [1, formula 6.6.5]. Therefore, the limit of the trivariate c -characteristic function of \mathbf{w}_k 's, as k approaches ∞ , is

$$\begin{aligned} \lim_{k \rightarrow \infty} g(\mathbf{t}; \mathbf{w}_k, c) &= \exp\left(\lim_{k \rightarrow \infty} \sum_{j=1}^k -\mu(B_{kj}) \ln(1 - \mathbf{it} \cdot \mathbf{g}_k(\mathbf{b}_{kj}))\right) \\ &= \exp\left(-\int_Y \ln(1 - \mathbf{it} \cdot \mathbf{h}(\mathbf{x})) d\mu(\mathbf{x})\right). \end{aligned}$$

In addition, by the Dominated Convergence Theorem, we have $\lim_{k \rightarrow \infty} \mathbf{w}_k = \mathbf{w}$. By [12, Theorem 2.4], we conclude that

$$g(\mathbf{t}; \mathbf{w}, c) = \exp\left(-\int_Y \ln(1 - \mathbf{it} \cdot \mathbf{h}(\mathbf{x})) d\mu(\mathbf{x})\right).$$

□

In the rest of this section, we study the random functional $\mathbf{u} = \int_X \mathbf{x} dU(\mathbf{x})$, where X is the unit sphere in \mathbb{R}^3 . We use Lemma 3 in the following theorem to first establish the trivariate c -characteristic function of \mathbf{u} .

Theorem 4. Let $X = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, and U be a Ferguson-Dirichlet process over X with uniform measure μ as its parameter, where $\mu(X) = c$. Then the trivariate c -characteristic

function of the random functional $\mathbf{u} = \int_X \mathbf{x} dU(\mathbf{x})$ can be expressed as

$$g(\mathbf{t}; \mathbf{u}, c) = \exp \left(\sum_{n=1}^{\infty} \frac{c}{2n(2n+1)} (-t_1^2 - t_2^2 - t_3^2)^n \right), \text{ where } \mathbf{t} = (t_1, t_2, t_3)'.$$

Proof. First, we give the following two equations, which are about Appell's notations and can be shown easily.

$$\Gamma(a+n) = \Gamma(a)(a, n), \tag{9}$$

$$(a, 2n) = 2^{2n} \left(\frac{a}{2}, n \right) \left(\frac{a+1}{2}, n \right). \tag{10}$$

By Lemma 3, we have

$$\begin{aligned} g(\mathbf{t}; \mathbf{u}, c) &= \exp \left(\frac{-c}{4\pi} \int_X \ln(1 - i\mathbf{t} \cdot \mathbf{x}) d\mathbf{x} \right) \\ &= \exp \left(\frac{-c}{4\pi} \int_0^\pi \int_0^{2\pi} \ln(1 - it_1 \cos \theta_1 - it_2 \sin \theta_1 \cos \theta_2 - it_3 \sin \theta_1 \sin \theta_2) \sin \theta_1 d\theta_2 d\theta_1 \right) \\ &= \exp \left(\frac{c}{4\pi} \sum_{n=1}^{\infty} \frac{i^n}{n} \int_0^\pi \int_0^{2\pi} (t_1 \cos \theta_1 + t_2 \sin \theta_1 \cos \theta_2 + t_3 \sin \theta_1 \sin \theta_2)^n \sin \theta_1 d\theta_2 d\theta_1 \right) \\ &= \exp \left(\frac{c}{4\pi} \sum_{n=1}^{\infty} \frac{i^n}{n} \sum_{k=0}^n \binom{n}{k} \int_0^\pi \int_0^{2\pi} (t_1 \cos \theta_1)^k \sin^{n-k+1} \theta_1 (t_2 \cos \theta_2 + t_3 \sin \theta_2)^{n-k} d\theta_2 d\theta_1 \right) \\ &= \exp \left(\frac{c}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \sum_{k=0}^n \binom{2n}{2k} \frac{(1/2, n-k)(t_2^2 + t_3^2)^{n-k}}{(n-k)!} t_1^{2k} B(n-k+1, k+1/2) \right) \\ &= \exp \left(\sum_{n=1}^{\infty} \frac{c}{2n(2n+1)} (-t_1^2 - t_2^2 - t_3^2)^n \right). \end{aligned}$$

The fifth identity can be obtained by Eqs. (2) and (3). The last identity follows from Eqs. (9) and (10). □

By [12, Lemma 2.2], Lemma 2, and Theorem 4, we can obtain the following corollary.

Corollary 5. *The probability density function of $\mathbf{u} = \int_X \mathbf{x} dU(\mathbf{x})$, where U is a Ferguson-Dirichlet process over the unit sphere X with uniform probability measure as its parameter, is*

$$f(\mathbf{u}) = \frac{-e}{4\pi^2 r} (1+r)^{-(1+r)/2} (1-r)^{-(1-r)/2} \left(-\pi \sin \frac{\pi r}{2} + \ln \frac{1-r}{1+r} \cos \frac{\pi r}{2} \right),$$

where $r = \sqrt{u_1^2 + u_2^2 + u_3^2}$ and $u_1^2 + u_2^2 + u_3^2 < 1$.

4 Conclusions

In this paper, we obtain the trivariate c -characteristic function expression for a random functional of a Ferguson-Dirichlet process over any finite three-dimensional space. We also obtain the probability density function of the random functional of a Ferguson-Dirichlet process with uniform probability measure parameter over the unit sphere. This generalizes [11, Theorem 2].

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