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SMALL TIME EXPANSIONS FOR TRANSITION PROBABILITIES OF SOME LÉVY PROCESSES

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Abstract
We show that there exist Lévy processes $(X_t, t \geq 0)$ and reals $y > 0$ such that for small t, the probability $\mathbb{P}(X_t > y)$ has an expansion involving fractional powers or more general functions of t. This constrats with previous results giving polynomial expansions under additional assumptions.

1 The Brownian case

1.1 Main result

Let $(X_t, t \geq 0)$ be a real-valued Lévy process with Lévy measure Π and let $y > 0$. It is well-known (see for example [B], Chapter 1) that when $t \to 0$,

$$\mathbb{P}(X_t \geq y) \sim t\overline{\Pi}(y) \tag{1}$$

whenever $\overline{\Pi}(y) > 0$ and $\overline{\Pi}$ is is continuous at y, where $\overline{\Pi}$ stands for the tail of Π: for every $z > 0$,

$$\overline{\Pi}(z) = \Pi([z, \infty))$$

It has been proved that under additional assumptions, which in particular include the smoothness of $\overline{\Pi}$, one gets more precise expansions of the probability $\mathbb{P}(X_t \geq y)$ and that these are polynomial in t. See [L, P, RW, FH2] among others.
The problem of relating Π to the marginals of the process have several applications. The paper [RW], as well as [FH1], is concerned with problems of mathematical finance. Applications of statistical nature can be found in [F]. From a more theoretical point of view, this relation plays an important role when studying small-time behaviour of Lévy processes, which involves fine properties of the Lévy measure (see for instance Section 4 in [BDM]).
Our goal is to exhibit some examples where this expansion involves more general functions of t, such as fractional powers, powers of the logarithm and so on. We shall focus on the case when X

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has the form $X_t = S_t + Y_t$ where $(Y_t, t \geq 0)$ is a compound Poisson process with Lévy measure Π and $(S_t, t \geq 0)$ is a stable process, S and Y being independent. Assume first that

$$X_t = B_t + Y_t$$

where $(B_t, t \geq 0)$ is a standard Brownian motion. Then we have:

Theorem 1. *(i) Suppose that Π has a continuous density f on $[y-\delta, y) \cup (y, y+\delta]$ for some $\delta > 0$. Suppose that*

$$f_+ := \lim_{x \to 0+} f(y+x) \neq f_- := \lim_{x \to 0-} f(y+x)$$

Then as $t \to 0$,

$$\mathbb{P}(X_t \geq y) - t\left[\overline{\Pi}(z) - \frac{\Pi(\{z\})}{2}\right] \sim \lambda t^{3/2}\left[\frac{(f_- - f_+)\mathbb{E}(G)}{2}\right]$$

where G is the absolute value of a standard gaussian random variable.
(ii) Define the functions $g_-, g_+ : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$g_+(x) = \Pi((y, y+x))$$

$$g_-(x) = \Pi((y-x, y))$$

Suppose that

$$x^2 = o(|g_+(x) - g_-(x)|)$$

as $x \to 0+$. Then as $t \to 0$,

$$\mathbb{P}(X_t \geq y) - t\left[\overline{\Pi}(z) - \frac{\Pi(\{z\})}{2}\right] \sim \frac{t\mathbb{E}[g_-(\sqrt{t}G) - g_+(\sqrt{t}G)]}{2}$$

Remarks
(i) Suppose that for small $x > 0$,

$$g_+(x) = ax + cx^\alpha |\log x|^\beta + o(x^\alpha |\log x|^\beta)$$

$$g_-(x) = ax + c' x^\gamma |\log x|^\delta + o(x^\gamma |\log x|^\delta)$$

with the conditions that $(c, \alpha, \beta) \neq (c'\gamma, \delta)$ and $1 < \min(\alpha, \gamma) < 2$. Then $x^2 = o(|g_+(x) - g_-(x)|)$ and the conclusion of (ii) applies. For example, if $\alpha < \gamma$, this gives the estimate

$$\mathbb{P}(X_t \geq y) - t\left[\overline{\Pi}(z) - \frac{\Pi(\{z\})}{2}\right] \sim -\frac{c\mathbb{E}(G^\alpha |\log G|^\beta)}{2} t^{1+(\alpha/2)} |\log t|^\beta$$

Of course, one could take any slowly varying function instead of the logarithm. On the other hand, if Π has a density that is twice differentiable in the neighbourhood of y, then $|g_+(x) - g_-(x)| = O(x^2)$ and (ii) does not apply.
(ii) For a fixed time t, adding B_t to Y_t has a smoothing effect on the probability measure $\mathbb{P}(X_t \in dx)$. In turn, if we fix y and consider the function $h_y : t \mapsto \mathbb{P}(X_t \geq y)$, the effect of adding B_t to Y_t is counter-regularizing. Indeed, h_y would be analytic in the absence of Brownian motion while it is not twice differentiable in the presence of Brownian motion. This is not very intuitive in our view.

Proof of Theorem 1

Let λ be the total mass of Π. For every $y>0$ one can write

$$\begin{aligned}\mathbb{P}(X_t \geq y) = e^{-\lambda t}\mathbb{P}(B_t \geq y) + \lambda t e^{-\lambda t}\mathbb{P}(B_t + Z_1 \geq y) \\ + \frac{(\lambda t)^2 e^{-\lambda t}}{2}\mathbb{P}(B_t + Z_1 + Z_2 \geq y) \\ + \dots \end{aligned} \tag{2}$$

where the random variables Z_n are iid with common law $\lambda^{-1}\Pi$. As $t \to 0$, for every integer $n \geq 0$, $\mathbb{P}(B_t \geq y) = o(t^n)$. Moreover,

$$\lambda t e^{-\lambda t}\mathbb{P}(B_t + Z_1 \geq y) = \lambda t \mathbb{P}(B_t + Z_1 \geq y) + O(t^2)$$

Hence, as $t \to 0$,

$$\mathbb{P}(X_t \geq y) = \lambda t \mathbb{P}(B_t + Z_1 \geq y) + O(t^2)$$

Since $\mathbb{P}(B_t + Z_1 \geq y) = \mathbb{P}(Z_1 \geq y - B_t)$, we have

$$\begin{aligned}\mathbb{P}(B_t + Z_1 \geq y) \quad = \quad & \lambda^{-1}\overline{\Pi}(y) + \mathbb{P}(Z_1 \in [y - B_t, y), B_t > 0) \\ & -\mathbb{P}(Z_1 \in [y, y + |B_t|), B_t < 0)\end{aligned}$$

The stability property $B_t \stackrel{d}{=} \sqrt{t}B_1$ entails

$$\mathbb{P}(B_t + Z_1 \geq y) - \lambda^{-1}\overline{\Pi}(y) = \frac{1}{2}\left[\mathbb{P}(Z_1 \in [y - \sqrt{t}G, y)) - \mathbb{P}(Z_1 \in [y, y + \sqrt{t}G))\right]$$

where G is the absolute value of a standard gaussian random variable. Under the assumptions of (i), as $t \to 0$,

$$\mathbb{P}(Z_1 \in [y - \sqrt{t}G, y)) = \lambda^{-1} f_- \sqrt{t}\mathbb{E}(G) + o(\sqrt{t})$$

and

$$\mathbb{P}(Z_1 \in [y, y + \sqrt{t}G)) = \lambda^{-1}\Pi(\{y\}) + \lambda^{-1} f_+ \sqrt{t}\mathbb{E}(G) + o(\sqrt{t})$$

Therefore

$$\mathbb{P}(B_t + Z_1 \geq y) - \lambda^{-1}\left[\overline{\Pi}(z) - \frac{\Pi(\{z\})}{2}\right] = \frac{\lambda^{-1}\left[f_-\sqrt{t}\mathbb{E}(G) - f_+\sqrt{t}\mathbb{E}(G)\right]}{2} + o(\sqrt{t})$$

and, together with (2), this entails (i). The proof of (ii) is similar. Remark that proving (ii) does not involve the existence of the expectation $\mathbb{E}(G)$. □

1.2 Additional remarks

As a slight generalization of Theorem 1, we have:

Proposition 1. *With the same notation as in Theorem 1, suppose that there exists an integer $n \geq 1$ such that for every $i < 2n$,*

$$f^{(i)}(y+) = f^{(i)}(y-)$$

but that

$$f^{(2n)}(y+) \neq f^{(2n)}(y-)$$

Then there exist some constants $c_k, 1 \leq k \leq 2n+2$ such that as $t \to 0$,

$$\mathbb{P}(X_t \geq y) = \sum_{k=1}^{n+1} c_k t^k + c_{n+2} t^{n+(3/2)} + o(t^{n+(3/2)})$$

Proof
The proof is exactly the same as in Theorem 1. The estimate

$$\begin{aligned}
&\lambda[\mathbb{P}(Z_1 \in [y - \sqrt{t}G, y)) - \mathbb{P}(Z_1 \in [y, y + \sqrt{t}G))] + \Pi(\{y\}) \\
&= \sum_{i=1}^{n} \frac{[f^{(2i-1)}(y-) + f^{(2i-1)}(y+)]\mathbb{E}(G^{2i})t^i}{(2i)!} \\
&\quad + \sum_{i=1}^{n} \frac{[f^{(2i)}(y-) - f^{(2i)}(y+)]\mathbb{E}(G^{2i+1})t^{i+(1/2)}}{(2n+1)!} \\
&\quad + o(t^{n+(1/2)}) \qquad (3)
\end{aligned}$$

shows that in (2), the term

$$\lambda t e^{-\lambda t}\mathbb{P}(B_t + Z_1 \geq y)$$

gives rise to a singularity as stated in the proposition. On the other hand, it is clear that the other terms in (2) yield polynomial terms of degree at least $n+2$ in the small t asymptotics. This proves the proposition. □

Thanks to the estimate (3), we can see that the expression of the coefficients c_k involves the successive derivatives of f at y. This fact was first observed by Figueroa and Houdré [FH2] in the more general context of a Lévy process whose Lévy measure may have infinite mass near 0. Our method enables us to recover their result in the particular case when X_t has the form $X_t = B_t + Y_t$. On the other hand, we do not assume any regularity of the Lévy measure Π outside a neighbourhood of y, in contrast to [FH2].
It appears that the function $h_y : t \mapsto \mathbb{P}(X_t \geq y)$ "feels" the irregularities of the derivatives of f of even order but *not* the irregularities of the derivatives of f of odd order. In particular, if Π has an atom of mass, say m at y but if the measure $\Pi - m\delta_y$ is smooth at y, then h_y is smooth at 0. Thus in that case, the largest possible irregularity of Π at y is not reflected by an irregularity of h_y. This may seem counter-intuitive.
Remark that the first-order estimate (1) does not enable us to detect the presence or absence of a Brownian part in the process X. In turn, looking at finer estimates, we can see that the presence of a Brownian part is felt either through the fact that for some y, the function $h_y : t \mapsto \mathbb{P}(X_t \geq y)$ is not smooth, or through the fact that the functions h_y are smooth for all y but that their expression involves the derivatives of f.
Our last remark concerns the case when Π has a Dirac mass at y. In that case, Theorem 1 states that

$$\mathbb{P}(X_t \geq z) \sim t\left[\overline{\Pi}(z) - \frac{\Pi(\{z\})}{2}\right]$$

and the function $z \mapsto \overline{\Pi}(z) - \Pi(\{z\})/2$ is discontinuous at y. However, since X has a Brownian component, the law of X_t has a smooth density for every $t > 0$ and so the function $z \mapsto \mathbb{P}(X_t \geq z)$ is continuous at y. The compatibility between these two observations is explained in the following:

Proposition 2. *With the same notation as in Theorem 1, suppose that for some $y > 0$, $\Pi(\{y\}) > 0$ and that Π has a continuous density f on $\mathbb{R} - \{y\}$. Then for every fixed $c > 0$, as $t \to 0$,*

$$\mathbb{P}(X_t \geq y + c\sqrt{t}) \sim t\left[\overline{\Pi}(y) - \frac{\Pi(\{y\})\mathbb{P}(G \leq c)}{2}\right]$$

Of course, a similar result holds for $c < 0$.
Proof
The same arguments as in the proof of Theorem 1 give

$$\mathbb{P}(X_t \geq y + c\sqrt{t}) - t\overline{\Pi}(y + c\sqrt{t}) \quad \sim \quad \frac{\lambda t}{2}\Big[\mathbb{P}(Z_1 \in [y + \sqrt{t}(c - G), y + \sqrt{t}c)) - \mathbb{P}(Z_1 \in [y + \sqrt{t}c, y + \sqrt{t}(c + G)))\Big]$$

Using the regularity of Π on $\mathbb{R} - \{y\}$, we get the estimates

$$\mathbb{P}(Z_1 \in [y + \sqrt{t}(c - G), y + \sqrt{t}c)) = \lambda^{-1}\Pi(\{y\})\mathbb{P}(G \geq c) + O(\sqrt{t}),$$

$$\mathbb{P}(Z_1 \in [y + \sqrt{t}c, y + \sqrt{t}(c + G))) = O(\sqrt{t})$$

and

$$\overline{\Pi}(y + c\sqrt{t}) = \overline{\Pi}(y) - \Pi(\{y\}) + O(\sqrt{t})$$

This gives the result. □

2 The stable case

Consider now the process

$$X_t = Y_t + S_t$$

where S is a stable process of index $\alpha \in (0, 2)$ and Y is an independent compound Poisson process with Lévy measure Π. Let ν be the Lévy measure of X and denote by $\overline{\nu}$ the tail of ν.

Theorem 2. *(i) Let g_+, g_- be as in Theorem 1. Suppose that when $t \to 0$,*

$$t = o\left(\mathbb{E}[g_-(t^{1/\alpha}S_1)\mathbf{1}_{\{S_1>0\}} - g_+(t^{1/\alpha}|S_1|\mathbf{1}_{\{S_1<0\}}]\right)$$

Then for small $t > 0$,

$$\mathbb{P}(X_t \geq y) - t\left[\overline{\nu}(y) - \mathbb{P}(S_1 < 0)\Pi(\{y\})\right] \sim t\mathbb{E}\left[g_-(t^{1/\alpha}S_1)\mathbf{1}_{\{S_1>0\}} - g_+(t^{1/\alpha}|S_1|\mathbf{1}_{\{S_1<0\}}\right]$$

(ii) Suppose that there exist $\beta > \alpha$, $a \in \mathbb{R}$, $b, \delta_0 > 0$ such that if $|x| < \delta_0$,

$$\left|\overline{\Pi}(y + x) - \overline{\Pi}(y) - ax\right| < bx^\beta \tag{4}$$

Then there exists a real c such that as $t \to 0$,

$$\mathbb{P}(X_t \geq y) = t\left[\overline{\nu}(y) - \mathbb{P}(S_1 < 0)\Pi(\{y\})\right] + ct^2 + o(t^2) \tag{5}$$

Remarks
(i) Suppose that $\alpha > 1$. Then Theorem 2 (i) applies for example when $g_+(x) \sim ax$, $g_-(x) \sim bx$ in the neighbourhood of 0, with $a \neq b$. Another instance is the case when

$$g_+(x) = ax + cx^\eta|\log x|^\beta + o(x^\eta|\log x|^\beta)$$

$$g_-(x) = ax + c'x^\gamma|\log x|^\delta + o(x^\gamma|\log x|^\delta)$$

with the conditions that $(c,\alpha,\beta)\neq(c'\gamma,\delta)$ and $1<\min(\eta,\gamma)<\alpha$.
(ii) Likewise, in the case when $\alpha<1$, choosing

$$g_+(x)=cx^\eta|\log x|^\beta+o(x^\eta|\log x|^\beta)$$

$$g_-(x)=c'x^\gamma|\log x|^\delta+o(x^\gamma|\log x|^\delta)$$

with $(c,\alpha,\beta)\neq(c'\gamma,\delta)$ and $\alpha/2<\min(\eta,\gamma)<\alpha$ provides an example in which the conditions of Theorem 2 (i) are satisfied. Remark that Π does not have a bounded density, which is not surprising. Indeed, Theorem 2.2 in [FH2] shows, in the general framework of a Lévy process with bounded variation, that if the Lévy measure is bounded outside a neighbourhood of 0, then an estimate of the form (5) always holds.
(iii) The examples provided for $\alpha<1$ also work when $\alpha=1$. Besides, when $\alpha=1$, consider the case when $y>1/2$, Π is supported on $[y-1/2,y+1/2]$ and for $0\le x\le 1/2$,

$$g_+(x)=\frac{ax}{(-1+\log x)^2}$$

$$g_-(x)=\frac{bx}{(-1+\log x)^2}$$

with $b\neq a$. Then it is easily seen that Π has bounded density and that the conditions of Theorem 2 (i) are satisfied. Of course, the difference with the case $\alpha<1$ is that when $\alpha=1$, the process has infinite variation.
(iv) Theorem 2 (ii) indicates that, loosely speaking, adding S_t instead of B_t to Y_t is more regularizing for the function $h_y:t\mapsto\mathbb{P}(X_t\ge y)$. Moreover, the smaller α is, the easier it is to satisfy (4).

Proof of Theorem 2
The proof of (i) is the same as the proof of Theorem 1 (ii). Recall that this proof does not use the existence of $\mathbb{E}(G)$, and thus can be mimicked even in the case when $\alpha\le 1$, in which $\mathbb{E}(S_1)$ does not exist. On the other hand, the proof of Proposition 1 cannot be reproduced in the stable case. Indeed, an analogue of (3) no longer holds, since one would have to replace G with $|S_1|$ but $E|S_1|^n=\infty$ if $n\ge 2$.
Let us prove (ii). To simplify the notation, we assume that Π has total mass 1. Recall that there exists a family (c_n) of reals such that for every $N\ge 1$,

$$\mathbb{P}(S_t\in dy)=\sum_{n=1}^{N}c_nt^ny^{-n\alpha-1}+o(t^N) \tag{6}$$

as $t\to 0$. See Zolotarev [Z], Chapter 2.5. As in the proof of Theorem 1,

$$\mathbb{P}(X_t\ge y)=e^{-t}\mathbb{P}(S_t\ge y)+te^{-t}\mathbb{P}(S_t+Z_1\ge y)+\frac{t^2e^{-t}}{2}\mathbb{P}(S_t+Z_1+Z_2\ge y)+o(t^2)$$

Remark that

$$te^{-t}\mathbb{P}(S_t+Z_1\ge y)=t\mathbb{P}(S_t+Z_1\ge y)-t^2\mathbb{P}(S_t+Z_1\ge y)+o(t^2)$$

and

$$\frac{t^2e^{-t}}{2}\mathbb{P}(S_t+Z_1+Z_2\ge y)=\frac{t^2}{2}\,P(S_t+Z_1+Z_2\ge y)+o(t^2)$$

Together with (6), this entails

$$\mathbb{P}(X_t \geq y) = At + Bt^2 + t\mathbb{P}(S_t + Z_1 \geq y) + o(t^2) \tag{7}$$

for some constants A and B. The key point is to show that

$$\mathbb{P}(S_t + Z_1 \geq y) = \overline{\Pi}(y) + Ct + o(t) \tag{8}$$

for some constant C. Let us first handle the case when $\alpha > 1$. As already seen,

$$\begin{aligned} \mathbb{P}(S_t + Z_1 \geq y) - \overline{\Pi}(y) &= \mathbb{P}(Z_1 \in [y - t^{1/\alpha}S_1, y), S_1 > 0) \\ &\quad -\mathbb{P}(Z_1 \in [y, y + |t^{1/\alpha}S_1|), S_1 < 0) \end{aligned}$$

Let us consider the first term of the right-hand side:

$$I_1 := \mathbb{P}(Z_1 \in [y - t^{1/\alpha}S_1, y), S_1 > 0) = \int_0^\infty g(x)\mathbb{P}(Z_1 \in [y - t^{1/\alpha}x, y))dx$$

where g denotes the density of S_1. Put

$$F(z) = \mathbb{P}(Z_1 \in [y - z, y)) - az \tag{9}$$

Then

$$I_1 = at^{1/\alpha}\int_0^\infty xg(x)dx + \int_0^\infty g(x)F(t^{1/\alpha}x)dx$$

Let $\delta > 0$ and cut the last integral as follows:

$$\int_0^\infty g(x)F(t^{1/\alpha}x)dx = \int_0^{\delta t^{-1/\alpha}} + \int_{\delta t^{-1/\alpha}}^\infty$$

By a change of variable, the second integral can be rewritten as

$$\int_{\delta t^{-1/\alpha}}^\infty g(x)F(t^{1/\alpha}x)dx = t^{-1/\alpha}\int_\delta^\infty g(zt^{-1/\alpha})F(z)dz$$

Using Zolotarev's estimate (6) yields $g(zt^{-1/\alpha}) \sim K(zt^{-1/\alpha})^{-1-\alpha}$ for some $K > 0$ and thus we get

$$\int_{\delta t^{-1/\alpha}}^\infty g(x)F(t^{1/\alpha}x)dx = Kt\int_\delta^\infty F(z)\frac{dz}{z^{1+\alpha}} + H_1(\delta, t)$$

where the function $H_1(\delta, t)$ depends on δ but in any case, $H_1(\delta, t) = o(t)$. Let us consider the other integral, namely

$$I(\delta) := \int_0^{\delta t^{-1/\alpha}} g(x)F(t^{1/\alpha}x)dx$$

Then if $\delta < \delta_0$, the assumption (4) entails that for every $x \in [0, \delta]$, $|F(x)| \leq bx^\beta$, whence

$$|I(\delta)| < b\int_0^{\delta t^{-1/\alpha}} t^{\beta/\alpha}x^\beta g(x)dx \tag{10}$$

Let us bound, for large M,

$$\int_0^M x^\beta g(x)dx = \mathbb{E}(S_1^\beta \mathbf{1}_{\{0<S_1<M\}})$$

Write

$$\begin{aligned}\mathbb{E}(S_1^\beta \mathbf{1}_{\{0<S_1<M\}}) &= \int_0^\infty \mathbb{P}(S_1^\beta > x, S_1 < M)dx \\ &= \int_0^{M^\beta} \mathbb{P}(x^{1/\beta} < S_1 < M)dx \\ &\leq \int_{\log M}^{M^\beta} \mathbb{P}(x^{1/\beta} < S_1)dx + \log M\end{aligned}$$

Using again (6), we get that if $x \geq \log M$,

$$\mathbb{P}(x^{1/\beta} < S_1) \leq \frac{K}{x^{\alpha/\beta}}\left(1+\left(\frac{c}{\log M}\right)^{\alpha/\beta}\right)$$

for some $c > 0$. Therefore there exists some $M_1 > 0$ such that if $M > M_1$,

$$\int_0^M x^\beta g(x)dx \leq \frac{2KM^{\beta-\alpha}}{1-(\alpha/\beta)}$$

Using this estimate together with (10) leads to:

$$|I(\delta)| \leq \frac{2bK\delta^{\beta-\alpha}t}{1-(\alpha/\beta)}$$

whenever $\delta < \delta_0$ and $\delta t^{-1/\alpha} > M_1$. Thus for δ, t satisfying these conditions,

$$\begin{aligned}&\Big|\mathbb{P}(Z_1 \in [y - t^{1/\alpha}S_1, y), S_1 > 0) \\ &-at^{1/\alpha}\int_0^\infty xg(x)dx - Kt\int_\delta^\infty \left[\mathbb{P}(Z_1 \in [y-z, y)) - az\right]\frac{dz}{z^{1+\alpha}}\Big| \\ &\leq \frac{2bK\delta^{\beta-\alpha}t}{1-(\alpha/\beta)} + H_1(\delta, t)\end{aligned}$$

Similarly,

$$\begin{aligned}&\Big|\mathbb{P}(Z_1 \in [y, y + |t^{1/\alpha}S_1|), S_1 < 0) - \mathbb{P}(S_1 < 0)\Pi(\{y\}) \\ &-at^{1/\alpha}\int_{-\infty}^0 |x|g(x)dx - Kt\int_\delta^\infty \left[\mathbb{P}(Z_1 \in (y, y+z)) - az\right]\frac{dz}{z^{1+\alpha}}\Big| \\ &\leq \frac{2bK\delta^{\beta-\alpha}t}{1-(\alpha/\beta)} + H_2(\delta, t)\end{aligned}$$

Remark that in the formula above, we have replaced the semi-open interval $[y, y+z)$ with the open interval $(y, y+z)$ and this accounts for presence of the term $\mathbb{P}(S_1 < 0)\Pi(\{y\})$. Since S is stable with index $\alpha > 1$,

$$\int_0^\infty xg(x)dx - \int_{-\infty}^0 |x|g(x)dx = \mathbb{E}(S_1) = 0 \tag{11}$$

and this entails

$$\begin{aligned} &\Bigg| \mathbb{P}(S_t + Z_1 \geq y) - \left[\overline{\Pi}(y) - \mathbb{P}(S_1 < 0)\Pi(\{y\})\right] \\ &-Kt\left(\int_\delta^\infty [\mathbb{P}(Z_1 \in [y-z, y)) - \mathbb{P}(Z_1 \in (y, y+z))]\frac{dz}{z^{1+\alpha}}\right)\Bigg| \\ &\leq \frac{4bK\delta^{\beta-\alpha}t}{1-(\alpha/\beta)} + H_1(\delta, t) + H_2(\delta, t) \end{aligned}$$

Because of the assumption (4),

$$\int_\delta^\infty [\mathbb{P}(Z_1 \in [y-z, y)) - \mathbb{P}(Z_1 \in (y, y+z))]\frac{dz}{z^{1+\alpha}}$$

has a limit as $\delta \to 0$. Put

$$L = \int_0^\infty [\mathbb{P}(Z_1 \in [y-z, y)) - \mathbb{P}(Z_1 \in (y, y+z]))]\frac{dz}{z^{1+\alpha}}$$

Now fix $\epsilon > 0$. There exists δ_1 such that if $\delta \leq \delta_1$,

$$\left| L - \left(\int_\delta^\infty [\mathbb{P}(Z_1 \in [y-z, y]) - \mathbb{P}(Z_1 \in (y, y+z])]\frac{dz}{z^{1+\alpha}}\right)\right| \leq \epsilon$$

Moreover, one can choose $\delta > 0$ such that $\delta \leq \inf(\delta_0, \delta_1)$ and that

$$\frac{4bK\delta^{\beta-\alpha}}{1-(\alpha/\beta)} \leq \epsilon$$

For such a choice of δ, if t satisfies $\delta t^{-1/\alpha} > M_1$, i.e. $t < (\delta/M)^\alpha$, we have

$$\begin{aligned} &|\mathbb{P}(S_t + Z_1 \geq y) - [\overline{\Pi}(y) - \mathbb{P}(S_1 < 0)\Pi(\{y\})] - KLt| \\ &\leq 2\epsilon t + H_1(\delta, t) + H_2(\delta, t) \end{aligned}$$

Finally, since $H_1(\delta, t) + H_2(\delta, t) = o(t)$, one may choose t small enough so that

$$H_1(\delta, t) + H_2(\delta, t) \leq \epsilon t$$

and thus we have proved that if t is small enough,

$$|\mathbb{P}(S_t + Z_1 \geq y) - [\overline{\Pi}(y) - \mathbb{P}(S_1 < 0)\Pi(\{y\})] - KLt| \leq 3\epsilon t$$

which proves (8) in the case $\alpha > 1$.

When $\alpha = 1$, we replace (9) with

$$F(z) = \mathbb{P}(Z_1 \in [y - z, y)) - az\mathbf{1}_{(|z|<1)}$$

The proof then goes along the same lines. The only difference is that (11) is replaced by the following equality:

$$\int_0^1 xg(x)dx - \int_{-1}^0 |x|g(x)dx$$

which uses the symmetry of S.
Finally, when $\alpha < 1$, starting again from (7), we can directly evaluate, using a change of variable together with (6),

$$\begin{aligned}\mathbb{P}(Z_1 \in [y - t^{1/\alpha}S_1, y), S_1 > 0) &= \int_0^\infty g(x)\mathbb{P}(Z_1 \in [y - t^{1/\alpha}x, y)) \\ &\sim Kt\int_0^\infty \frac{dz}{z^{1+\alpha}}\mathbb{P}(Z_1 \in [y - z, y))\end{aligned}$$

The latter integral is convergent at 0 thanks to the assumptions of the theorem and this concludes the proof in the case $\alpha < 1$. □

Finally, let us state the analogue of Proposition 2 in the case when $X_t = S_t + Y_t$:

Proposition 3. *Suppose that for some $y > 0$, $\Pi(\{y\}) > 0$ and that Π has a continuous density on $\mathbb{R} - \{y\}$. Then for every fixed $c > 0$, as $t \to 0$,*

$$\mathbb{P}(X_t \geq y + ct^{1/\alpha}) \sim t\left[\overline{\Pi}(y) - \mathbb{P}(0 < S_1 \leq c)\Pi(\{y\})\right]$$

Here again, a similar result holds for $c < 0$.

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