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ON MEAN NUMBERS OF PASSAGE TIMES IN SMALL BALLS OF DISCRETIZED ITÔ PROCESSES

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Abstract

The aim of this note is to prove estimates on mean values of the number of times that Itô processes observed at discrete times visit small balls in $\mathbb{R}^d$. Our technique, in the infinite horizon case, is inspired by Krylov's arguments in [2, Chap.2]. In the finite horizon case, motivated by an application in stochastic numerics, we discount the number of visits by a locally exploding coefficient, and our proof involves accurate properties of last passage times at 0 of one dimensional semimartingales.

1 Introduction

The present work is motivated by two convergence rate analyses concerning discretization schemes for diffusion processes whose generators have non smooth coefficients: Bernardin et al. [1] study discretization schemes for stochastic differential equations with multivalued drift coefficients; Martinez and Talay [4] study discretization schemes for diffusion processes whose infinitesimal generators are of divergence form with discontinuous coefficients. In both cases, to have reasonable accuracies, a scheme needs to mimic the local behaviour of the exact solution in small neighborhoods of the discontinuities of the coefficients, in the sense that the expectations of the total times spent by the scheme (considered as a piecewise constant process) and the exact solution in these neighborhoods need to be close to each other. The objective of this paper is to provide two estimates for the expectations of such total times spent by fairly general Itô processes observed at discrete times; these estimates concern in particular the discretization schemes studied in the above references (see section 7 for more detailed explanations). To this end, we carefully modify the technique developed by Krylov [2] to prove inequalities of the type

$$\mathbb{E}\int_0^\infty e^{-\lambda t} f(X_t)dt \le C\|f\|_{L^d}$$

for positive Borel functions f and $\mathbb{R}^d$ valued continuous controlled diffusion processes (X_t) under some strong ellipticity condition (see also Krylov and Lipster [3] for extensions in the degenerate case). The difficulties here come from the facts that time is discrete and, in the finite horizon case, we discount by a locally exploding function of time the number of times that the discrete time process visits a given small ball in $\mathbb{R}^d$. Another difficulty comes from the fact that, because of the convergence rate analyses which motivate this work, we aim to get accurate estimates in terms of the time discretization step. Our estimates seem interesting in their own right: they contribute to show the robustness of Krylov's estimates.
Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(W_t)_{t\ge 0}$ be a m-dimensional standard Brownian motion on this space. Consider two progressively measurable processes $(b_t)_{t\ge 0}$ and $(\sigma_t)_{t\ge 0}$ taking values respectively in $\mathbb{R}^d$ and in the space $\mathscr{M}_{d,m}(\mathbb{R})$ of real $d\times m$ dimensional matrices. For X_0 in $\mathbb{R}^d$, consider the Itô process

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s. \tag{1}$$

We assume the following hypotheses:

Assumption 1.1. *There exists a positive number $K \ge 1$ such that, $\mathbb{P}$-a.s.,*

$$\forall t \ge 0,\ \|b_t\| \le K, \tag{2}$$

and

$$\forall\, 0 \le s \le t,\ \frac{1}{K^2}\int_s^t \psi(u)du \le \int_s^t \psi(u)\|\sigma_u\sigma_u^*\|du \le K^2\int_s^t \psi(u)du \tag{3}$$

for all positive locally integrable map $\psi : \mathbb{R}^+ \to \mathbb{R}^+$.

Notice that (3) is satisfied when (σ_t) is a bounded continuous process; our slightly more general framework is motivated by the reduction of our study, without loss of generality, to one dimensional Itô processes: see section 2.
The aim of this note is to prove the following two results.
Our first result concerns discounted expectations of the total number of times that an Itô process observed at discrete times visits small neighborhoods of an arbitrary point in $\mathbb{R}^d$.

Theorem 1.1 (Infinite horizon). *Let (X_t) be as in (1). Suppose that the hypotheses 1.1 are satisfied. Let $h > 0$. There exists a constant C, depending only on K, such that, for all $\xi \in \mathbb{R}^d$ and $0 < \varepsilon < 1/2$, there exists $h_0 > 0$ (depending on ε) satisfying*

$$\forall\, h \le h_0,\ h\sum_{p=0}^{\infty} e^{-ph}\mathbb{P}\left(\|X_{ph} - \xi\| \le \delta_h\right) \le C\delta_h, \tag{4}$$

where $\delta_h := h^{1/2-\varepsilon}$.

Our second result concerns expectations of the number of times that an Itô process, observed at discrete times between 0 and a finite horizon T, visits small neighborhoods of an arbitrary point ξ in $\mathbb{R}^d$. In this setting, the expectations are discounted with a function $f(t)$ which may tend to infinity when t tends to T. Such a discount factor induces technical difficulties which might seem artificial to the reader. However, this framework is necessary to analyze the convergence rate of simulation methods for Markov processes whose generators are of divergence form with discontinuous coefficients: see section 7 and Martinez and Talay [4]. More precisely, we consider functions f satisfying the following hypotheses:

Assumption 1.2.

1. *f is positive and increasing,*
2. *f belongs to $C^1([0,T);\mathbb{R}^+)$,*
3. *f^α is integrable on $[0,T)$ for all $1 \le \alpha < 2$,*
4. *there exists $1 < \beta < 1+\eta$, where $\eta := \frac{1}{4K^4}$, such that*

$$\int_0^T f^{2\beta-1}(v) f'(v) \frac{(T-v)^{1+\eta}}{v^\eta} \, dv < +\infty. \tag{5}$$

Remember that we have supposed $K \ge 1$. An example of a suitable function f is the function $t \to \frac{1}{\sqrt{T-t}}$ (for which one can choose $\beta = 1 + \frac{1}{8K^4}$).

Theorem 1.2 (Finite horizon). *Let (X_t) be as in (1). Suppose that the hypotheses 1.1 are satisfied. Let f be a function on $[0,T)$ satisfying the hypotheses 1.2. There exists a constant C, depending only on β, K and T, such that, for all $\xi \in \mathbb{R}^d$ and $0 < \varepsilon < 1/2$, there exists $h_0 > 0$ (depending on ε) satisfying*

$$\forall\, h \le h_0,\ h \sum_{p=0}^{N_h} f(ph) \mathbb{P}\left(\|X_{ph} - \xi\| \le \delta_h \right) \le C\delta_h, \tag{6}$$

where $N_h := \lfloor T/h \rfloor - 1$ and where $\delta_h := h^{1/2-\varepsilon}$.

Remark 1.1. Since all the norms in $\mathbb{R}^d$ are equivalent, without loss of generality we may choose $\|Y\| = \sup_{1\le i\le d} \left|Y^i\right|$, Y^i denoting the i^{th} component of Y.

Remark 1.2. Changing X_0 into $X_0 - \xi$ allows us to limit ourselves to the case $\xi = 0$ without loss of generality, which we actually do in the sequel.

2 Reduction to one dimensional (X_t)'s

In order to prove Theorem 1.1, it suffices to prove that, for all $1 \le i \le d$ and $h \le h_0$,

$$h \sum_{p=0}^{\infty} e^{-ph} \mathbb{P}\left(\left|X^i_{ph}\right| \le \delta_h \right) \le C\delta_h,$$

or, similarly, for Theorem 1.2

$$h \sum_{p=0}^{N_h} f(ph) \mathbb{P}\left(|X^i_{ph}| \le \delta_h \right) \le C\delta_h.$$

Set

$$M_t^i := \sum_{j=1}^{m} \left(\int_0^t \sigma_s^{ij} dW_s^j \right).$$

The martingale representation theorem implies that there exist a process $(f_t^i)_{t\geq 0}$ and a standard $\mathscr{F}_t$-Brownian motion $(B_t)_{t\geq 0}$ on an extension of the original probability space, such that

$$M_t^i = \int_0^t f_s^i dB_s,$$

and

$$\forall t > 0, \int_0^t (f_\theta^i)^2 d\theta = \sum_{j=1}^{m} \int_0^t \left(\sigma_\theta^{ij}\right)^2 d\theta.$$

Therefore it suffices to prove Theorems 1.1 and 1.2 for all one dimensional Itô process

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dB_s \tag{7}$$

such that, $\exists K \geq 1$, $\mathbb{P}$-a.s.,

$$\forall\, t \geq 0,\ |b_t| \ \leq K, \tag{8}$$

and

$$\forall\, 0 \leq s \leq t,\ \frac{1}{K^2} \int_s^t \psi(u) du \leq \int_s^t \psi(u) \sigma_u^2 du \leq K^2 \int_s^t \psi(u) du \tag{9}$$

for all positive locally integrable map $\psi : \mathbb{R}^+ \to \mathbb{R}^+$.

3 An estimate for a localizing stopping time

In this section we prove a key proposition which is a variant of Theorem 4 in [2, Chap.2]: here we need to consider time indices and stopping times which take values in a mesh with step-size h. We carefully modify Krylov's arguments. We start by introducing two non-decreasing sequences of random times $(\theta_p^h)_p$ and $(\tau_p^h)_p$ taking values in $h\overline{\mathbb{N}}$ as

$$\begin{cases} \theta_0^h & = \tau_0^h = 0 \\ \theta_{p+1}^h & = \inf\{t \geq \tau_p^h,\, t \in h\mathbb{N},\, \left|X_t\right| \leq \delta_h\},\ p \geq 0 \\ \tau_p^h & = \inf\{t \geq \theta_p^h,\, t \in h\mathbb{N},\, \left|X_t - X_{\theta_p^h}\right| \geq 1\},\ p \geq 1. \end{cases} \tag{10}$$

Our goal is then to prove the following equality :

$$\mathbb{E}\left(h \sum_{p=0}^{+\infty} e^{-ph} \mathbf{1}_{[-\delta_h,\delta_h]}(X_{ph}) \right) = \sum_{p=0}^{+\infty} \mathbb{E}\left(\mathbf{1}_{\theta_p^h < +\infty} \int_{\theta_p^h}^{\tau_p^h} e^{-\eta_s} \mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s}) ds \right), \tag{11}$$

where

$$\eta_s := h \lfloor s/h \rfloor \ \text{ for all } s \geq 0. \tag{12}$$

It is easy to see that (11) holds if the sequence $\left(\theta_p^h\right)_p$ takes an infinite value in a finite time, or converges to infinity when p tends to infinity, which is a consequence of Lemma 3.1 (see below). For the proof of Lemma 3.1, we need the following

Proposition 3.1. *Let (X_t) be as in (7). Suppose that conditions (8) and (9) hold true. Let $\theta : \Omega \to h\overline{\mathbb{N}}$ be a $(\mathscr{F}_t)$-stopping time. Then $\tau := \inf\{t \in h\mathbb{N},\, t \geq \theta,\, \left|X_t - X_\theta\right| \geq 1\}$ is a $(\mathscr{F}_t)$-stopping time, and*

$$\mathbb{E}\left(\mathbf{1}_{\tau<+\infty} e^{-\tau} \middle| \mathscr{F}_\theta \right) \leq \frac{1}{\mathrm{ch}(\mu)} e^{-\theta} \mathbf{1}_{\theta<+\infty} \text{ a.s.}, \tag{13}$$

where

$$\mu > 0 \text{ and } 1 - K\mu - (1/2)K^2\mu^2 = 0.$$

Proof. Set $\pi(z) := \cosh(\mu z)$ for all z in $\mathbb{R}$ and

$$L_u\pi(z) := b_u \pi'(z) + \frac{1}{2}\sigma_u^2 \pi''(z).$$

Let $A \in \mathscr{F}_\theta$. Suppose that we have proven that, for all $t > 0$,

$$e^{-t\wedge\theta} \geq \mathbb{E}\left(e^{-\tau} \pi(X_{t\wedge\tau} - X_{t\wedge\theta}) \middle| \mathscr{F}_{t\wedge\theta} \right). \tag{14}$$

Then we would have

$$\begin{aligned}
\mathbb{E}\left(\mathbf{1}_A\, \mathbf{1}_{\theta<+\infty}\, e^{-\theta}\right) &= \lim_{t\to+\infty} \mathbb{E}\left(\mathbf{1}_A\, \mathbf{1}_{\theta\leq t}\, e^{-\theta\wedge t}\right) \\
&\geq \lim_{t\to+\infty} \mathbb{E}\left(\mathbf{1}_A\, \mathbf{1}_{\theta\leq t}\, \mathbb{E}\left(e^{-\tau}\pi(X_{t\wedge\tau} - X_{t\wedge\theta}) \middle| \mathscr{F}_{t\wedge\theta}\right)\right) \\
&= \lim_{t\to+\infty} \mathbb{E}\left(\mathbf{1}_A\, \mathbf{1}_{\theta\leq t}\, \mathbb{E}\left(e^{-\tau}\pi(X_{t\wedge\tau} - X_{t\wedge\theta}) \middle| \mathscr{F}_{\theta}\right)\right) \\
&= \lim_{t\to+\infty} \mathbb{E}\left(\mathbf{1}_A\, \mathbf{1}_{\theta\leq t}\, e^{-\tau}\pi(X_{t\wedge\tau} - X_{t\wedge\theta})\right) \\
&= \lim_{t\to+\infty} \mathbb{E}\left(\mathbf{1}_A\, \mathbf{1}_{\theta<+\infty}\, e^{-\tau}\pi(1)\right) \\
&\geq \mathbb{E}\left(\mathbf{1}_A\, \cosh(\mu)\mathbf{1}_{\tau<+\infty} e^{-\tau}\right),
\end{aligned}$$

and thus the desired result would have been obtained. We now prove the inequality (14).
Fix $t \geq 0$ arbitrarily and set

$$B_1 := \{t \leq \theta\} \cap \{t \leq \tau\} = \{t \leq \theta\},$$

and

$$C_1 := \{t > \theta\} \cap \{t \leq \tau\} = \{t > \theta\} \cap \left\{\forall s \in [\theta, t] \cap h\mathbb{N},\, \left|X_s - X_\theta\right| < 1\right\}.$$

The sets B_1 and C_1 are $\mathscr{F}_t$-measurable. Consequently the set $\{t \leq \tau\}$ is $\mathscr{F}_t$-measurable and thus τ is a $(\mathscr{F}_t)$-stopping time.
To show (14), we start with checking that, $\mathbb{P}$-a.s, for all $0 \leq s \leq t$ and $z \in \mathbb{R}$,

$$\int_s^t \psi(u)\left(\pi(z) - L_u\pi(z)\right) du \geq \cosh(\mu z)\left(1 - K\mu - \frac{1}{2}K^2\mu^2\right)\int_s^t \psi(u) du = 0 \tag{15}$$

for all positive locally bounded map $\psi : \mathbb{R}^+ \to \mathbb{R}^+$. Indeed, for all $z \in \mathbb{R}$ and $u \geq 0$,

$$\pi(z) - L_u\pi(z) = \cosh(\mu z) - \mu\, b_u \sinh(\mu z) - \frac{1}{2}\mu^2\sigma_u^2 \cosh(\mu z).$$

As $\sinh(\mu\,|z|) \leq \cosh(\mu z)$, we get (15) by using (8) and (9).

Now, we observe that $\sup_{\theta\leq t} e^{\gamma X_\theta}$ is integrable for all $\gamma > 0$ since by (9), (b_t) and $(\langle X\rangle_t)$ are bounded. This justifies the following equality deduced from the Itô's formula : for all $y \in \mathbb{R}$,

$$e^{-t\wedge\theta}\pi(X_{t\wedge\theta}+y) = \mathbb{E}\left(e^{-t\wedge\tau}\pi(X_{t\wedge\tau}+y) \Big| \mathscr{F}_{t\wedge\theta}\right) + \mathbb{E}\left(\int_{t\wedge\theta}^{t\wedge\tau} e^{-s}(\pi(X_s+y) - L_s\pi(X_s+y))ds \middle| \mathscr{F}_{t\wedge\theta}\right) \text{ a.s..}$$

Therefore, in view of (15), there exists a set $\mathscr{N}_y$ with $\mathbb{P}(\mathscr{N}_y) = 0$ such that

$$e^{-t\wedge\theta}\pi(X_{t\wedge\theta}+y) \geq \mathbb{E}\left(e^{-\tau}\pi(X_{t\wedge\tau}+y) \Big| \mathscr{F}_{t\wedge\theta}\right) \quad \text{on } \Omega - \mathscr{N}_y. \tag{16}$$

We now use the continuity of π in order to replace y by $-X_{t\wedge\theta}$. For all positive integer q, let $k_q(y)$ be the unique integer ℓ such that $\ell/2^q \leq y < (\ell+1)/2^q$. Then

$$\pi(X_{t\wedge\tau} - X_{t\wedge\theta}) = \lim_{q\to+\infty} \pi\left(X_{t\wedge\tau} - \frac{k_q(X_{t\wedge\theta})}{2^q}\right).$$

Therefore, by Fatou's lemma and (16),

$$\begin{aligned}\mathbb{E}\left(e^{-\tau}\pi(X_{t\wedge\tau} - X_{t\wedge\theta}) \Big| \mathscr{F}_{t\wedge\theta}\right) &\leq \liminf_{q\to+\infty} \mathbb{E}\left(e^{-\tau}\pi\left(X_{t\wedge\tau} + \frac{k_q(-X_{t\wedge\theta})}{2^q}\right) \middle| \mathscr{F}_{t\wedge\theta}\right) \\ &\leq \liminf_{q\to+\infty} e^{-t\wedge\theta}\pi\left(X_{t\wedge\theta} + \frac{k_q(-X_{t\wedge\theta})}{2^q}\right) \\ &= e^{-t\wedge\theta}.\end{aligned}$$

□

We end this section by the following

Lemma 3.1. *For all $p \geq 0$,*

$$\mathbb{E}\left(\mathbf{1}_{\theta_p^h<+\infty} e^{-\theta_p^h}\right) \leq \left(\frac{1}{\cosh(\mu)}\right)^p, \tag{17}$$

where μ is defined as in Proposition 3.1.

Proof. Apply Proposition 3.1 to the stopping times θ_p^h and τ_p^h. It comes

$$\mathbb{E}\left(\mathbf{1}_{\tau_p^h<+\infty} e^{-\tau_p^h} \middle| \mathscr{F}_{\theta_p^h}\right) \leq \frac{1}{\cosh(\mu)} e^{-\theta_p^h} \mathbf{1}_{\theta_p^h<+\infty} \; \mathbb{P}-a.s.,$$

from which

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{\theta_{p+1}^h<+\infty} e^{-\theta_{p+1}^h}\right) \leq \mathbb{E}\left(\mathbf{1}_{\tau_p^h<+\infty} e^{-\tau_p^h}\right) &= \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\tau_p^h<+\infty} e^{-\tau_p^h} \middle| \mathscr{F}_{\theta_p^h}\right)\right) \\ &\leq \frac{1}{\cosh(\mu)} \mathbb{E}\left(\mathbf{1}_{\theta_p^h<+\infty} e^{-\theta_p^h}\right) \leq \cdots \leq \left(\frac{1}{\cosh(\mu)}\right)^{p+1}.\end{aligned}$$

□

In the next section we carefully estimate the right-hand side of (11).

4 Estimate for the localized problem in the infinite horizon setting

In this section we aim to prove the following lemma.

Lemma 4.1. *Consider a one dimensional Itô process (7) satisfying (8) and (9). Let $\theta : \Omega \to h\overline{\mathbb{N}}$ be a $\mathcal{F}_t$-stopping time such that $X_\theta \in [-\delta_h, \delta_h]$ almost surely. Let τ be the stopping time $\tau := \inf\{t \geq \theta,\ t \in h\mathbb{N},\ \left|X_t - X_\theta\right| \geq 1\}$. There exists a constant C, independent of θ and τ, such that, for all h small enough,*

$$\mathbb{E}\left(\mathbf{1}_{\theta<+\infty}\int_\theta^\tau e^{-\eta_s}\mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s})ds\right) \leq C\delta_h\sqrt{\mathbb{E}\left(\mathbf{1}_{\theta<+\infty}e^{-\theta}\right)}. \tag{18}$$

Proof. To start with, we construct a function I whose second derivative is $\mathbf{1}_{[-2\delta_h,2\delta_h]}$ in order to be able to apply Itô's formula. Let $I : \mathbb{R} \to \mathbb{R}$ be defined as

$$I(z) = \begin{cases} 0 & \text{if } z \leq -2\delta_h, \\ \frac{1}{2}z^2 + 2\delta_h z + 2\delta_h^2 & \text{if } -2\delta_h \leq z \leq 2\delta_h, \\ 4\delta_h z & \text{if } 2\delta_h \leq z. \end{cases} \tag{19}$$

The map I is of class $C^1(\mathbb{R})$ and of class $C^2(\mathbb{R} \setminus (\{-2\delta_h\} \cup \{2\delta_h\}))$. In addition, for all $z \in \mathbb{R}$, $I''(z) = \mathbf{1}_{[-2\delta_h,2\delta_h]}(z)$ (we set $I''(-2\delta_h) = I''(2\delta_h) = 1$). For h small enough, there exists a constant C independent of h such that

$$\sup_{-1-\delta_h\leq z\leq 1+\delta_h} |I(z)| \leq C\delta_h, \qquad \sup_{z\in\mathbb{R}}\left|I'(z)\right| \leq C\delta_h. \tag{20}$$

Set

$$q_h := \mathbf{1}_{\theta<+\infty}\int_\theta^\tau e^{-\eta_s}\mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s})ds, \tag{21}$$

and observe that, for all $s \geq 0$,

$$\begin{aligned}&\{X_{\eta_s} \in [-\delta_h, \delta_h]\} \\ &\quad = \left(\{X_{\eta_s} \in [-\delta_h,\delta_h]\} \cap \{X_s \in [-2\delta_h, 2\delta_h]\}\right) \cup \left(\{X_{\eta_s} \in [-\delta_h,\delta_h]\} \cap \{X_s \notin [-2\delta_h, 2\delta_h]\}\right),\end{aligned}$$

so that

$$\mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s}) \leq \mathbf{1}_{[-2\delta_h,2\delta_h]}(X_s) + \mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s})\mathbf{1}_{\mathbb{R}\setminus[-2\delta_h,2\delta_h]}(X_s). \tag{22}$$

From (21) and (22),

$$q_h \leq A^h + B^h, \tag{23}$$

where

$$A^h = \mathbf{1}_{\theta<+\infty}\int_\theta^\tau e^{-\eta_s}\mathbf{1}_{[-2\delta_h,2\delta_h]}(X_s)ds \tag{24}$$

and

$$B^h = \mathbf{1}_{\theta<+\infty}\int_\theta^\tau e^{-\eta_s}\mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s})\mathbf{1}_{\mathbb{R}\setminus[-2\delta_h,2\delta_h]}(X_s)ds. \tag{25}$$

An upper bound for A^h. In view of (9), one has

$$A^h \leq \mathbf{1}_{\theta<+\infty} K^2 e^{-h} \int_\theta^\tau e^{-s} \mathbf{1}_{[-2\delta_h, 2\delta_h]}(X_s) \sigma_s^2 ds = \mathbf{1}_{\theta<+\infty} K^2 e^{-h} \int_\theta^\tau e^{-s} I''(X_s) \sigma_s^2 ds.$$

Notice that $(\int_0^t e^{-s} I'(X_s) \sigma_s dW_s)$ is a uniformly square integrable martingale. Thus, the limit $\int_0^\infty e^{-s} I'(X_s) \sigma_s dW_s$ exists a.s. As I'' is continuous except at points $-2\delta_h$ and $2\delta_h$, an extended Itô's formula (see, e.g., Protter [5, Thm.71,Chap.IV]) implies

$$\begin{aligned} A^h \leq \mathbf{1}_{\theta<+\infty} K^2 e^{-h} \Big\{ & e^{-\tau} I\left(X_\tau\right) - e^{-\theta} I\left(X_\theta\right) + \int_\theta^\tau e^{-s} I(X_s) ds \\ & - \int_\theta^\tau e^{-s} I'(X_s) b_s ds - \int_\theta^\tau e^{-s} I'(X_s) \sigma_s dW_s \Big\}. \end{aligned} \quad (26)$$

In view of (20) we have $|I(X_s)| \leq C\delta_h$ even on the random set $\{\tau = +\infty\}$. In addition,

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\tau<\infty} \int_0^\tau e^{-s} I(X_s) ds \right] \quad \leq \quad & \mathbb{E}\left[\mathbf{1}_{\tau<\infty} \int_0^\tau e^{-s} |I(X_{\eta_s})| ds \right] \\ & + \mathbb{E}\left[\mathbf{1}_{\tau<\infty} \int_0^{+\infty} e^{-s} |I(X_s) - I(X_{\eta_s})| ds \right]. \end{aligned}$$

The first expectation is bounded from above by $C\delta_h$ since, for all integer k such that $\theta \leq kh \leq \tau - h$, one has $|X_{kh}| \leq 1 + \delta_h$. The second one is obviously bounded from above by $C\sqrt{h}$. Thus,

$$\mathbb{E}\left(A^h\right) \leq C\delta_h \mathbb{E}\left(\mathbf{1}_{\theta<+\infty} e^{-\theta}\right). \quad (27)$$

An upper bound for B^h. Setting $M_s := \int_0^s \sigma_u dW_u$, one has

$$\begin{aligned} B^h &\leq \mathbf{1}_{\theta<+\infty} \int_\theta^\tau e^{-\eta_s} \mathbf{1}_{[\delta_h, +\infty)}(\left|X_s - X_{\eta_s}\right|) ds \\ &= \mathbf{1}_{\theta<+\infty} e^{-\theta} \int_0^{\tau-\theta} e^{-\eta_s} \mathbf{1}_{[\delta_h, +\infty)}(\left|X_{s+\theta} - X_{\eta_s+\theta}\right|) ds \\ &\leq \mathbf{1}_{\theta<+\infty} e^{-\theta} \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h, +\infty)} \left(\left| \int_{\eta_s+\theta}^{s+\theta} b_u du + M_{s+\theta} - M_{\eta_s+\theta} \right| \right) ds \\ &\leq \mathbf{1}_{\theta<+\infty} e^{-\theta} \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h - Kh, +\infty)} \left(\left| M_{s+\theta} - M_{\eta_s+\theta} \right| \right) ds \\ &\leq \mathbf{1}_{\theta<+\infty} e^{-\theta} \int_0^{+\infty} e^{-\eta_s} \mathbf{1}_{[\delta_h/2, +\infty)} \left(\left| M_{s+\theta} - M_{\eta_s+\theta} \right| \right) ds, \end{aligned}$$

where we have used (8) and h is assumed small enough. We now successively use the Cauchy-Schwarz, Jensen and Bernstein inequalities (see, e.g., Revuz and Yor [6, Ex.3.16,Chap.IV] for the

Bernstein inequality for martingales):

$$
\begin{aligned}
\mathbb{E}B^h &\leq \sqrt{\mathbb{E}\left(\mathbf{1}_{\theta<+\infty}e^{-\theta}\right)^2}\sqrt{\mathbb{E}\left(\left(\int_0^{+\infty} e^{-\eta_s}\mathbf{1}_{[\delta_h/2,+\infty)}\left(\left|M_{s+\theta}-M_{\eta_s+\theta}\right|\right)ds\right)^2\right)} \\
&\leq \sqrt{\mathbb{E}\left(\mathbf{1}_{\theta<+\infty}e^{-\theta}\right)}\sqrt{\mathbb{E}\left(\left(\int_0^{+\infty} e^{-\eta_s}\mathbf{1}_{[\delta_h/2,+\infty)}\left(\left|M_{s+\theta}-M_{\eta_s+\theta}\right|\right)ds\right)\right)} \\
&\leq \sqrt{\mathbb{E}\left(\mathbf{1}_{\theta<+\infty}e^{-\theta}\right)}\sqrt{\int_0^{+\infty} e^{-\eta_s}\mathbb{P}\left(\sup_{\eta_s+\theta\leq u\leq s+\theta}\left|M_u-M_{\eta_s+\theta}\right|\geq\frac{\delta_h}{2}\right)ds} \\
&\leq C\sqrt{\mathbb{E}\left(\mathbf{1}_{\theta<+\infty}e^{-\theta}\right)}\sqrt{\int_0^{+\infty} e^{-\eta_s}\exp\left(-\frac{1}{8K^2}\frac{\delta_h^2}{h}\right)ds} \\
&\leq C\sqrt{\mathbb{E}\left(\mathbf{1}_{\theta<+\infty}e^{-\theta}\right)}\exp\left(-\frac{1}{16K^2}h^{-2\varepsilon}\right).
\end{aligned}
\tag{28}
$$

One then deduces (18) from (21), (23), (27) and (28). □

5 End of the proof of Theorem 1.1

We apply Lemmas 4.1 and 3.1 and the equality (11) and deduce

$$
\begin{aligned}
h\mathbb{E}\left(\sum_{p=0}^{+\infty} e^{-ph}\mathbf{1}_{[-\delta_h,\delta_h]}(X_{ph})\right) &= \sum_{p=0}^{+\infty}\mathbb{E}\left(\mathbf{1}_{\theta_p^h<+\infty}\int_{\theta_p^h}^{\tau_p^h} e^{-\eta_s}\mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s})ds\right) \\
&\leq C\delta_h\sum_{p=0}^{+\infty}\sqrt{\mathbb{E}\left(\mathbf{1}_{\theta_p^h<+\infty}e^{-\theta_p^h}\right)} \\
&\leq C\delta_h\sum_{p=0}^{+\infty}\left(\frac{1}{\cosh(\mu)}\right)^{\frac{p}{2}},
\end{aligned}
$$

which ends the proof of Theorem 1.1.

6 Proof of Theorem 1.2

In the finite horizon setting, we do not need to localize by means of an appropriate sequence of stopping times as in the preceding section. We may start from

$$
\mathbb{E}\left(h\sum_{p=0}^{N_h} f(ph)\mathbf{1}_{[-\delta_h,\delta_h]}(X_{ph})\right) = \sum_{p=0}^{N_h}\mathbb{E}\left(\int_{ph}^{(p+1)h} f(\eta_s)\mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s})ds\right). \tag{29}
$$

In view of (22) we have

$$\begin{aligned}\mathbb{E}\left(h\sum_{p=0}^{N_h} f(ph)\mathbf{1}_{[-\delta_h,\delta_h]}(X_{ph})\right) &\leq \mathbb{E}\left(\int_0^T f(\eta_s)\mathbf{1}_{[-2\delta_h,2\delta_h]}(X_s)ds\right)+\sum_{p=0}^{N_h}\mathbb{E}\left(D_h^p\right)\\ &=: A^h(T)+\sum_{p=0}^{N_h}\mathbb{E}\left(D_p^h\right),\end{aligned} \tag{30}$$

where for all $p \in \{0,\dots,N_h\}$

$$D_p^h := \int_{ph}^{(p+1)h} f(\eta_s)\mathbf{1}_{[-\delta_h,\delta_h]}(X_{\eta_s})\mathbf{1}_{\mathbb{R}\setminus[-2\delta_h,2\delta_h]}(X_s)ds. \tag{31}$$

An upper bound for $A^h(T)$.
In view of (9), as the function f is increasing, one has

$$A^h(T) \leq K^2\mathbb{E}\left(\int_0^T f(s)\mathbf{1}_{[-2\delta_h,2\delta_h]}(X_s)\sigma_s^2 ds\right).$$

Since we may have $\lim_{s\uparrow T} f(s) = +\infty$, the proof of Lemma 4.1 does not apply.
Set $\tau_y := \inf\{s > 0 \,:\, X_s - y = 0\}$. The generalized occupation time formula (see, e.g., Revuz and Yor [6, Ex.1.15,Chap.VI]) implies that

$$\begin{aligned}\mathbb{E}\left(\int_0^T f(s)\mathbf{1}_{[-2\delta_h,2\delta_h]}(X_s)\sigma_s^2 ds\right) &= \int_{\mathbb{R}} dy\,\mathbf{1}_{[-2\delta_h,2\delta_h]}(y)\mathbb{E}\left(\int_0^T f(s)dL_s^y(X)\right)\\ &= \int_{\mathbb{R}} dy\,\mathbf{1}_{[-2\delta_h,2\delta_h]}(y)\mathbb{E}\left(\mathbf{1}_{\tau_y\leq T}\int_{\tau_y}^T f(s)dL_s^y(X)\right),\end{aligned} \tag{32}$$

where $\{L_s^y(X) \,:\, s \in [0,T], y \in \mathbb{R}\}$ denotes a bi-continuous càdlàg version of the local time of the process $(X_t)_{t\geq 0}$.
Our aim now is to bound from above $\mathbb{E}\left(\mathbf{1}_{\tau_y\leq T}\int_{\tau_y}^T f(s)dL_s^y(X)\right)$ uniformly in y in $[-2\delta_h, 2\delta_h]$. It clearly suffices to show that there exists a constant C_T depending only β, K and T, such that

$$\mathbb{E}\left(\int_0^T f(s)dL_s^0(Y)\right) \leq C_T, \tag{33}$$

where the semimartingale Y is defined as $Y := X - y$.
Because f is not square integrable over $[0,T)$, we arbitrarily choose $\gamma > 0$. Now f is square integrable on $[0,T-\gamma]$ and the function $t \mapsto \int_0^t f(s)dL_s^0(Y)$ is the local time of the local semimartingale

$$\{f(g_t)Y_t,\ t \in [0,T-\gamma]\},$$

where $g_t := \sup\{s < t \;:\; Y_s = 0\}$ (see, e.g., Revuz and Yor [6, Sec.4,Chap.VI]). The Stieltjes measure induced by the function $s \mapsto f(g_s)$ has support $\mathscr{Z} := \{s \geq 0 \,:\, Y_s = 0\}$, and therefore

$$\int_0^{T-\gamma} |Y_\theta|df(g_\theta) = 0,$$

so that Itô-Tanaka formula implies

$$\mathbb{E}\left(\int_0^{T-\gamma} f(s)dL_s^0(Y)\right) = \mathbb{E}\left(|f(g_{T-\gamma})Y_{T-\gamma}|\right) - f(g_0)|Y_0| - \mathbb{E}\left(\int_0^{T-\gamma} f(g_s)\mathrm{sgn}((Y_s))dY_s\right). \quad (34)$$

Apply Theorem 4.2 of Chap.VI in [6]. It comes:

$$\mathbb{E}\left(|f(g_{T-\gamma})Y_{T-\gamma}|\right) = \mathbb{E}\left(|\, f(0)Y_0 + \int_0^{T-\gamma} f(g_s)dY_s\,|\right),$$

from which

$$\begin{aligned}\mathbb{E}\left(|f(g_{T-\gamma})Y_{T-\gamma}|\right) &\leq |\, f(0)Y_0\,| + \mathbb{E}\left(|\int_0^{T-\gamma} f(g_s)\sigma_s dW_s\,|\right) + \mathbb{E}\left(|\int_0^{T-\gamma} f(s)b_s ds\,|\right) \\ &\leq C_T + K^2\left(\int_0^T \mathbb{E}\left(f^2(g_s)\right)ds\right)^{\frac{1}{2}}.\end{aligned} \quad (35)$$

Coming back to (34) we deduce

$$\mathbb{E}\left(\int_0^{T-\gamma} f(s)dL_s^0(Y)\right) \leq C_T + K^2\left(\int_0^T \mathbb{E}\left(f^2(g_s)\right)ds\right)^{\frac{1}{2}}.$$

Thus, as $\beta > 1$ by assumption, (33) will be proven if we can show that

$$\int_0^T \tilde{\mathbb{E}}\left[f^{2\beta}(g_s)\right] ds < +\infty, \quad (36)$$

where $\tilde{\mathbb{P}}$ is the probability measure defined by a Girsanov transformation removing the drift of $(Y_t, 0 \leq t \leq T)$.

Set $\Lambda(t) := \langle Y \rangle_t$. Observe that

$$\sup\{\theta \leq s;\ Y_\theta = 0\} = \Lambda^{-1}\left(\sup\{\theta \leq \Lambda(s);\ Y_{\Lambda^{-1}(\theta)} = 0\}\right).$$

In addition, the Dubins-Schwarz theorem implies that there exists a $\tilde{\mathbb{P}}$-Brownian Motion $(\tilde{B}_t)_{t\geq 0}$ such that $Y_s = \tilde{B}_{\Lambda(s)}$. Therefore,

$$\begin{aligned}\tilde{\mathbb{P}}\left[g_s \geq u\right] &= \tilde{\mathbb{P}}\left[\Lambda^{-1}_{\sup\{t<\Lambda(s)\,:\,X_{\Lambda^{-1}(t)}=0\}} \geq u\right] \\ &= \tilde{\mathbb{P}}\left[\sup\{t < \Lambda(s)\ :\ \tilde{B}_t = 0\} \geq \Lambda(u)\right] \\ &= \tilde{\mathbb{P}}\left[\inf\{t > \Lambda(u)\ :\ \tilde{B}_t = 0\} \leq \Lambda(s)\right] \\ &\leq \tilde{\mathbb{P}}\left[\inf\{t > 0\ :\ Y_u + \tilde{B}_{u+t} - \tilde{B}_u = 0\} \leq K^2(s-u)\right],\end{aligned}$$

from which

$$\begin{aligned}\tilde{\mathbb{P}}\left[g_s \geq u\right] &\leq \tilde{\mathbb{E}}\left[\int_0^{K^2(s-u)} \frac{1}{\sqrt{2\pi\theta^3}}|Y_u|\exp\left(-\frac{Y_u^2}{2\theta}\right)d\theta\right] \\ &\leq C\tilde{\mathbb{E}}\left[\exp\left(-\frac{Y_u^2}{4K^2(s-u)}\right)\right].\end{aligned}$$

For $0 \leq v < s$ set

$$m_v := -\int_0^v \frac{Y_\theta}{2K^2(s-\theta)} dY_\theta.$$

Itô's formula applied to $h(v,y) = \dfrac{y^2}{s-v}$ yields

$$\frac{Y_v^2}{s-v} = \frac{Y_0^2}{s} + \int_0^v \frac{2Y_\theta}{s-\theta} dY_\theta + \int_0^v \frac{Y_\theta^2}{(s-\theta)^2} d\theta + \int_0^v \frac{1}{s-\theta} d\langle Y\rangle_\theta,$$

and then

$$-\frac{Y_v^2}{4K^2(s-v)} \leq m_v - \frac{1}{4K^2}\int_0^v \frac{Y_\theta^2}{(s-\theta)^2} d\theta - \frac{1}{4K^2}\int_0^v \frac{1}{s-\theta} d\langle Y\rangle_\theta,$$

Therefore, using (9),

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\exp\left(-\frac{Y_v^2}{4K^2(s-v)}\right)\right] &= \tilde{\mathbb{E}}\left[\exp\left\{\left(m_v - \frac{1}{2}\langle m\rangle_v\right) + \frac{1}{2}\langle m\rangle_v\right\}\right.\\
&\qquad \left.\exp\left\{-\int_0^v \frac{Y_\theta^2}{4K^2(s-\theta)^2} d\theta - \int_0^v \frac{d\langle Y\rangle_\theta}{4K^2(s-\theta)}\right\}\right]\\
&\leq \tilde{\mathbb{E}}\left[\exp\left\{\left(m_v - \frac{1}{2}\langle m\rangle_v\right) + \frac{1}{2}\int_0^v \frac{Y_\theta^2}{4K^4(s-\theta)^2} d\langle Y\rangle_\theta\right\}\right.\\
&\qquad \left.\exp\left\{-\int_0^v \frac{Y_\theta^2}{4K^2(s-\theta)^2} d\theta - \int_0^v \frac{d\theta}{4K^4(s-\theta)}\right\}\right]\\
&\leq \tilde{\mathbb{E}}\left[\exp\left\{\left(m_v - \frac{1}{2}\langle m\rangle_v\right) + \int_0^v \frac{Y_\theta^2}{8K^2(s-\theta)^2} d\theta\right.\right.\\
&\qquad \left.\left. - \int_0^v \frac{Y_\theta^2}{4K^2(s-\theta)^2} d\theta\right\}\right] \exp\left(-\int_0^v \frac{d\theta}{4K^4(s-\theta)}\right)\\
&\leq \tilde{\mathbb{E}}\left[\exp\left(m_v - \frac{1}{2}\langle m\rangle_v\right)\right] \exp\left(-\int_0^v \frac{d\theta}{4K^4(s-\theta)}\right),
\end{aligned}$$

from which

$$\tilde{\mathbb{E}}\left[\exp\left(-\frac{Y_v^2}{4K^2(s-v)}\right)\right] \leq \exp\left(-\frac{1}{4K^4}\int_0^v \frac{d\theta}{s-\theta}\right) = \left(\frac{s-v}{s}\right)^{\frac{1}{4K^4}}.$$

As f is increasing, we deduce

$$\tilde{\mathbb{E}}\left[f^{2\beta}(g_s)\right] = \int_{f^{2\beta}(0)}^{f^{2\beta}(s)} \tilde{\mathbb{P}}\left[f^{2\beta}(g_s) \geq u\right] du + f^{2\beta}(0)$$
$$= \int_{f^{2\beta}(0)}^{f^{2\beta}(s)} \tilde{\mathbb{P}}\left[g_s \geq (f^{2\beta})^{-1}(u)\right] du + f^{2\beta}(0)$$
$$= \int_0^s \tilde{\mathbb{P}}\left[g_s \geq v\right] 2\beta f^{2\beta-1}(v) f'(v) dv + f^{2\beta}(0)$$
$$\leq C_T \int_0^s \left(\frac{s-v}{s}\right)^{\frac{1}{4K^4}} f^{2\beta-1}(v) f'(v) dv + f^{2\beta}(0).$$

Therefore, using the Fubini-Tonelli theorem for positive functions and the hypothesis (5), we get

$$\begin{aligned}\int_0^T \tilde{\mathbb{E}}\left(f^{2\beta}(g_s)\right) ds &\leq C_T \int_0^T dv f^{2\beta-1}(v) f'(v) \int_v^T \left(\frac{s-v}{s}\right)^{\frac{1}{4K^4}} ds + C_T T f^{2\beta}(0) \\ &\leq C_T \int_0^T \frac{dv}{v^{\frac{1}{4K^4}}} f^{2\beta-1}(v) f'(v) (T-v)^{1+\frac{1}{4K^4}} + C_T T f^{2\beta}(0) \\ &< +\infty.\end{aligned} \tag{37}$$

In view of (32), (33), (36), and (37), we deduce that

$$A^h(T) \leq C_T\, \delta_h. \tag{38}$$

An upper bound for D_p^h. We proceed as in the proof of Lemma 4.1. We recall that $M_s := \int_0^s \sigma_u dW_u$. The main modification is the following one: As

$$D_p^h \leq \int_0^h f(ph) \mathbf{1}_{[\delta_h/2,+\infty)}\left(\left|M_{s+ph} - M_{ph}\right|\right) ds,$$

we deduce

$$\begin{aligned}\mathbb{E} D_h^p &\leq \mathbb{E}\left(\int_0^h f(ph) \mathbf{1}_{\sup_{0\leq u\leq s}\left|M_{u+ph}-M_{ph}\right| \geq \delta_h/2} ds\right) \\ &\leq f(ph) \mathbb{E}\left(\int_0^h \mathbf{1}_{\sup_{0\leq u\leq s}\left|M_{u+ph}-M_{ph}\right| \geq \delta_h/2} ds\right) \\ &\leq C f(ph) \exp\left(-\frac{1}{16K^2} \frac{\delta_h^2}{h}\right) \\ &\leq C h f(ph) \delta_h.\end{aligned} \tag{39}$$

From (30), (38), and (39), we deduce that

$$
\begin{aligned}
h\mathbb{E}\left(\sum_{p=0}^{N_h} f(ph)\mathbf{1}_{[-\delta_h,\delta_h]}(X_{ph})\right) &\leq C_T\delta_h + C\delta_h\sum_{p=0}^{N_h} hf(ph)\\
&\leq C_T\delta_h + C\delta_h\int_0^T f(s)ds\\
&\leq C_T\delta_h.
\end{aligned}
$$

This ends the proof of Theorem 1.2.

7 Appendix

In this section, we shortly explain why our sharp estimates are useful in the convergence rate analysis of discretization schemes for stochastic differential equations with irregular coefficients.
Consider a standard stochastic differential equation with smooth coefficients b and σ. Denote by P_t the transition operator of the solution (X_t). Let (X_t^h) be the Euler discretization scheme with step-size h. A now classical result (see Talay [7]) is that, for all smooth function f, or all measurable function f under an hypoellipticity condition,

$$
\exists C>0,\ \forall n\geq 1,\ \mathbb{E}f(X_T^h)-\mathbb{E}f(X_T)=Ch+\mathcal{O}(h^2).
$$

A preliminary step in the proof consists in getting the following expansion:

$$
\mathbb{E}f(X_T^h)-\mathbb{E}f(X_T)=h^2\sum_{p=0}^{N_h}\mathbb{E}\Psi\left(ph,X_{ph}^h\right)+\sum_{p=0}^{N_h}R_p^h; \tag{40}
$$

here we have set $N_h := \lfloor T/h\rfloor - 1$ and

$$
R_{N_h}^h := \mathbb{E}f(X_T^h)-\mathbb{E}(P_hf)(X_{T-h}^h);
$$

for all $p<n-1$, the function $\Psi(ph,\cdot)$ can be expressed in terms of the functions b, σ, $u(ph,\cdot) := P_{T-ph}f(\cdot)$ and their derivatives up to order 5; R_p^h is a sum of terms, each one of the form

$$
\mathbb{E}\left[\varphi_\alpha^\natural(X_{ph}^h)\int_{ph}^{(p+1)h}\int_{ph}^{s_1}\int_{ph}^{s_2}(\varphi_\alpha^\sharp(X_{s_3}^h)\partial_\alpha u(s_3,X_{s_3}^h)+\varphi_\alpha^\flat(X_{s_3})\partial_\alpha u(s_3,X_{s_3}))ds_3ds_2ds_1\right],
$$

with $|\alpha|\leq 6$; the functions $\varphi_\alpha^\natural$'s, $\varphi_\alpha^\sharp$'s, $\varphi_\alpha^\flat$'s are products of functions which are partial derivatives up to order 3 of b and σ.
A next step in the proof consists in proving that

$$
\exists C>0,\ \forall h<1,\ \max_{0\leq p\leq N_h-1}|R_p^h|\leq Ch^3,
$$

either by using the smoothness up to T of the function $u(s,x)$ when f is smooth, or by using Malliavin calculus techniques when f is measurable only. In both cases, the smoothness of b and σ is necessary.
When b or σ are locally discontinuous, the derivatives of $u(s,x)$ may explode around their singularities. In order to prove that, however, the convergence rate of Euler type schemes is not worse

than $h^{1/2-\epsilon}$, one needs to modify (40) by considering the event $A_p^h := [X_{ph}^h \in \mathscr{C}^h]$, where $\mathscr{C}^h$ is a neighborhood of the singularities whose size depends on h: to summarize long and technical calculations, one expands $u((p+1)h, X_{(p+1)h}^h) - u(ph, X_{ph}^h)$ only when $A_{(p+1)}^h$ or A_p^h do not occur; if $A_{(p+1)}^h$ and A_p^h occur, one needs to take into account sharp estimates on $\mathbb{P}(A_p^h)$. In the case where the generator of (X_t) is a divergence form operator with a discontinuous coefficient, one even has to get sharp estimates on

$$\sum_{p=0}^{N_h} \frac{1}{\sqrt{T-ph}} \mathbb{P}(A_p^h)$$

where the extra factor $\dfrac{1}{\sqrt{T-ph}}$ comes from the explosion rate in time of the spatial derivatives of $u(T-t,x)$ near the singularities of σ (see Martinez and Talay [4] for details).

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