*Elect. Comm. in Probab.* **14** (2009), 358–371

**ELECTRONIC COMMUNICATIONS in PROBABILITY**

# A NOTE ON NEW CLASSES OF INFINITELY DIVISIBLE DISTRIBUTIONS ON $\mathbb{R}^d$

MAKOTO MAEJIMA
*Department of Mathematics, Faculty and Science and Technology, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan*
email: maejima@math.keio.ac.jp

GENTA NAKAHARA
*Department of Mathematics, Faculty and Science and Technology, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan*
email: genta.nakahara@gmail.com

*Submitted* January 3, 2009, *accepted in final form* August 19, 2009

AMS 2000 Subject classification: 60E07
Keywords: infinitely divisible distribution on $\mathbb{R}^d$, stochastic integral representation.

*Abstract*
This paper introduces and studies a family of new classes of infinitely divisible distributions on $\mathbb{R}^d$ with two parameters. Depending on parameters, these classes connect the Goldie–Steutel–Bondesson class and the class of generalized type $G$ distributions, connect the Thorin class and the class $M$, and connect the class $M$ and the class of generalized type $G$ distributions. These classes are characterized by stochastic integral representations with respect to Lévy processes.

## 1 Introduction

Let $I(\mathbb{R}^d)$ be the class of all infinitely divisible distributions on $\mathbb{R}^d$. $\widehat{\mu}(z), z \in \mathbb{R}^d$, denotes the characteristic function of $\mu \in I(\mathbb{R}^d)$ and $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. We use the Lévy-Khintchine triplet $(A, \nu, \gamma)$ of $\mu \in I(\mathbb{R}^d)$ in the sense that

$$\widehat{\mu}(z) = \exp\left\{-2^{-1}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle(1+|x|^2)^{-1}\right)\nu(dx)\right\}, \quad z \in \mathbb{R}^d,$$

where $A$ is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and $\nu$ is a measure (called the Lévy measure) on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \ \text{ and } \ \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty.$$

The following polar decomposition is a basic result on the Lévy measure of $\mu \in I(\mathbb{R}^d)$. Let $\nu$ be the Lévy measure of some $\mu \in I(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \le \infty$. Then there exist a measure $\lambda$ on

$S = \{x \in \mathbb{R}^d : |x| = 1\}$ with $0 < \lambda(S) \le \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in $\xi$ for each $B \in \mathscr{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \le \infty$ for each $\xi \in S$, and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr),\ B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}). \tag{1.1}$$

Here $\lambda$ and $\{\nu_\xi\}$ are uniquely determined by $\nu$ up to multiplication of measurable functions $c(\xi)$ and $\frac{1}{c(\xi)}$, respectively, with $0 < c(\xi) < \infty$. We say that $\nu$ has the polar decomposition $(\lambda, \nu_\xi)$ and $\nu_\xi$ is called the radial component of $\nu$. (See, e.g., Barndorff-Nielsen et al. (2006), Lemma 2.1.)
A real-valued function $f$ defined on $(0,\infty)$ is said to be completely monotone if it has derivatives $f^{(n)}$ of all orders and for each $n = 0, 1, 2, ..., (-1)^n f^{(n)}(r) \ge 0, r > 0$. Bernstein's theorem says that $f$ on $(0,\infty)$ is completely monotone if and only if there exists a (not necessarily finite) measure $Q$ on $[0,\infty)$ such that $f(r) = \int_{[0,\infty)} e^{-ru}Q(du)$. (See, e.g., Feller (1966), p.439.)
In this paper, we introduce and study the following classes.

**Definition 1.1.** *(The class $J_{\alpha,\beta}(\mathbb{R}^d)$.) Let $\alpha < 2$ and $\beta > 0$. We say that $\mu \in I(\mathbb{R}^d)$ belongs to the class $J_{\alpha,\beta}(\mathbb{R}^d)$ if $\nu = 0$ or $\nu \ne 0$ and, in case $\nu \ne 0$, $\nu_\xi$ in (1.1) has expression*

$$\nu_\xi(dr) = r^{-\alpha-1} g_\xi(r^\beta)dr,\ r > 0, \tag{1.2}$$

*where $g_\xi(x)$ is measurable in $\xi$, is completely monotone in $x$ on $(0,\infty)$ $\lambda$-a.e. $\xi$, not identically zero and $\lim_{x\to\infty} g_\xi(x) = 0$ $\lambda$-a.e. $\xi$.*

**Remark 1.2.** *If $\alpha \le 0$, then automatically $\lim_{x\to\infty} g_\xi(x) = 0$ $\lambda$-a.e. $\xi$, because of the finiteness of $\int_{|x|>1} \nu(dx)$. So, when we consider the classes $B(\mathbb{R}^d), G(\mathbb{R}^d), T(\mathbb{R}^d)$ and $M(\mathbb{R}^d)$ appearing later, we do not have to write this condition explicitly.*

**Remark 1.3.** *The integrability condition of the Lévy measure $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$ implies that*

$$\int_0^\infty (r^2 \wedge 1)r^{-\alpha-1} g_\xi(r^\beta)dr < \infty,\ \lambda\text{-a.e. } \xi, \tag{1.3}$$

*so we do not have to assume (1.3) in the definition. It is automatically satisfied.*

**Remark 1.4.** *The classes $J_{\alpha,1}(\mathbb{R}^d), \alpha < 2$, are studied in Sato (2006b).*

Before mentioning our motivation of this study, we state a general result on the relations among the classes $J_{\alpha,\beta}(\mathbb{R}^d), \alpha < 2, \beta > 0$.

**Theorem 1.5.** $(i)$ *Fix $\alpha < 2$ and let $0 < \beta_1 < \beta_2$. Then*

$$J_{\alpha,\beta_1}(\mathbb{R}^d) \subset J_{\alpha,\beta_2}(\mathbb{R}^d).$$

$(ii)$ *Fix $\beta > 0$ and let $\alpha_1 < \alpha_2 < 2$. Then*

$$J_{\alpha_2,\beta}(\mathbb{R}^d) \subset J_{\alpha_1,\beta}(\mathbb{R}^d).$$

*Proof.* For the proof of (i), we need the following lemma.

**Lemma 1.6.** (See Feller (1966), p.441, Corollary 2.) *Let $\phi$ be a completely monotone function on $(0,\infty)$ and let $\psi$ be a nonnegative function on $(0,\infty)$ whose derivative is completely monotone. Then $\phi(\psi)$ is completely monotone.*

Let $h_\xi(x) = g_\xi(x^{\beta_1/\beta_2}), x > 0$, where $g_\xi$ is the one in (1.2), which is completely monotone on $(0,\infty)$. Since $\psi(x) = x^{\beta_1/\beta_2}, x > 0$, has a completely monotone derivative, it follows from Lemma 1.6 that $h_\xi(x)$ is completely monotone. Suppose $\mu \in J_{\alpha,\beta_1}(\mathbb{R}^d)$ and let $g_\xi$ be the one in (1.2). Since $g_\xi(r^{\beta_1}) = h_\xi(r^{\beta_2})$, where $h_\xi$ is completely monotone as has been just shown above, we have $\mu \in J_{\alpha,\beta_2}(\mathbb{R}^d)$. This proves (i).
To prove (ii), suppose that $\mu \in J_{\alpha_2,\beta}(\mathbb{R}^d)$. Then $\nu_\xi(dr) = r^{-\alpha_2-1}g_\xi(r^\beta)dr$, $r > 0$, as in (1.2), where $g_\xi$ is completely monotone on $(0,\infty)$ $\lambda$-a.e. $\xi$. Note that

$$h_\xi(x) = x^{-(\alpha_2-\alpha_1)/\beta}g_\xi(x)$$

is completely monotone, because $x^{-p}, p > 0$, is completely monotone and the product of two completely monotone functions is also completely monotone. We now have

$$\nu_\xi(dr) = r^{-\alpha_2-1}g_\xi(r^\beta)dr = r^{-\alpha_1-1}h_\xi(r^\beta)dr,$$

and thus $\mu$ also belongs to $J_{\alpha_1,\beta}(\mathbb{R}^d)$. This proves (ii). □

The motivations for studying the classes $J_{\alpha,\beta}(\mathbb{R}^d)$ are the following.

**I.** The classes connecting the Goldie–Steutel–Bondesson class and the class of generalized type $G$ distributions.

Let $\alpha = -1$ and consider the classes $J_{-1,\beta}(\mathbb{R}^d)$, $\beta > 0$. A distribution $\mu \in I(\mathbb{R}^d)$ is said to be of generalized type $G$ if $\nu_\xi$ in (1.2) has expression $\nu_\xi(dr) = g_\xi(r^2)dr$ for some completely monotone function $g_\xi$ on $(0,\infty)$, and denote by $G(\mathbb{R}^d)$ the class of all generalized type $G$ distributions on $\mathbb{R}^d$. Let $I_{\mathrm{sym}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) \,:\, \mu \text{ is symmetric in the sense that } \mu(B) = \mu(-B), B \in \mathscr{B}(\mathbb{R}^d)\}$.

**Remark 1.7.** *A distribution $\mu \in G(\mathbb{R}^d) \cap I_{\mathrm{sym}}(\mathbb{R}^d)$ is a so-called type G distribution, which is, in one dimension, a variance mixture of the standard normal distribution with a positive infinitely divisible mixing distribution.*

**Remark 1.8.** $G(\mathbb{R}^d) = J_{-1,2}(\mathbb{R}^d)$.

**Remark 1.9.** *The Goldie-Steutel-Bondesson class denoted by $B(\mathbb{R}^d)$ is $J_{-1,1}(\mathbb{R}^d)$. (For details on $B(\mathbb{R}^d)$, see Barndorff-Nielsen et al. (2006).)*

Therefore, by Theorem 1.5 (i) with $\alpha = -1$, for $1 < \beta < 2$,

$$B(\mathbb{R}^d) \subset J_{-1,\beta}(\mathbb{R}^d) \subset G(\mathbb{R}^d),$$

and hence $\{J_{-1,\beta}(\mathbb{R}^d), 1 \le \beta \le 2\}$ is a family of classes of infinitely divisible distributions on $\mathbb{R}^d$ connecting $B(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$ with continuous parameter $\beta \in [1,2]$.

**II.** The classes connecting the Thorin class and the class $M(\mathbb{R}^d)$.

Let $\alpha = 0$ and consider the classes $J_{0,\beta}(\mathbb{R}^d)$, $\beta > 0$.

**Remark 1.10.** *The Thorin class denoted by $T(\mathbb{R}^d)$ is $J_{0,1}(\mathbb{R}^d)$. (For details on $T(\mathbb{R}^d)$, see also Barndorff-Nielsen et al. (2006).)*

**Remark 1.11.** *The class $M(\mathbb{R}^d)$ is defined by $J_{0,2}(\mathbb{R}^d)$. (The class $M(\mathbb{R}^d) \cap I_{\mathrm{sym}}(\mathbb{R}^d)$ is studied in Aoyama et al. (2008).)*

By Theorem 1.5 (i) with $\alpha = 0$, for $1 < \beta < 2$,

$$T(\mathbb{R}^d) \subset J_{0,\beta}(\mathbb{R}^d) \subset M(\mathbb{R}^d),$$

and hence $\{J_{0,\beta}(\mathbb{R}^d), 1 \le \beta \le 2\}$ is a family of classes of infinitely divisible distributions on $\mathbb{R}^d$ connecting $T(\mathbb{R}^d)$ and $M(\mathbb{R}^d)$ with continuous parameter $\beta \in [1,2]$.

**III.** The classes connecting the classes $M(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$.

Let $\beta = 2$ and consider the classes $J_{\alpha,2}(\mathbb{R}^d)$, $\alpha < 2$. Then, by Theorem 1.5 (ii) with $\beta = 2$, for $-1 \le \alpha \le 0$

$$M(\mathbb{R}^d) \subset J_{\alpha,2}(\mathbb{R}^d) \subset G(\mathbb{R}^d),$$

and hence $\{J_{\alpha,2}(\mathbb{R}^d), -1 \le \alpha \le 0\}$ is a family of classes of infinitely divisible distributions on $\mathbb{R}^d$ connecting $M(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$ with continuous parameter $\alpha \in [-1,0]$.

**IV.** The classes connecting the classes $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$.

Let $\beta = 1$ and consider the classes $J_{\alpha,1}(\mathbb{R}^d)$, $\alpha < 2$. Then, by Theorem 1.5 (ii) with $\beta = 1$, for $-1 \le \alpha \le 0$

$$T(\mathbb{R}^d) \subset J_{\alpha,1}(\mathbb{R}^d) \subset B(\mathbb{R}^d),$$

and hence $\{J_{\alpha,1}(\mathbb{R}^d), -1 \le \alpha \le 0\}$ is a family of classes of infinitely divisible distributions on $\mathbb{R}^d$ connecting $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$ with continuous parameter $\alpha \in [-1,0]$. (This fact is already mentioned in Sato (2006b).)

## 2 Stochastic integral characterizations for $J_{\alpha,\beta}(\mathbb{R}^d)$

The purpose of this paper is to characterize the classes $J_{\alpha,\beta}(\mathbb{R}^d)$ by stochastic integral representations. For that, we first define mappings from $I(\mathbb{R}^d)$ into $I(\mathbb{R}^d)$ and investigate the domains of those mappings.

We introduce the following function $G_{\alpha,\beta}(u)$. For $\alpha < 2$ and $\beta > 0$, let

$$G_{\alpha,\beta}(u) = \int_u^\infty x^{-\alpha-1} e^{-x^\beta} dx, \quad u \ge 0,$$

and let $G^*_{\alpha,\beta}(t)$ be the inverse function of $G_{\alpha,\beta}(u)$, that is, $t = G_{\alpha,\beta}(u)$ if and only if $u = G^*_{\alpha,\beta}(t)$. Let $\{X_t^{(\mu)}\}$ be a Lévy process on $\mathbb{R}^d$ with the law $\mu \in I(\mathbb{R}^d)$ at $t = 1$. We consider the stochastic integrals

$$\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t) dX_t^{(\mu)}, \quad \text{where} \quad G_{\alpha,\beta}(0) = \begin{cases} \beta^{-1}\Gamma(-\alpha\beta^{-1}), & \text{if } \alpha < 0, \\ \infty, & \text{if } \alpha \ge 0. \end{cases}$$

As to the definition of stochastic integrals of non-random measurable functions $f$ which are $\int_0^T f(t) dX_t^{(\mu)}, T < \infty, \mu \in I(\mathbb{R}^d)$, we follow the definition in Sato (2004, 2006a), whose idea is to define a stochastic integral with respect to $\mathbb{R}^d$-valued independently scatted random measure induced by a Lévy process on $\mathbb{R}^d$. The improper stochastic integral $\int_0^\infty f(t) dX_t^{(\mu)}$ is defined as the limit in probability of $\int_0^T f(t)dX_t^{(\mu)}$ as $T\to\infty$ whenever the limit exists. See also Sato (2006b). In the following, $\mathscr{L}(X)$ stands for "the law of $X$". If we write

$$\Psi_{\alpha,\beta}(\mu)=\mathscr{L}\left(\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t^{(\mu)}\right),$$

then $\Psi_{\alpha,\beta}$ can be considered as a mapping with domain $\mathfrak{D}(\Psi_{\alpha,\beta})$ being the class of $\mu\in I(\mathbb{R}^d)$ for which $\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t^{(\mu)}$ is definable.

**Theorem 2.1.** *If $\alpha<0$, then $\mathfrak{D}(\Psi_{\alpha,\beta})=I(\mathbb{R}^d)$.*

*Proof.* By Proposition 3.4 in Sato (2006a), since $G_{\alpha,\beta}(0)<\infty$ for $\alpha<0$, if $\int_0^{G_{\alpha,\beta}(0)}\left(G^*_{\alpha,\beta}(t)\right)^2dt<\infty$, then $\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t^{(\mu)}$ is well-defined. Actually,

$$\int_0^{G_{\alpha,\beta}(0)}\left(G^*_{\alpha,\beta}(t)\right)^2dt=-\int_0^\infty u^2dG_{\alpha,\beta}(u)=\int_0^\infty u^{1-\alpha}e^{-u^\beta}du<\infty.$$

□

To determine the domain of $\Psi_{\alpha,\beta},\alpha\ge 0$, we need the following result by Sato (2006b). In the following, $a(t)\sim b(t)$ means that $\lim_{t\to\infty}a(t)/b(t)=1$, $a(t)\asymp b(t)$ means that $0<\liminf_{t\to\infty}a(t)/b(t)\le\limsup_{t\to\infty}a(t)/b(t)<\infty$ and $I_{\log}(\mathbb{R}^d)=\{\mu\in I(\mathbb{R}^d):\int_{\mathbb{R}^d}\log^+|x|\mu(dx)<\infty\}$, where $\log^+|x|=(\log|x|)\vee 0$.

**Proposition 2.2.** (Sato (2006b), Theorems 2.4 and 2.8.) *Let $p\ge 0$. Denote*

$$\Phi_{\varphi_p}(\mu)=\mathscr{L}\left(\int_0^\infty \varphi_p(t)dX_t^{(\mu)}\right).$$

*Suppose that $\varphi_p$ is locally square-integrable with respect to Lebesgue measure on $[0,\infty)$ and satisfies*
*(1) $\varphi_0(t)\asymp e^{-ct}$ as $t\to\infty$ with some $c>0$,*
*(2) $\varphi_p(t)\asymp t^{-1/p}$ as $t\to\infty$ for $p\in(0,1)\cup(1,\infty)$,*
*(3) $\varphi_1(t)\asymp t^{-1}$ as $t\to\infty$ and for some $t_0>0,c>0$ and $\psi(t),\varphi_1(t)=t^{-1}\psi(t)$ for $t>t_0$ with $\int_{t_0}^\infty t^{-1}|\psi(t)-c|dt<\infty$.*
*Then*
*$(i)$ If $p=0$, then $\mathfrak{D}(\Phi_{\varphi_0})=I_{\log}(\mathbb{R}^d)$.*
*$(ii)$ If $0<p<1$, then $\mathfrak{D}(\Phi_{\varphi_p})=\{\mu\in I(\mathbb{R}^d):\int_{\mathbb{R}^d}|x|^p\mu(dx)<\infty\}=:I_p(\mathbb{R}^d)$.*
*$(iii)$ If $p=1$, then $\mathfrak{D}(\Phi_{\varphi_1})=\{\mu\in I(\mathbb{R}^d):\int_{\mathbb{R}^d}|x|\mu(dx)<\infty$*
*$\lim_{T\to\infty}\int_{t_0}^T t^{-1}dt\int_{|x|>t}x\nu(dx)$ exists in $\mathbb{R}^d$, $\int_{\mathbb{R}^d}x\mu(dx)=0\}=:I_1^*(\mathbb{R}^d)$.*
*$(iv)$ If $1<p<2$, then $\mathfrak{D}(\Phi_{\varphi_p})=\{\mu\in I(\mathbb{R}^d):\int_{\mathbb{R}^d}|x|^p\mu(dx)<\infty,\int_{\mathbb{R}^d}x\mu(dx)=0\}$*
*$=:I_p^0(\mathbb{R}^d)$.*
*$(v)$ If $p\ge 2$, then $\mathfrak{D}(\Phi_{\varphi_p})=\{\delta_0\}$, where $\delta_0$ is the distribution with the total mass at*
0.

We apply Proposition 2.2 to our problem. First we note that when $\alpha<2$, $G^*_{\alpha,\beta}(t)$ is locally square-integrable with respect to Lebesgue measure on $[0,\infty)$.

**Theorem 2.3.** (*Case* $\alpha = 0$.) $\mathfrak{D}(\Psi_{0,\beta}) = I_{\log}(\mathbb{R}^d)$.

*Proof.* Note that $t(= G_{\alpha,\beta}(u)) \uparrow \infty$ if and only if $u(= G^*_{\alpha,\beta}(t)) \downarrow 0$, when $\alpha \geq 0$. It is enough to show that for some $C_1 \in (0,\infty)$, $u \sim C_1 e^{-t}$ as $t \to \infty$. We have

$$\begin{aligned}
\frac{u}{e^{-t}} &= \frac{u}{\exp\{-G_{0,\beta}(u)\}} = \exp\{G_{0,\beta}(u) + \log u\} = \exp\left\{\int_u^\infty x^{-1}e^{-x^\beta}dx + \log u\right\} \\
&= \exp\left\{\beta^{-1}\int_{u^\beta}^\infty y^{-1}e^{-y}dy - \beta^{-1}\int_{u^\beta}^1 y^{-1}dy\right\} \\
&= \exp\left\{\beta^{-1}\int_{u^\beta}^1 y^{-1}(e^{-y}-1)dy + \beta^{-1}\int_1^\infty y^{-1}e^{-y}dy\right\} \to C_1,
\end{aligned}$$

say, as $u \downarrow 0$. Hence $u \sim C_1 e^{-t}$ as $t \to \infty$, and the condition (1) of Proposition 2.2 is satisfied. Thus Proposition 2.2 (i) gives us the assertion. $\square$

**Theorem 2.4.** (*Case* $\alpha \in (0,\infty)$.)
($i$) *If* $0 < \alpha < 1$, *then* $\mathfrak{D}(\Psi_{\alpha,\beta}) = I_\alpha(\mathbb{R}^d)$.
($ii$) *If* $\alpha = 1$, *then* $\mathfrak{D}(\Psi_{1,\beta}) = I_1^*(\mathbb{R}^d)$.
($iii$) *If* $1 < \alpha < 2$, *then* $\mathfrak{D}(\Psi_{\alpha,\beta}) = I_\alpha^0(\mathbb{R}^d)$.
($iv$) *If* $\alpha \geq 2$, *then* $\mathfrak{D}(\Psi_{\alpha,\beta}) = \{\delta_0\}$.

*Proof.* (i) and (iii). It is enough to show that $u \sim C_2 t^{-1/\alpha}$ as $t \to \infty$ for some $C_2 \in (0,\infty)$. We have, as $t \to \infty$ (equivalently $u \downarrow 0$), for some $C_3 \in (0,\infty)$,

$$\frac{u}{t^{-1/\alpha}} = \frac{u}{\left(G_{\alpha,\beta}(u)\right)^{-1/\alpha}} = \frac{u}{\left(\beta^{-1}\int_{u^\beta}^\infty y^{-(\alpha/\beta)-1}e^{-y}dy\right)^{-1/\alpha}} \sim \frac{u}{\left(C_3 u^{-\alpha}\right)^{-1/\alpha}} = C_3^{1/\alpha} =: C_2,$$

and the condition (2) of Proposition 2.3 is satisfied. Thus Proposition 2.3 (ii) and (iv) give us the assertions.
(ii). Suppose $\beta \neq 1$. (The case $\beta = 1$ is proved in Sato (2006b).) We first have

$$\begin{aligned}
G_{1,\beta}(u) &= \int_u^\infty x^{-2}e^{-x^\beta}dx = \int_u^\infty x^{-2}dx + \int_u^\infty x^{-2}(e^{-x^\beta}-1)dx \\
&= \int_u^\infty x^{-2}dx + \int_u^1 x^{-2}(e^{-x^\beta}-1+x^\beta)du - \int_u^1 x^{-2+\beta}dx + \int_1^\infty x^{-2}(e^{-x^\beta}-1)dx \\
&= u^{-1} + (\beta-1)^{-1}u^{-1+\beta} + O(1),\ u \downarrow 0.
\end{aligned}$$

Thus

$$t = G^*_{1,\beta}(t)^{-1} + (\beta-1)^{-1}G^*_{1,\beta}(t)^{-1+\beta} + O(1),\ t \to \infty.$$

Therefore,

$$G^*_{1,\beta}(t) = t^{-1} + (\beta-1)^{-1}t^{-1}G^*_{1,\beta}(t)^\beta + O(t^{-1}G^*_{1,\beta}(t)),\ t \to \infty. \tag{2.1}$$

We have shown in (i) and (iii) that $u \sim C_2 t^{-1/\alpha}$, but this is also true for $\alpha = 1$. Hence

$$u = G^*_{1,\beta}(t) = C_2 t^{-1}(1 + o(1)),\ t \to \infty. \tag{2.2}$$

By substituting (2.2) into (2.1), we have

$$\begin{aligned} G^*_{1,\beta}(t) &= t^{-1} + C_2^\beta(\beta-1)^{-1}t^{-1-\beta} + t^{-1}a(t),\ t\to\infty, \\ &= t^{-1}\left(1 + C_2^\beta(\beta-1)^{-1}t^{-\beta} + a(t)\right),\ t\to\infty, \end{aligned}$$

where

$$a(t) = \begin{cases} o(t^{-\beta}),\ t\to\infty, & \text{when } 0<\beta<1, \\ O(t^{-1}),\ t\to\infty, & \text{when } \beta>1. \end{cases}$$

Thus

$$G^*_{1,\beta}(t) = t^{-1}\psi(t),$$

where

$$\psi(t) := 1 + C_2^\beta(\beta-1)^{-1}t^{-\beta} + a(t),$$

and

$$\int_1^\infty t^{-1}|\psi(t)-1|dt = \int_1^\infty t^{-1}|C_2^\beta(\beta-1)^{-1}t^{-\beta} + a(t)|dt < \infty.$$

Thus the condition (3) of Proposition 2.2 is satisfied with $t_0 = 1$ and $c = 1$, and Proposition 2.2 (iii) gives us the assertion (iii).
(iv) The same as in Sato (2006b). □

We now calculate the Lévy measure of $\widetilde{\mu} = \Psi_{\alpha,\beta}(\mu)$, and note that the mapping $\Psi_{\alpha,\beta}$ is one-to-one.

**Lemma 2.5.** *Let $\alpha < 2$ and $\beta > 0$. Let $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta})$ and $\widetilde{\mu} = \Psi_{\alpha,\beta}(\mu)$, and let $\nu$ and $\widetilde{\nu}$ be the Lévy measures of $\mu$ and $\widetilde{\mu}$, respectively.*
(1) *We have*

$$\widetilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)s^{-\alpha-1}e^{-s^\beta}ds, \quad B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}). \tag{2.3}$$

(2) *If $\nu \neq 0$, and $\nu$ has polar decomposition $(\lambda, \nu_\xi)$, then a polar decomposition of $\widetilde{\nu} = (\widetilde{\lambda}, \widetilde{\nu}_\xi)$ is given by $\widetilde{\lambda} = \lambda$ and $\widetilde{\nu}_\xi(dr) = r^{-\alpha-1}\widetilde{g}_\xi(r^\beta)dr$, where*

$$\widetilde{g}_\xi(u) = \int_0^\infty r^\alpha e^{-u/r^\beta}\nu_\xi(dr). \tag{2.4}$$

(3) *$\widetilde{g}_\xi$ in (2.4) satisfies the requirements of $g_\xi$ in (1.2).*

*Proof.* Suppose $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta})$ and $\widetilde{\mu} = \Psi_{\alpha,\beta}(\mu)$.
(1) We see that (by using Proposition 2.6 of Sato (2006b)),

$$\begin{aligned} \widetilde{\nu}(B) &= \int_0^{G_{\alpha,\beta}(0)} dt \int_{\mathbb{R}^d} 1_B(xG^*_{\alpha,\beta}(t))\nu(dx) = -\int_0^\infty dG_{\alpha,\beta}(s)\int_{\mathbb{R}^d} 1_B(xs)\nu(dx) \\ &= \int_0^\infty s^{-\alpha-1}e^{-s^\beta}ds \int_{\mathbb{R}^d} 1_{s^{-1}B}(x)\nu(dx) = \int_0^\infty \nu(s^{-1}B)s^{-\alpha-1}e^{-s^\beta}ds, \end{aligned}$$

which is (2.3).

(2) Next assume that $\nu \neq 0$ and $\nu$ has polar decomposition $(\lambda, \nu_\xi)$. Then, we have

$$\begin{aligned}
\widetilde{\nu}(B) &= \int_0^\infty s^{-\alpha-1} e^{-s^\beta} ds \int_S \lambda(d\xi) \int_0^\infty 1_{s^{-1}B}(r\xi) \nu_\xi(dr) \\
&= \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(dr) r^{-1} \int_0^\infty (u/r)^{-\alpha-1} e^{-(u/r)^\beta} 1_B(u\xi) du \\
&= \int_S \lambda(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} \widetilde{g}_\xi(u^\beta) du,
\end{aligned}$$

where $\widetilde{\lambda} = \lambda$ and

$$\widetilde{g}_\xi(u) = \int_0^\infty r^\alpha e^{-u/r^\beta} \nu_\xi(dr), \tag{2.5}$$

which is (2.4). The finiteness of $\widetilde{g}_\xi$ is trivial for $\alpha \leq 0$. For $\alpha > 0$, since $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta})$, we have that $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty$. When $\alpha > 0$, note that $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty$ implies $\int_1^\infty r^\alpha \nu_\xi(dr) < \infty$, (see, e.g. Sato (1999), Theorem 25.3). Hence the integral $\widetilde{g}_\xi$ exists.
(3) If we put

$$\widetilde{Q}(B) = \int_0^\infty r^\alpha 1_B(r^{-\beta}) \nu_\xi(dr),$$

then it follows that $\widetilde{g}_\xi(u) = \int_0^\infty e^{-uy} \widetilde{Q}(dy)$, and thus $\widetilde{g}_\xi$ is completely monotone by Bernstein's theorem. If $\alpha \leq 0$, then automatically $\lim_{u\to\infty} \widetilde{g}_\xi(u) = 0$ $\lambda$-a.e. $\xi$, since

$$\infty > \int_{|x|>1} \widetilde{\nu}(dx) = \int_S \lambda(d\xi) \int_1^\infty u^{-\alpha-1} \widetilde{g}_\xi(u^\beta) du.$$

When $\alpha > 0$, since $\int_1^\infty r^\alpha \nu_\xi(dr) < \infty$, the assertion that $\lim_{u\to\infty} \widetilde{g}_\xi(u) = 0$ $\lambda$-a.e. $\xi$ also follows from (2.5) by the dominated convergence theorem.
The proof of the lemma is thus concluded. □

**Remark 2.6.** *(2.3) can be written as, by introducing a transformation $\Upsilon_{\alpha,\beta}$ of Lévy measures as $\widetilde{\nu} = \Upsilon_{\alpha,\beta}(\nu)$. Then this $\Upsilon_{\alpha,\beta}$ is a generalized Upsilon transformation discussed in Barndorff-Nielsen et al. (2008) with the dilation measure $\tau(ds) = s^{-\alpha-1} e^{-s^\beta} ds$.*

**Theorem 2.7.** *For each $\alpha < 2$ and $\beta > 0$, the mapping $\Psi_{\alpha,\beta}$ is one-to-one.*

The proof is carried out in the same way as for Proposition 4.1 of Sato (2006b).

We are now ready to discuss stochastic integral characterizations of the classes $J_{\alpha,\beta}(\mathbb{R}^d)$, by showing that $J_{\alpha,\beta}(\mathbb{R}^d)$ is the range of the mapping $\Psi_{\alpha,\beta}$. However, in this paper, we restrict ourselves to the case $\alpha < 1$, because in the case $1 \leq \alpha < 2$, $J_{\alpha,\beta}(\mathbb{R}^d)$ is strictly bigger than the range $\Psi_{\alpha,\beta}(\mathfrak{D}(\Psi_{\alpha,\beta}))$ and more deep calculations would be needed. (See, e.g., Sato (2006b) and Maejima et al. (2009).) Also, the classes appearing in our motivation of introducing the classes $J_{\alpha,\beta}(\mathbb{R}^d)$ are restricted to the case $\alpha \leq 0$.

**Theorem 2.8.** *Let $\alpha < 1$ and $\beta > 0$. The range of the mapping $\Psi_{\alpha,\beta}$ equals $J_{\alpha,\beta}(\mathbb{R}^d)$, that is,*

$$J_{\alpha,\beta}(\mathbb{R}^d) = \Psi_{\alpha,\beta}(\mathfrak{D}(\Psi_{\alpha,\beta})).$$

**Remark 2.9.** *This theorem is already known for $\alpha = -1, 0$ and $\beta = 1$ in Theorems A and C of Barndorff-Nielsen et al. (2006) and for $\alpha < 1$ and $\beta = 1$ in Theorem 4.2 of Sato (2006b).*

*Proof of Theorem 2.8.* We first show that $\Psi_{\alpha,\beta}(\mathfrak{D}(\Psi_{\alpha,\beta})) \subset J_{\alpha,\beta}(\mathbb{R}^d)$. Suppose $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta})$ and $\widetilde{\mu} = \Psi_{\alpha,\beta}(\mu)$, and let $\nu$ and $\widetilde{\nu}$ be the Lévy measures of $\mu$ and $\widetilde{\mu}$, respectively. Thus, if $\nu = 0$, then $\widetilde{\nu} = 0$ and $\widetilde{\mu} \in J_{\alpha,\beta}(\mathbb{R}^d)$. When $\nu \neq 0$, it follows from Lemma 2.5 that $\widetilde{\mu} \in J_{\alpha,\beta}(\mathbb{R}^d)$.
Next we show that $J_{\alpha,\beta}(\mathbb{R}^d) \subset \Psi_{\alpha,\beta}(\mathfrak{D}(\Psi_{\alpha,\beta}))$. Suppose $\widetilde{\mu} \in J_{\alpha,\beta}(\mathbb{R}^d)$ with the Lévy-Khintchine triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$. If $\widetilde{\nu} = 0$, then $\widetilde{\mu} = \Psi_{\alpha,\beta}(\mu)$ for some $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta})$. Thus, suppose that $\widetilde{\nu} \neq 0$. Then, in a polar decomposition $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ of $\widetilde{\nu}$, we have $\widetilde{\nu}_\xi(dr) = r^{-\alpha-1}\widetilde{g}_\xi(r^\beta)dr$, where $\widetilde{g}_\xi(v)$ is completely monotone in $v > 0$ $\widetilde{\lambda}$-a.e. $\xi$, and is measurable in $\xi$. Thus by Bernstein's theorem, there are measures $\widetilde{Q}_\xi$ on $[0, \infty)$ such that

$$\widetilde{g}_\xi(v) = \int_{[0,\infty)} e^{-vu}\widetilde{Q}_\xi(du).$$

In general, $\widetilde{Q}_\xi$ is a measure on $[0, \infty)$, but since $\lim_{v\to\infty} \widetilde{g}_\xi(v) = 0$ $\widetilde{\lambda}$-a.e. $\xi$, $\widetilde{Q}_\xi$ does not have a point mass at 0, and hence $\widetilde{Q}_\xi$ is a measure on $(0, \infty)$. We see that

$$\begin{aligned}\widetilde{\nu}(B) &= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(r\xi)r^{-\alpha-1}\widetilde{g}_\xi(r^\beta)dr \\ &= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(r\xi)r^{-\alpha-1}dr\int_0^\infty e^{-r^\beta u}\widetilde{Q}_\xi(du).\end{aligned} \tag{2.6}$$

Since $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\widetilde{\nu}(dx) < \infty$, we have

$$\int_S \widetilde{\lambda}(d\xi)\int_0^1 r^{1-\alpha}dr\int_1^\infty e^{-r^\beta u}\widetilde{Q}_\xi(du) + \int_S \widetilde{\lambda}(d\xi)\int_1^\infty r^{-\alpha-1}dr\int_0^1 e^{-r^\beta u}\widetilde{Q}_\xi(du) < \infty.$$

Hence, we have, by the change of variables $r \to v$ by $r^\beta u = v$,

$$\begin{aligned}\int_0^1 r^{1-\alpha}dr\int_1^\infty e^{-r^\beta u}\widetilde{Q}_\xi(du) &= \int_1^\infty \widetilde{Q}_\xi(du)\int_0^1 r^{1-\alpha}e^{-r^\beta u}dr \\ &= \beta^{-1}\int_1^\infty u^{(\alpha-2)/\beta}\widetilde{Q}_\xi(du)\int_0^u v^{-1+(2-\alpha)/\beta}e^{-v}dv \geq C_4\int_1^\infty u^{(\alpha-2)/\beta}\widetilde{Q}_\xi(du),\end{aligned}$$

where

$$C_4 = \beta^{-1}\int_0^1 v^{-1+(2-\alpha)/\beta}e^{-v}dv \in (0, \infty).$$

Thus

$$\int_S \widetilde{\lambda}(d\xi)\int_1^\infty u^{(\alpha-2)/\beta}\widetilde{Q}_\xi(du) < \infty. \tag{2.7}$$

We also have for any $\alpha < 1$,

$$\int_1^\infty r^{-\alpha-1}dr\int_0^1 e^{-r^\beta u}\widetilde{Q}_\xi(du) = \int_0^1 \widetilde{Q}_\xi(du)\int_1^\infty r^{-\alpha-1}e^{-r^\beta u}dr \tag{2.8}$$

$$= \beta^{-1}\int_0^1 u^{\alpha/\beta}\widetilde{Q}_\xi(du)\int_u^\infty v^{-1-(\alpha/\beta)}e^{-v}dv \ge C_5\int_0^1 u^{\alpha/\beta}\widetilde{Q}_\xi(du),$$

where

$$C_5 = \beta^{-1}\int_1^\infty v^{-1-(\alpha/\beta)}e^{-v}dv \in (0,\infty).$$

Thus

$$\int_S \widetilde{\lambda}(d\xi)\int_0^1 u^{\alpha/\beta}\widetilde{Q}_\xi(du) < \infty. \tag{2.9}$$

In addition, if $\alpha = 0$, (2.8) is turned out to be

$$\begin{aligned}\int_1^\infty r^{-1}dr\int_0^1 e^{-r^\beta u}\widetilde{Q}_\xi(du) &= \beta^{-1}\int_0^1 \widetilde{Q}_\xi(du)\int_u^1 v^{-1}e^{-v}dv\\ &\ge (\beta e)^{-1}\int_0^1 \widetilde{Q}_\xi(du)\int_u^1 v^{-1}dv = (\beta e)^{-1}\int_0^1(-\log u)\widetilde{Q}_\xi(du).\end{aligned}$$

Thus, when $\alpha = 0$,

$$\int_S \widetilde{\lambda}(d\xi)\int_0^1(-\log u)\widetilde{Q}_\xi(du) < \infty. \tag{2.10}$$

Furthermore,

$$\int_1^\infty r^{-\alpha-1}dr\int_0^1 e^{-r^\beta u}\widetilde{Q}(du) \ge \int_1^\infty r^{-\alpha-1}e^{-r^\beta}dr\int_0^1 \widetilde{Q}_\xi(du) = C_6\int_0^1 \widetilde{Q}_\xi(du),$$

where

$$C_6 := \int_1^\infty r^{-\alpha-1}e^{-r^\beta}dr \in (0,\infty).$$

Thus we have

$$\int_S \widetilde{\lambda}(d\xi)\int_0^1 \widetilde{Q}_\xi(dr) < \infty. \tag{2.11}$$

Define

$$\nu_\xi(B) = \int_0^\infty u^{\alpha/\beta}1_B\left(u^{-1/\beta}\right)\widetilde{Q}_\xi(du), \quad B \in \mathscr{B}((0,\infty)). \tag{2.12}$$

Then, it follows from (2.7) and (2.9) that

$$\begin{aligned}\int_S \widetilde{\lambda}(d\xi)\int_0^\infty (r^2\wedge 1)\nu_\xi(dr) &= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty u^{\alpha/\beta}(u^{-2/\beta}\wedge 1)\widetilde{Q}_\xi(du) \qquad (2.13)\\ &= \int_S \widetilde{\lambda}(d\xi)\left(\int_0^1 u^{\alpha/\beta}\widetilde{Q}_\xi(du) + \int_1^\infty u^{(\alpha-2)/\beta}\widetilde{Q}_\xi(du)\right) < \infty.\end{aligned}$$

Define $\nu$ by

$$\nu(B) = \int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(r\xi)\nu_\xi(dr). \tag{2.14}$$

Then, by (2.13), $\nu$ is the Lévy measure of some infinitely divisible distribution $\mu$, and $\mu$ belongs to $\mathfrak{D}(\Psi_{\alpha,\beta})$ and satisfies

$$\widetilde{\nu}(B) = \int_0^{G_{\alpha,\beta}(0)} \nu((G^*_{\alpha,\beta}(t))^{-1}B)dt. \tag{2.15}$$

To show (2.15), by (2.6), (2.12) and (2.14), we have

$$\begin{aligned}
\widetilde{\nu}(B) &= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(r\xi)r^{-\alpha-1}dr\int_0^\infty e^{-r^\beta u}\widetilde{Q}_\xi(du)\\
&= \int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(u^{-1/\beta}s\xi)s^{-\alpha-1}e^{-s^\beta}ds\int_0^\infty u^{\alpha/\beta}\widetilde{Q}_\xi(du)\\
&= \int_0^\infty s^{-\alpha-1}e^{-s^\beta}ds\int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(u^{-1/\beta}s\xi)u^{\alpha/\beta}\widetilde{Q}_\xi(du)\\
&= \int_0^\infty s^{-\alpha-1}e^{-s^\beta}ds\int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(rs\xi)\nu_\xi(dr)\\
&= \int_0^\infty s^{-\alpha-1}e^{-s^\beta}ds\int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_{s^{-1}B}(r\xi)\nu_\xi(dr)\\
&= \int_0^\infty \nu(s^{-1}B)s^{-\alpha-1}e^{-s^\beta}ds = -\int_0^\infty \nu(s^{-1}B)dG_{\alpha,\beta}(s)\\
&= \int_0^{G_{\alpha,\beta}(0)} \nu((G^*_{\alpha,\beta}(t))^{-1}B)dt.
\end{aligned}$$

To show that $\mu \in \mathfrak{D}(\Psi_{\alpha,\beta})$, it is enough to show that $\int_{|x|>1}|x|^\alpha\nu(dx) < \infty$, which is if and only if $\mu \in I_\alpha(\mathbb{R}^d)$, when $0 < \alpha < 1$, and $\int_{|x|>1}\log|x|\nu(dx) < \infty$, which is if and only if $\mu \in I_{\log}(\mathbb{R}^d)$, when $\alpha = 0$, (see Sato (1999), Theorem 25.3). Note that by (2.12) we see, for any nonnegative measurable function $f$ on $(0,\infty)$,

$$\int_0^\infty f(r)\nu_\xi(dr) = \int_0^\infty u^{\alpha/\beta}f(u^{-1/\beta})\widetilde{Q}_\xi(du).$$

Thus if we choose $f(r) = I[r>1]r^\alpha$, where $I[A]$ is the indicator function of the set $A$, then $\nu$ in (2.14) satisfies that for $\alpha > 0$

$$\int_{|x|>1}|x|^\alpha\nu(dx) = \int_S \widetilde{\lambda}(d\xi)\int_1^\infty r^\alpha\nu_\xi(dr) = \int_S \widetilde{\lambda}(d\xi)\int_0^1 \widetilde{Q}_\xi(du) < \infty \tag{2.16}$$

due to (2.11). When $\alpha = 0$,

$$\begin{aligned}
&\int_{|x|>1}\log|x|\nu(dx) = \int_S \widetilde{\lambda}(d\xi)\int_1^\infty \log r\,\nu_\xi(dr) \qquad (2.17)\\
&= \int_S \widetilde{\lambda}(d\xi)\int_0^1 \log u^{-1/\beta}\widetilde{Q}_\xi(du) = \beta^{-1}\int_S \widetilde{\lambda}(d\xi)\int_0^1 (-\log u)\widetilde{Q}_\xi(du) < \infty
\end{aligned}$$

due to (2.10).

Notice again that

$$\int_0^{G_{\alpha,\beta}(0)} \left(G^*_{\alpha,\beta}(t)\right)^2 dt = -\int_0^\infty u^2 dG_{\alpha,\beta}(u) = \int_0^\infty u^{1-\alpha} e^{-u^\beta} du < \infty.$$

Define $A$ and $\gamma$ by

$$\widetilde{A} = \left(\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)^2 dt\right) A \tag{2.18}$$

and

$$\widetilde{\gamma} = \int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dt\left(\gamma + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|G^*_{\alpha,\beta}(t)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right). \tag{2.19}$$

Here we have to check the finiteness of this integral. We first have

$$\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dt = -\int_0^\infty u dG_{\alpha.\beta}(u) = \int_0^\infty u^{-\alpha} e^{-u^\beta} du < \infty,$$

since $\alpha < 1$. Below, $C_7, C_8 \in (0,\infty)$ are suitable constants. Recall $\alpha < 1$. When $\alpha \neq 0$, we have

$$\begin{aligned}
&\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dt \int_{\mathbb{R}^d} |x| \left|\frac{1}{1+|G^*_{\alpha,\beta}(t)x|^2} - \frac{1}{1+|x|^2}\right| \nu(dx) \\
&= \int_0^\infty u^{-\alpha} e^{-u^\beta} du \int_{\mathbb{R}^d} |x| \left|\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2}\right| \nu(dx) \\
&\le \int_0^\infty u^{-\alpha}(1+u^2) e^{-u^\beta} du \int_{\mathbb{R}^d} \frac{|x|^3}{(1+|ux|^2)(1+|x|^2)} \nu(dx) \\
&\le \int_0^\infty u^{-\alpha}(1+u^2) e^{-u^\beta} du \\
&\quad \times \left(\int_{|x|\le 1} |x|^2 \nu(dx) + \int_{|x|>1, |ux|\le 1} |x| \nu(dx) + \int_{|x|>1, |ux|>1} \frac{|x|}{|ux|^2} \nu(dx)\right) \\
&= C_7 + \int_{|x|>1} |x| \nu(dx) \int_0^{1/|x|} u^{-\alpha}(1+u^2) e^{-u^\beta} du \\
&\qquad + \int_{|x|>1} \nu(dx) \int_{1/|x|}^\infty u^{-\alpha-1}(1+u^2) e^{-u^\beta} du \\
&\le C_7 + \int_{|x|>1} |x| \nu(dx) \int_0^{1/|x|} 2u^{-\alpha} du \\
&\qquad + \int_{|x|>1} \nu(dx) \left\{\left(\int_{1/|x|}^1 + \int_1^\infty\right) u^{-\alpha-1}(1+u^2) e^{-u^\beta} du\right\} \\
&\le C_7 + 2(1-\alpha)^{-1} \int_{|x|>1} |x|^\alpha \nu(dx) \\
&\qquad + \int_{|x|>1} \nu(dx) \left\{\int_{1/|x|}^1 2u^{-\alpha-1} du + \int_1^\infty u^{-\alpha-1}(1+u^2) e^{-u^\beta} du\right\}
\end{aligned} \tag{2.20}$$

$$
\begin{aligned}
&= C_7 + 2(1-\alpha)^{-1}\int_{|x|>1} |x|^\alpha \nu(dx) \\
&\qquad + \int_{|x|>1} \nu(dx)\left\{-2\alpha^{-1}(1-|x|^\alpha) + C_8\right\} \\
&= C_7 + 2(1-\alpha)^{-1}\int_{|x|>1} |x|^\alpha \nu(dx) \\
&\qquad + 2\alpha^{-1}\int_{|x|>1} |x|^\alpha \nu(dx) + (C_8 - 2\alpha^{-1})\int_{|x|>1} \nu(dx) < \infty, \qquad (2.21)
\end{aligned}
$$

by (2.16). When $\alpha = 0$, since

$$
\int_{1/|x|}^1 u^{-\alpha-1}du = \int_{1/|x|}^1 u^{-1}du = \log|x|,
$$

in (2.20), we have

$$
\int_{|x|>1} \log|x|\nu(dx) \qquad (2.22)
$$

instead of $\int_{|x|>1} |x|^\alpha \nu(dx)$ in (2.21) in the calculation above. The finiteness of (2.22) is assured by (2.17).

Thus $\gamma$ can be defined. Hence, if we denote by $\mu$ an infinitely divisible distribution having the Lévy-Khintchine triplet $(A, \nu, \gamma)$ above, then by (2.15), (2.18) and (2.19), we see that

$$
\widetilde{\mu} = \mathscr{L}\left(\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t^{(\mu)}\right),
$$

concluding that $\widetilde{\mu} \in \Psi_{\alpha,\beta}(\mathfrak{D}(\Psi_{\alpha,\beta}))$. This completes the proof. $\square$

**Acknowledgment** The authors greatly appreciate the referee's detailed comments, which improved the paper very much.

## References

[1] T. Aoyama, M. Maejima and J. Rosiński (2008). A subclass of type $G$ selfdecomposable distributions, *J. Theoret. Probab.*, **21**, 14–34. MR2384471 MR2384471

[2] O.E. Barndorff-Nielsen, M. Maejima and K. Sato (2006). Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations, *Bernoulli*, **12**, 1–33. MR2202318 MR2202318

[3] O.E. Barndorff-Nielsen, J. Rosiński and S. Thorbjørnsen, (2008): General $\Upsilon$ transformations. *ALEA Lat. Am. J. Probab. Math. Statist.* **4**, 131-165. MR2421179 MR2421179

[4] W. Feller (1966). *An Introduction to Probability Theory and Its Applications, Vol. II, 2nd ed.*, John Wiley & Sons. MR0210154 MR0210154

[5] M. Maejima, M. Matsui and M. Suzuki (2009). Classes of infinitely divisible distributions on $\mathbb{R}^d$ related to the class of selfdecomposable distributions, to appear in *Tokyo J. Math.*

[6] K. Sato (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press. MR1739520 MR1739520

[7] K. Sato (2004) Stochastic integrals in additive processes and application to semi-Lévy processes, *Osaka J. Math.* **41**, 211–236. MR2040073 MR2040073

[8] K. Sato (2006a). Additive processes and stochastic integrals, *Illinois J. Math.* **50**, 825–851. MR2247848 MR2247848

[9] K. Sato (2006b). Two families of improper stochastic integrals with respect to Lévy processes, *ALEA Lat. Am. J. Probab. Math. Statist.* **1** 47–87. MR2235174 MR2235174