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SMALL TIME ASYMPTOTICS OF ORNSTEIN-UHLENBECK DENSITIES IN HILBERT SPACES

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Abstract
We show that Varadhan's small time asymptotics for densities of the solution of a stochastic differential equation in $\mathbb{R}^n$ [8] carries over to a Hilbert space-valued Ornstein-Uhlenbeck process whose transition semigroup is strongly Feller and symmetric. In the Hilbert space setting, densities are with respect to a Gaussian invariant measure.

1 Introduction

Varadhan [8] investigated the small time asymptotics of the probability densities of an $\mathbb{R}^n$-valued diffusion process $(z_\zeta(t))_{t\geq 0}$ with initial point $\zeta \in \mathbb{R}^n$. Denoting the density of $z_\zeta(t)$ by $p(t,\zeta,\cdot)$, Varadhan showed that

$$\lim_{t\to 0} t \ln p(t,\zeta,y) = -\frac{1}{2}d^2(\zeta,y) \tag{1}$$

uniformly for ζ and y in any bounded subset of $\mathbb{R}^n$. In equality (1)

$$d(\zeta,y) := \inf\left\{\int_0^1 \sqrt{\langle \dot{u}(\tau),\, a^{-1}(u(\tau))\dot{u}(\tau)\rangle_{\mathbb{R}^n}}\, d\tau \,:\, u:[0,1]\to\mathbb{R}^n \text{ is absolutely continuous with derivative } \dot{u} \text{ and } u(0)=\zeta \text{ and } u(1)=y\right\},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ is the scalar product in $\mathbb{R}^n$ and a is the diffusion matrix in the stochastic differential equation which $(z_\zeta(t))_{t\geq 0}$ solves.

The small time asymptotics formula for densities (1) has been shown to hold in many different settings, for example Norris [6] showed that the formula holds in a finite dimensional Lipschitz Riemannian manifold, with the definition of the distance function d depending on the manifold.

In the setting of an infinite dimensional separable Hilbert space H, let $(X_x(t))_{t\geq 0}$ be the mild solution of the stochastic initial value problem

$$\left.\begin{array}{rcll} dX &=& AX\,dt + dW & t>0 \\ X(0) &=& x\in H, & \end{array}\right\} \tag{2}$$

where A is a linear operator on H and W is a (possibly cylindrical) Wiener process on H. Only in special situations is the distribution of $X_x(t)$ absolutely continuous with respect to a natural reference measure on H at all times $t > 0$. We consider one such situation, namely when an invariant measure μ exists and the transition semigroup is strongly Feller and symmetric on $L^2(H, \mu)$. Under these conditions we obtain the small time limiting behaviour of the probability density of $X_x(t)$ with respect to μ. The continuous density $k(t, x, \cdot)$ of $X_x(t)$ in Proposition 1 is valid whenever the transition semigroup is strongly Feller and symmetric; we have the small time limit in Proposition 2 when Assumption 4 also holds. Assumption 4 is rather restrictive, nevertheless it is interesting that the limit in equation (10) is of the same form as that in equation (1).

In the next section we present the main results and their proofs and finish with an example.

2 Small time limiting behaviour of densities

Let $(H, \langle\cdot,\cdot\rangle, |\cdot|)$ be a separable infinite dimensional Hilbert space. Let $A : D(A) \subset H \to H$ be the infinitesimal generator of the strongly continuous semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on H. Let Q be a symmetric and positive definite bounded linear operator on H, in particular $\ker Q = \{0\}$. Define the Hilbert space $(H_W := Q^{\frac{1}{2}}(H), |\cdot|_{H_W} := |Q^{-\frac{1}{2}}\cdot|)$. Let $(W(t))_{t\geq 0}$ be a Hilbert space-valued Wiener process defined on a probability space $(\Omega, \mathscr{F}, P)$ and such that the distribution of $W(1)$ has reproducing kernel Hilbert space H_W. If Q is a trace class operator then $(W(t))_{t\geq 0}$ is a H-valued Wiener process, otherwise it is a cylindrical Wiener process on H (see [3, Proposition 4.11]). In this article Q need not be trace class. The embedding of H_W into H is denoted by

$$i : H_W \hookrightarrow H.$$

We use the symbol $\mathscr{N}(m, C)$ to denote a Gaussian measure on the Borel sets of H, with mean m and covariance operator C.

Assumption 1 *A trace class operator on H is defined by*

$$Q_\infty x := \int_0^\infty S(t)QS^*(t)x\,dt, \quad x \in H.$$

Set $\mu := \mathscr{N}(0, Q_\infty)$. For each $t > 0$ the operator

$$Q_t x := \int_0^t S(s)QS^*(s)x\,ds, \quad x \in H,$$

is trace class and $\ker Q_t = \{0\}$. The mild solution of the initial value problem (2) at positive times t,

$$X_x(t) := S(t)x + \int_0^t S(t-s)i\,dW(s), \tag{3}$$

has distribution $\mathscr{N}(S(t)x, Q_t)$ and μ is an invariant measure for the equation in (2).
Define the strongly continuous transition semigroup $(R_t)_{t\geq 0}$ on $L^2(H, \mu)$ by

$$(R_t\phi)(x) := \int_H \phi(y)\,d\mathscr{N}(S(t)x, Q_t)(y) \quad \text{for } \mu \text{ a.e. } x \in H$$

and for all $\phi \in L^2(H, \mu)$.

Assumption 2 *Semigroup $(R_t)_{t\geq 0}$ is strongly Feller, that is, $S(t)(H) \subset Q_t^{\frac{1}{2}}(H)$ for all positive times t.*
Chojnowska-Michalik and Goldys have shown in [1, Proposition 2] that

$$S_0(t) := Q_\infty^{-\frac{1}{2}} S(t) Q_\infty^{\frac{1}{2}}, \quad t \geq 0,$$

defines a strongly continuous semigroup of contractions on H. Some consequences of Assumption 2 are that for each $t > 0$

1. $Q_\infty^{\frac{1}{2}}(H) = Q_t^{\frac{1}{2}}(H)$, which is equivalent to $\|S_0(t)\|_{L(H,H)} < 1$ and
2. $S_0(t)$ is Hilbert-Schmidt.

As shown in [4, Lemma 10.3.3], it follows that for each $t > 0$ and each $x \in H$ the distribution of $X_x(t)$, $\mathcal{N}(S(t)x, Q_t)$, is absolutely continuous with respect to μ and its Radon-Nikodym derivative $\frac{d\mathcal{N}(S(t)x,Q_t)}{d\mu}$ is

$$\begin{aligned}\frac{d\mathcal{N}(S(t)x,Q_t)}{d\mu}(y) \quad = \quad & (\det(I_H - \Theta_t))^{-\frac{1}{2}} \exp\Bigg[-\frac{1}{2}\langle (I_H-\Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t)x, Q_\infty^{-\frac{1}{2}} S(t)x\rangle \\ & + \langle (I_H-\Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t)x, Q_\infty^{-\frac{1}{2}} y\rangle \\ & - \frac{1}{2}\langle \Theta_t (I_H-\Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} y, Q_\infty^{-\frac{1}{2}} y\rangle\Bigg] \end{aligned} \tag{4}$$

for μ a.e. $y \in H$, where I_H is the identity operator on H and $\Theta_t := S_0(t)S_0^*(t)$. The second and third terms in the argument of the exponential function in equation (4) are defined for only μ a.e. y, in terms of limits (see for example [4, Proposition 1.2.10]).

Assumption 3 *The operators R_t are symmetric for all $t \geq 0$.*

Chojnowska-Michalik and Goldys [2, Lemma 2.2] have shown that symmetry of R_t is equivalent to symmetry of $S_0(t)$ and this allows us to prove that there is a continuous version of the Radon-Nikodym derivative in equation (4).

Proposition 1. *Under Assumptions 1 to 3, there is a continuous version of the Radon-Nikodym derivative $\frac{d\mathcal{N}(S(t)x,Q_t)}{d\mu}$, which we denote by $k(t,x,\cdot)$:*

$$\begin{aligned} k(t,x,y) \quad := \quad & (\det(I_H - S_0(2t)))^{-\frac{1}{2}} \times \\ & \exp\left[-\frac{1}{2}|Q_t^{-\frac{1}{2}} S(t)x|^2 + \langle Q_t^{-\frac{1}{2}} S(t/2)x, Q_t^{-\frac{1}{2}} S(t/2)y\rangle - \frac{1}{2}|Q_t^{-\frac{1}{2}} S(t)y|^2\right] \end{aligned} \tag{5}$$

for all $y \in H$.

Proof. Define the bounded linear bijections

$$J(t) := Q_\infty^{-\frac{1}{2}} Q_t^{\frac{1}{2}}, \quad t > 0.$$

The identity $Q_\infty = Q_t + S(t)Q_\infty S^*(t)$ yields

$$\begin{aligned} J(t)J^*(t) = I_H - S_0(t)S_0^*(t) \quad &= \quad I_H - \Theta_t \quad \text{for } t > 0 \text{ and} \\ (I_H - \Theta_t)^{-1} \quad &= \quad (J^{-1}(t))^* J^{-1}(t) \quad \text{for } t > 0. \end{aligned} \tag{6}$$

From equality (6) we have

$$\begin{aligned}\langle (I_H-\Theta_t)^{-1}Q_\infty^{-\frac{1}{2}}S(t)x,\, Q_\infty^{-\frac{1}{2}}S(t)x\rangle &= \langle J^{-1}(t)Q_\infty^{-\frac{1}{2}}S(t)x,\, J^{-1}(t)Q_\infty^{-\frac{1}{2}}S(t)x\rangle \\ &= |Q_t^{-\frac{1}{2}}S(t)x|^2. \qquad (7)\end{aligned}$$

The other two terms in the argument of the exponential in equation (4) are defined in terms of limits. Let (f_k) be an orthonormal basis of H made up of eigenvectors of Q_∞. For each $n\in\mathbb{N}$ define P_n to be the orthogonal projection onto the linear span of $\{f_1,\dots,f_n\}$. In the following expressions (n_k) denotes some strictly increasing sequence of natural numbers. We have

$$\begin{aligned}\langle \Theta_t(I_H-\Theta_t)^{-1}Q_\infty^{-\frac{1}{2}}y,\, Q_\infty^{-\frac{1}{2}}y\rangle &= \lim_{k\to\infty}\langle \Theta_t(I_H-\Theta_t)^{-1}Q_\infty^{-\frac{1}{2}}P_{n_k}y,\, Q_\infty^{-\frac{1}{2}}P_{n_k}y\rangle\,, \quad \mu \text{ a.e. } y\in H,\\ &= \lim_{k\to\infty}\langle (I_H-\Theta_t)^{-1}\Theta_t^{\frac{1}{2}}Q_\infty^{-\frac{1}{2}}P_{n_k}y,\, \Theta_t^{\frac{1}{2}}Q_\infty^{-\frac{1}{2}}P_{n_k}y\rangle \\ &= \lim_{k\to\infty}\langle (I_H-\Theta_t)^{-1}Q_\infty^{-\frac{1}{2}}S(t)P_{n_k}y,\, Q_\infty^{-\frac{1}{2}}S(t)P_{n_k}y\rangle \\ &= |Q_t^{-\frac{1}{2}}S(t)y|^2. \qquad (8)\end{aligned}$$

We have

$$\begin{aligned}\langle (I_H-\Theta_t)^{-1}Q_\infty^{-\frac{1}{2}}S(t)x,\, Q_\infty^{-\frac{1}{2}}y\rangle &= \lim_{k\to\infty}\langle (I_H-\Theta_t)^{-1}Q_\infty^{-\frac{1}{2}}S(t)x,\, Q_\infty^{-\frac{1}{2}}P_{n_k}y\rangle\,, \quad \mu \text{ a.e. } y\in H,\\ &= \lim_{k\to\infty}\langle (I_H-\Theta_t)^{-1}Q_\infty^{-\frac{1}{2}}S(t/2)x,\, S_0(t/2)Q_\infty^{-\frac{1}{2}}P_{n_k}y\rangle \\ &= \langle Q_t^{-\frac{1}{2}}S(t/2)x,\, Q_t^{-\frac{1}{2}}S(t/2)y\rangle. \qquad (9)\end{aligned}$$

Substituting the expressions from equalities (7), (8) and (9) into the right hand side of equation (4), we get the formula for $k(t,x,y)$ shown in equation (5). □

When x and y belong to $Q^{\frac{1}{2}}(H)$ we can write $k(t,x,y)$ in terms of the eigenvalues of A_0, the infinitesimal generator of $(S_0(t))_{t\geq 0}$; then it is straightforward to find $\lim_{t\to 0} t\ln k(t,x,y)$. The results obtained in this way can be of interest only if $\mu(Q^{\frac{1}{2}}(H))=1$. We now introduce a further assumption to ensure that $\mu(Q^{\frac{1}{2}}(H))=1$. Chojnowska-Michalik and Goldys [2, Theorems 2.7 and 2.9] showed that the symmetry of R_t, $t>0$, implies that

$$S_Q(t) := Q^{-\frac{1}{2}}S(t)Q^{\frac{1}{2}}\,, \quad t\geq 0,$$

defines a strongly continuous semigroup of symmetric contractions on H and there is an isometric isomorphism $U: H\to H$ such that

$$S_Q(t) = US_0(t)U^{-1} \quad \text{for all } t\geq 0.$$

Hence, like $S_0(t)$, $S_Q(t)$ is a Hilbert-Schmidt strict contraction for each $t>0$. Since $(S_Q(t))$ is a semigroup of compact, symmetric contractions, its infinitesimal generator A_Q is self-adjoint and its spectrum consists of real eigenvalues

$$0 > -\alpha_1 \geq -\alpha_2 \geq -\alpha_3 \geq \cdots$$

where $-\alpha_j \to -\infty$ as $j \to \infty$ (see [5, Theorem 13 in chapter 34] and [7, Theorems 2.3 and 2.4 in chapter 2]). By [7, Theorem 3.3 in chapter 2], A_Q^{-1} is compact as well as symmetric and hence there is an orthonormal basis (g_k) of H composed of eigenvectors of A_Q:

$$A_Q g_k = -\alpha_k g_k \quad \text{for all } k \in \mathbb{N}.$$

Assumption 4 *A_Q^{-1} is trace class, that is, $\sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty$.*

Chojnowska-Michalik and Goldys [2, Theorem 5.1] showed that $\mu(Q^{\frac{1}{2}}(H)) = 1$ if and only if $\int_0^\infty \|S_Q(t)\|^2_{L_2(H,H)}\, dt < \infty$, where $\|\cdot\|_{L_2(H,H)}$ denotes the Hilbert-Schmidt norm. We have

$$\int_0^\infty \|S_Q(t)\|^2_{L_2(H,H)}\, dt = \sum_{k=1}^{\infty} \int_0^\infty \mathrm{e}^{-2\alpha_k t}\, dt = \sum_{k=1}^{\infty} \frac{1}{2\alpha_k}.$$

Thus Assumption 4 is equivalent to the assumption that $\mu(Q^{\frac{1}{2}}(H)) = 1$.

Proposition 2. *Under Assumptions 1 to 4 we have for all x and y in $Q^{\frac{1}{2}}(H)$*

$$\lim_{t\to 0} t \ln k(t,x,y) = -\frac{1}{2}|Q^{-\frac{1}{2}}(x-y)|^2 \tag{10}$$

and convergence is uniform for $Q^{-\frac{1}{2}}x$ and $Q^{-\frac{1}{2}}y$ in any compact subset of H.

Remark In the example following the proof we show that equality (10) does not necessarily hold if $x-y$ is in $Q^{\frac{1}{2}}(H)$ but x and y are in $H \backslash Q^{\frac{1}{2}}(H)$.

Proof. Assumption 4 is sufficient (but not necessary) to ensure that

$$\lim_{t\to 0} t \ln \det(I_H - S_0(2t)) = 0.$$

We have for $t > 0$:

$$t \ln \det(I_H - S_0(2t)) = t \ln \prod_{j=1}^{\infty}(1-\mathrm{e}^{-2\alpha_j t}) = \sum_{j=1}^{\infty} t \ln(1-\mathrm{e}^{-2\alpha_j t}).$$

By L'Hôpital's rule

$$\lim_{t\to 0} t \ln(1-\mathrm{e}^{-2\alpha_j t}) \quad = \quad 0 \quad \text{for each } j \in \mathbb{N}; \tag{11}$$

thus, since the function $x \in (0,\infty) \mapsto x \ln(1-\mathrm{e}^{-x})$ is bounded we have

$$\begin{aligned} t \ln \det(I_H - S_0(2t)) \quad &= \quad \sum_{j=1}^{\infty} \frac{2\alpha_j t \ln(1-\mathrm{e}^{-2\alpha_j t})}{2\alpha_j} \\ &\to \quad 0 \quad \text{as } t \to 0. \end{aligned} \tag{12}$$

It remains to find the limit of t times the argument of the exponential function in equation (5). The key to this is equality (17), which we now derive.

Let $t > 0$. We have

$$\begin{aligned} Q_t x &= \int_0^t S(2r)Qx\,dr \\ &= Q^{\frac{1}{2}} \int_0^t A_Q S_Q(2r) A_Q^{-1} Q^{\frac{1}{2}} x\,dr \\ &= \frac{1}{2} Q^{\frac{1}{2}} (S_Q(2t) A_Q^{-1} Q^{\frac{1}{2}} x - A_Q^{-1} Q^{\frac{1}{2}} x) \\ &= \frac{1}{2} Q^{\frac{1}{2}} (I_H - S_Q(2t))(-A_Q)^{-1} Q^{\frac{1}{2}} x\,, \quad x \in H. \end{aligned}$$

Substituting $x = Q^{-\frac{1}{2}} y$ into this equation, where $y \in Q^{\frac{1}{2}}(H)$, we have

$$Q^{-\frac{1}{2}} Q_t Q^{-\frac{1}{2}} y = \frac{1}{2}(I_H - S_Q(2t))(-A_Q)^{-1} y \quad \text{for } y \in Q^{\frac{1}{2}}(H). \tag{13}$$

By [2, Proposition 2.10]

$$Q_t^{\frac{1}{2}}(H) = Q^{\frac{1}{2}}(D(\sqrt{-A_Q})) \quad \text{for } t > 0, \tag{14}$$

therefore $Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}}$ is a bounded linear operator with range $D(\sqrt{-A_Q})$. Since $Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}}$ is one to one and has a dense range, its adjoint $(Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}})^*$ has the same properties. From equation (13) we have

$$Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}} (Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}})^* = \frac{1}{2}(I_H - S_Q(2t))(-A_Q)^{-1}; \tag{15}$$

notice that, since $\|S_Q(2t)\|_{L(H,H)} < 1$, $(I_H - S_Q(2t))$ is invertible and the range of the operator in equation (15) is $D(A_Q)$. Taking inverses on both sides of equation (15) we have

$$((Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}})^{-1})^* Q_t^{-\frac{1}{2}} Q^{\frac{1}{2}} x = -2(I_H - S_Q(2t))^{-1} A_Q x\,, \quad x \in D(A_Q). \tag{16}$$

Let $r > 0$. Then since A_Q is self-adjoint,

$$S_Q(r)(H) \subset D(A_Q).$$

Hence for $u, v \in H$ equation (16) yields

$$-2\langle (I_H - S_Q(2t))^{-1} A_Q S_Q(r) u, S_Q(r) v\rangle = \langle Q_t^{-\frac{1}{2}} S(r) Q^{\frac{1}{2}} u, Q_t^{-\frac{1}{2}} S(r) Q^{\frac{1}{2}} v\rangle. \tag{17}$$

The expression on the right hand side of equality (17) appears in equation (5) when x and y are both in $Q^{\frac{1}{2}}(H)$. The expression on the left hand side of equality (17) can be written in terms of the eigenvalues $(-\alpha_j)$ of A_Q.

Recall that (g_k) is an orthonormal basis of H such that $A_Q g_k = -\alpha_k g_k$ for each $k \in \mathbb{N}$. Setting $u_k := \langle u, g_k\rangle$ and $v_k := \langle v, g_k\rangle$ for $k \in \mathbb{N}$, we have from equality (17):

$$\begin{aligned} t\langle Q_t^{-\frac{1}{2}} S(t/2) Q^{\frac{1}{2}} u, Q_t^{-\frac{1}{2}} S(t/2) Q^{\frac{1}{2}} v\rangle &= -2t\langle (I_H - S_Q(2t))^{-1} A_Q S_Q(t/2) u, S_Q(t/2) v\rangle \\ &= \sum_{k=1}^{\infty} \frac{2\alpha_k t}{\mathrm{e}^{\alpha_k t} - \mathrm{e}^{-\alpha_k t}} u_k v_k \qquad (18) \\ &\to \sum_{k=1}^{\infty} u_k v_k = \langle u, v\rangle \quad \text{as } t \to 0, \qquad (19) \end{aligned}$$

and the convergence is uniform for u and v in any compact subset of H. The uniform convergence on compact sets is because for any compact set $K \subset H$ we have $\sup\{\sum_{j=n}^{\infty} \langle u, g_j \rangle^2 : u \in K\} \to 0$ as n goes to infinity.
Similarly we have

$$t|Q_t^{-\frac{1}{2}} S(t) Q^{\frac{1}{2}} u|^2 = \sum_{k=1}^{\infty} \frac{2\alpha_k t}{\mathrm{e}^{2\alpha_k t} - 1} u_k^2 \to \sum_{k=1}^{\infty} u_k^2 = |u|^2 \quad \text{as } t \to 0, \tag{20}$$

and the convergence is uniform for u in any compact subset of H.
Using the results in (12), (19) and (20), we have for x and y in $Q^{\frac{1}{2}}(H)$:

$$\begin{aligned} \lim_{t\to 0} t \ln k(t,x,y) &= \lim_{t\to 0} -\frac{1}{2}(t|Q_t^{-\frac{1}{2}} S(t)x|^2 - 2t\langle Q_t^{-\frac{1}{2}} S(t/2)x, Q_t^{-\frac{1}{2}} S(t/2)y\rangle + t|Q_t^{-\frac{1}{2}} S(t)y|^2) \\ &= -\frac{1}{2}|Q^{-\frac{1}{2}}x - Q^{-\frac{1}{2}}y|^2 , \end{aligned}$$

and the convergence is uniform for $Q^{-\frac{1}{2}}x$ and $Q^{-\frac{1}{2}}y$ in any compact subset of H. □

Example. Let $H = L^2((0,\pi))$ with the usual inner product $\langle u, v\rangle := \int_0^\pi u(t)v(t)\,dt$ for all u and $v \in H$. Define the operator $(A, D(A))$ on H by

$$\begin{aligned} Au &:= u'' \quad \text{for all } u \in D(A) \text{ where} \\ D(A) &:= \Big\{ u \in L^2((0,\pi)) : u \text{ and } u' \text{ are absolutely continuous and} \\ &\qquad u'' \in L^2((0,\pi)) \text{ and } \lim_{t\to 0} u(t) = \lim_{t\to\pi} u(t) = 0 \Big\}. \end{aligned}$$

As shown in [9, Proposition 1 of section 3.1], $(A, D(A))$ is a self-adjoint operator on H and generates the strongly continuous semigroup of operators:

$$S(t)u := \sum_{m=1}^{\infty} \mathrm{e}^{-m^2 t} \langle u, g_m \rangle g_m , \quad u \in H , \quad t \geq 0 , \tag{21}$$

where $\{g_m(y) := \sqrt{\frac{2}{\pi}} \sin(my),\ y \in (0,\pi)\ : m \in \mathbb{N}\}$ is an orthonormal basis of H. Moreover we have $Ag_m = -m^2 g_m$ for all $m \in \mathbb{N}$. Define

$$Qu := \sum_{m=1}^{\infty} \frac{1}{m^2} \langle u, g_m \rangle g_m , \quad u \in H . \tag{22}$$

Straightforward computations show that Assumptions 1 to 4 are satisfied. Criteria for checking that (R_t) is strongly Feller and symmetric are [3, Proposition B.1] and [2, Theorem 2.4], respectively.
We shall show that equality (10) does not necessarily hold if $x - y \in Q^{\frac{1}{2}}(H)$ but x and y are in $H \backslash Q^{\frac{1}{2}}(H)$. We have

$$\begin{aligned} &\lim_{t\to 0} t \ln k(t,x,y) \\ &= \lim_{t\to 0} \left[-\frac{t}{2}|Q_t^{-\frac{1}{2}} S(t)(x-y)|^2 + t\langle Q_t^{-\frac{1}{2}} S(t/2)x, Q_t^{-\frac{1}{2}} S(t/2)y\rangle - t\langle Q_t^{-\frac{1}{2}} S(t)x, Q_t^{-\frac{1}{2}} S(t)y\rangle \right] . \end{aligned}$$

Take $x = y = \sum_{k=1}^{\infty} \frac{1}{k} g_k$. Clearly $x \notin Q^{\frac{1}{2}}(H)$. Proceeding as in equation (18), we have

$$\begin{aligned} t\langle Q_t^{-\frac{1}{2}} S(t/2)x, Q_t^{-\frac{1}{2}} S(t/2)y\rangle - t\langle Q_t^{-\frac{1}{2}} S(t)x, Q_t^{-\frac{1}{2}} S(t)y\rangle &= \sum_{k=1}^{\infty} \frac{2k^2 t \mathrm{e}^{-k^2 t}}{1+\mathrm{e}^{-k^2 t}} \\ &\geq \int_0^{\infty} r^2 t \mathrm{e}^{-r^2 t}\, dr - 2\mathrm{e}^{-1} \\ &= \frac{1}{4}\sqrt{\frac{\pi}{t}} - 2\mathrm{e}^{-1}; \end{aligned}$$

hence in this case we have $x = y$ and $\lim_{t\to 0} t \ln k(t, x, y) = \infty$.

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