

## STOCHASTIC INTEGRAL REPRESENTATION OF THE $L^2$ MODULUS OF BROWNIAN LOCAL TIME AND A CENTRAL LIMIT THEOREM

YAOZHONG HU

Department of Mathematics, University of Kansas, Lawrence, Kansas, 66045  
email: hu@math.ku.edu

DAVID NUALART<sup>1</sup>

Department of Mathematics, University of Kansas, Lawrence, Kansas, 66045  
email: nualart@math.ku.edu

Submitted August 19, 2009, accepted in final form November 11, 2009

AMS 2000 Subject classification: 60H07, 60F05, 60J55, 60J65

Keywords: Malliavin calculus, Clark-Ocone formula, Brownian local time, Knight theorem, central limit theorem, Tanaka formula.

### Abstract

The purpose of this note is to prove a central limit theorem for the  $L^2$ -modulus of continuity of the Brownian local time obtained in [3], using techniques of stochastic analysis. The main ingredients of the proof are an asymptotic version of Knight's theorem and the Clark-Ocone formula for the  $L^2$ -modulus of the Brownian local time.

## 1 Introduction

Let  $B = \{B_t, t \geq 0\}$  be a standard Brownian motion, and denote by  $\{L_t(x), t \geq 0, x \in \mathbb{R}\}$  its local time. In [3] the authors have proved the following central limit theorem for the  $L^2$  modulus of continuity of the local time:

**Theorem 1.** For each fixed  $t > 0$ ,

$$h^{-\frac{3}{2}} \left( \int_{\mathbb{R}} (L_t(x+h) - L_t(x))^2 dx - 4th \right) \xrightarrow{\mathcal{L}} 8\sqrt{\frac{\alpha_t}{3}} \eta, \quad (1.1)$$

as  $h$  tends to zero, where

$$\alpha_t = \int_{\mathbb{R}} (L_t(x))^2 dx, \quad (1.2)$$

and  $\eta$  is a  $N(0, 1)$  random variable independent of  $B$ .

---

<sup>1</sup>D. NUALART IS SUPPORTED BY THE NSF GRANT DMS0604207.

We make use of the notation

$$G_t(h) = \int_{\mathbb{R}} (L_t(x+h) - L_t(x))^2 dx. \quad (1.3)$$

It is proved in [3, Lemma 8.1] that  $\mathbb{E}(G_t(h)) = 4th + O(h^2)$ . Therefore, we can replace the term  $4th$  in (1.1) by  $\mathbb{E}(G_t(h))$ .

The proof of Theorem 1 is done in [3] by the method of moments. The purpose of this paper is to provide a simple proof of this result. Our method is based on an asymptotic version of Knight's theorem (see Revuz and Yor [7], Theorem (2.3), page 524) combined with the techniques of stochastic analysis and Malliavin calculus. The main idea is to apply the Clark-Ocone stochastic integral representation formula to express  $G_t(h) - \mathbb{E}(G_t(h))$  as a stochastic integral. Then, by means of simple estimates using Hölder's inequality, it is proved that the leading term is a martingale, to which we can apply an asymptotic version of Knight's theorem. An important ingredient is to show the convergence of the quadratic variation of this martingale, which will be derived by using Tanaka's formula and backward Itô stochastic integrals.

The paper is organized as follows. In the next section we recall some preliminaries on Malliavin calculus and we establish a stochastic integral representation for the random variable  $G_t(h)$ . Then, Section 3 is devoted to the proof of Theorem 1.

## 2 Stochastic integral representation of the $L^2$ -modulus of continuity

Let us introduce some basic facts on the Malliavin calculus with respect to the Brownian motion  $B = \{B_t, t \geq 0\}$ . We refer to [4] for a complete presentation of these notions. We assume that  $B$  is defined on a complete probability space  $(\Omega, \mathcal{F}, P)$  such that  $\mathcal{F}$  is generated by  $B$ . Consider the set  $\mathcal{S}$  of smooth random variables of the form

$$F = f(B_{t_1}, \dots, B_{t_n}), \quad (2.4)$$

where  $t_1, \dots, t_n \geq 0$ ,  $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$  (the space of bounded functions which have bounded derivatives of all orders) and  $n \in \mathbb{N}$ . The derivative operator  $D$  on a smooth random variable of the form (2.4) is defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, \dots, B_{t_n}) I_{[0, t_i]}(t),$$

which is an element of  $L^2(\Omega \times [0, \infty))$ . We denote by  $\mathbb{D}^{1,2}$  the completion of  $\mathcal{S}$  with respect to the norm  $\|F\|_{1,2}$  given by

$$\|F\|_{1,2}^2 = \mathbb{E}[F^2] + \mathbb{E}\left(\int_0^\infty (D_t F)^2 dt\right).$$

The classical Itô representation theorem asserts that any square integrable random variable can be expressed as

$$F = \mathbb{E}[F] + \int_0^\infty u_t dB_t,$$

where  $u = \{u_t, t \geq 0\}$  is a unique adapted process such that  $\mathbb{E} \left( \int_0^\infty u_t^2 dt \right) < \infty$ . If  $F$  belongs to  $\mathbb{D}^{1,2}$ , then  $u_t = \mathbb{E}[\mathbb{D}_t F | \mathcal{F}_t]$ , where  $\{\mathcal{F}_t, t \geq 0\}$  is the filtration generated by  $B$ , and we obtain the Clark-Ocone formula (see [5])

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[\mathbb{D}_t F | \mathcal{F}_t] dB_t. \tag{2.5}$$

The random variable  $G_t(h) = \int_{\mathbb{R}} (L_t(x+h) - L_t(x))^2 dx$  can be expressed in terms of the self-intersection local time of Brownian motion. In fact,

$$\begin{aligned} G_t(h) &= \int_{\mathbb{R}} \left[ \int_0^t \delta(B_u + x + h) du - \int_0^t \delta(B_u + x) du \right]^2 dx \\ &= -2 \left[ \int_0^t \int_0^t \delta(B_v - B_u + h) dudv - \int_0^t \int_0^t \delta(B_v - B_u) dudv \right] \\ &= -2 \int_0^t \int_0^v (\delta(B_v - B_u + h) + \delta(B_v - B_u - h) - 2\delta(B_v - B_u)) dudv. \end{aligned}$$

The rigorous justification of above argument can be made easily by approximating the Dirac delta function by the heat kernel  $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/2\varepsilon}$  as  $\varepsilon$  tends to zero. That is,  $G_t(h)$  is the limit in  $L^2(\Omega)$  as  $\varepsilon$  tends to zero of

$$G_t^\varepsilon(h) = -2 \int_0^t \int_0^v (p_\varepsilon(B_v - B_u + h) + p_\varepsilon(B_v - B_u - h) - 2p_\varepsilon(B_v - B_u)) dudv. \tag{2.6}$$

Applying Clark-Ocone formula we can derive the following stochastic integral representation for  $G_t(h)$ .

**Proposition 2.** *The random variable  $G_t(h)$  defined in (1.3) can be expressed as*

$$G_t(h) = \mathbb{E}(G_t(h)) + \int_0^t u_{t,h}(r) dB_r,$$

where

$$\begin{aligned} u_{t,h}(r) &= 4 \int_0^r \int_0^h (p_{t-r}(B_r - B_u - \eta) - p_{t-r}(B_r - B_u + \eta)) d\eta du \\ &\quad + 4 \int_0^r (I_{[0,h]}(B_u - B_r) - I_{[0,h]}(B_r - B_u)) du. \end{aligned} \tag{2.7}$$

*Proof* For any  $u < v$  and any  $h \in \mathbb{R}$  we can write

$$D_r p_\varepsilon(B_v - B_u + h) = p'_\varepsilon(B_v - B_u + h) I_{[u,v]}(r),$$

and for any  $u < r < v$

$$\begin{aligned} \mathbb{E} \left( D_r p'_\varepsilon(B_v - B_u + h) | \mathcal{F}_r \right) &= \mathbb{E} p'_\varepsilon(\sqrt{v-r}\eta + B_r - B_u + h) \\ &= p'_{v-r+\varepsilon}(B_r - B_u + h), \end{aligned}$$

where  $\eta$  denotes a  $N(0, 1)$  random variable independent of  $B$ . Therefore, from Clark-Ocone formula (2.5) and Equation (2.6) we obtain

$$G_t^\varepsilon(h) = \mathbb{E}(G_t^\varepsilon(h)) + \int_0^t u_{t,h}^\varepsilon(r) dB_r,$$

where

$$\begin{aligned} u_{t,h}^\varepsilon(r) = & -2 \int_r^t \int_0^r \left( p'_{v-r+\varepsilon}(B_r - B_u + h) + p'_{v-r+\varepsilon}(B_r - B_u - h) \right. \\ & \left. + 2p'_{v-r+\varepsilon}(B_r - B_u) \right) dudv. \end{aligned}$$

This expression can be written as

$$u_{t,h}^\varepsilon(r) = -2 \int_r^t \int_0^r \int_0^h \left( (p''_{v-r+\varepsilon}(B_r - B_u + \eta) - p''_{v-r+\varepsilon}(B_r - B_u - \eta)) \right) d\eta dudv.$$

Using the fact that  $p_t''(x) = 2 \frac{\partial p_t}{\partial t}(x)$  we obtain

$$\begin{aligned} u_{t,h}^\varepsilon(r) = & -4 \left( \int_0^r \int_0^h \left( (p_{t-r+\varepsilon}(B_r - B_u + \eta) - p_{t-r+\varepsilon}(B_r - B_u - \eta)) \right) d\eta du \right. \\ & \left. - \int_0^r \int_0^h \left( (p_\varepsilon(B_r - B_u + \eta) - p_\varepsilon(B_r - B_u - \eta)) \right) d\eta du \right). \end{aligned}$$

Letting  $\varepsilon$  tend to zero we get that  $u_{t,h}^\varepsilon(r)$  converges in  $L^2(\Omega \times [0, t])$  to  $u_{t,h}(r)$  as  $h$  tends to zero, which implies the desired result.

From Proposition 2 we can make the following decomposition

$$u_{t,h}(r) = \hat{u}_{t,h}(r) + 4\Psi_h(r),$$

where

$$\begin{aligned} \hat{u}_{t,h}(r) = & -4 \int_0^r \int_0^h \left( p_{t-r}(B_r - B_u + \eta) - p_{t-r}(B_r - B_u - \eta) \right) d\eta du \\ = & -4 \int_0^r \int_0^h \int_{-\eta}^\eta p'_{t-r}(B_r - B_u + \xi) d\xi d\eta du \end{aligned} \tag{2.8}$$

and

$$\Psi_h(r) = - \int_0^r \left( I_{[0,h]}(B_r - B_u) - I_{[0,h]}(B_u - B_r) \right) du. \tag{2.9}$$

As a consequence, we finally obtain

$$G_t(h) - \mathbb{E}(G_t(h)) = \int_0^t \hat{u}_{t,h}(r) dB_r + 4 \int_0^t \Psi_h(r) dB_r. \tag{2.10}$$

### 3 Proof of Theorem 1

The proof will be done in several steps. Along the proof we will denote by  $C$  a generic constant, which may be different from line to line.

*Step 1* We claim that the stochastic integral  $\int_0^t \hat{u}_{t,h}(r) dB_r$  makes no contribution to the limit (1.1). That is,

$$h^{-3/2} \int_0^t \hat{u}_{t,h}(r) dB_r$$

converges in  $L^2(\Omega)$  to zero as  $h$  tends to zero. This is a consequence of the next proposition.

**Proposition 3.** *There is a constant  $C > 0$  such that*

$$\mathbb{E} \left( \int_0^t |\hat{u}_{t,h}(r)|^2 dr \right) \leq Ch^4,$$

for all  $h > 0$ .

*Proof* From (2.8) we can write

$$\begin{aligned} \mathbb{E} (|\hat{u}_{t,h}(r)|^2) &= \int_0^r \int_0^r \int_0^h \int_0^h \int_{-\eta_1}^{\eta_1} \int_{-\eta_2}^{\eta_2} \mathbb{E}(p'_{t-r}(B_r - B_{u_1} + \xi_1) \\ &\quad \times p'_{t-r}(B_r - B_{u_2} + \xi_2)) d\xi_1 d\xi_2 d\eta_1 d\eta_2 du_1 du_2. \end{aligned}$$

By a symmetry argument, it suffices to integrate in the region  $0 < u_1 < u_2 < r$ . Set

$$\Phi(u_1, u_2, \xi_1, \xi_2) = \mathbb{E} (p'_{t-r}(B_r - B_{u_1} + \xi_1) p'_{t-r}(B_r - B_{u_2} + \xi_2)).$$

Then,

$$\begin{aligned} \Phi(u_1, u_2, \xi_1, \xi_2) &= \mathbb{E} (p'_{t-r}(B_r - B_{u_2} + B_{u_2} - B_{u_1} + \xi_1) p'_{t-r}(B_r - B_{u_2} + \xi_2)) \\ &= \mathbb{E} (p'_{t-r+u_2-u_1}(B_r - B_{u_2} + \xi_1) p'_{t-r}(B_r - B_{u_2} + \xi_2)) \\ &= \int_{\mathbb{R}} p_{r-u_2}(z) p'_{t-r+u_2-u_1}(z + \xi_1) p'_{t-r}(z + \xi_2) dz \\ &\leq \|p_{r-u_2}\|_{p_1} \|p'_{t-r+u_2-u_1}\|_{p_2} \|p'_{t-r}\|_{p_3}, \end{aligned}$$

where  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ . It is easy to see that

$$\begin{aligned} \|p_{r-u_2}\|_{p_1} &\leq C(r-u_2)^{-\frac{1}{2} + \frac{1}{2p_1}}, \\ \|p'_{t-r+u_2-u_1}\|_{p_2} &\leq C(t-r+u_2-u_1)^{-1 + \frac{1}{2p_2}} \leq C(u_2-u_1)^{-1 + \frac{1}{2p_2}}, \\ \|p'_{t-r}\|_{p_3} &\leq C(t-r)^{-1 + \frac{1}{2p_3}}, \end{aligned}$$

for some constant  $C > 0$ . Thus

$$\begin{aligned} \mathbb{E} (|\hat{u}_{t,h}(r)|^2) &\leq C \int_0^r \int_0^{u_2} \int_0^h \int_0^h \int_{-\eta_1}^{\eta_1} \int_{-\eta_2}^{\eta_2} (r-u_2)^{-\frac{1}{2} + \frac{1}{2p_1}} \\ &\quad \times (u_2-u_1)^{-1 + \frac{1}{2p_2}} (t-r)^{-1 + \frac{1}{2p_3}} d\xi_1 d\xi_2 d\eta_1 d\eta_2 du_1 du_2 \\ &\leq Ch^4. \end{aligned}$$

This proves the proposition.

*Step 2* Taking into account Proposition 3 and Equation (2.10), in order to show Theorem 1 it suffices to show the following convergence in law:

$$h^{-\frac{3}{2}} \int_0^t \Psi_h(r) dB_r \xrightarrow{\mathcal{L}} 2\eta \sqrt{\frac{\alpha_t}{3}},$$

where  $\eta$  is a standard normal random variable independent of  $B$ ,  $\alpha_t$  has been defined in (1.2), and  $\Psi_h(r)$  is given by (2.9). Notice that

$$M_t^h = h^{-\frac{3}{2}} \int_0^t \Psi_h(r) dB_r$$

is a martingale with quadratic variation

$$\langle M^h \rangle_t = h^{-3} \int_0^t \Psi_h^2(r) dr.$$

From the asymptotic version of Knight's theorem (see Revuz and Yor [7], Theorem (2.3) page. 524) it suffices to show the following convergences in probability.

$$h^{-3} \int_0^t \Psi_h^2(r) dr \rightarrow \frac{4}{3} \alpha_t, \quad (3.11)$$

and

$$\langle M^h, B \rangle_t = h^{-3/2} \int_0^t \Psi_h(r) dr \rightarrow 0, \quad (3.12)$$

as  $h$  tends to zero, where the convergence (3.12) is uniform in compact sets. In fact, let  $B^h$  be the Brownian motion such that  $M_t^h = B_{\langle M^h \rangle_t}^h$ . Then, from Theorem (2.3) pag. 524 in [7], and the convergences (3.11) and (3.12), we deduce that  $(B, B^h, \langle M^h \rangle_t)$  converges in distribution to  $(B, \beta, \frac{4}{3} \alpha_t)$ , where  $\beta$  is a Brownian motion independent of  $B$ . This implies that  $M_t^h = B_{\langle M^h \rangle_t}^h$  converges in distribution to  $\beta_{\frac{4}{3} \alpha_t}$ , which yields the desired result.

Before proving (3.11) and (3.12) we will express  $\Psi_h(r)$  using Tanaka's formula. By the occupation formula for the Brownian motion we can write

$$\begin{aligned} \Psi_h(r) &= - \int_{\mathbb{R}} (I_{[0,h]}(B_r - x) - I_{[0,h]}(x - B_r)) L_r(x) dx \\ &= \int_0^h (L_r(B_r + y) - L_r(B_r - y)) dy. \end{aligned}$$

We can express the difference  $L_r(B_r - y) - L_r(B_r + y)$  by means of Tanaka's formula for the Brownian motion  $\{B_r - B_s, 0 \leq s \leq r\}$ :

$$\begin{aligned} L_r(B_r + y) - L_r(B_r - y) &= y + (B_r - y)^+ - (B_r + y)^+ \\ &\quad - \int_0^r (I_{B_r - B_s + y > 0} - I_{B_r - B_s - y > 0}) d\widehat{B}_s, \end{aligned}$$

where  $d\widehat{B}_s$  denote the backward stochastic Itô integral and  $y > 0$ . Integrating in the variable  $y$  yields

$$\begin{aligned}\Psi_h(r) &= \frac{h^2}{2} - \int_0^h [(B_r + y)^+ - (B_r - y)^+] dy \\ &\quad - \int_0^h \left( \int_0^r I_{y > |B_r - B_s|} d\widehat{B}_s \right) dy.\end{aligned}\quad (3.13)$$

By stochastic Fubini's theorem

$$\int_0^h \left( \int_0^r I_{y > |B_r - B_s|} d\widehat{B}_s \right) dy = \int_0^r (h - |B_r - B_s|)^+ d\widehat{B}_s.\quad (3.14)$$

Hence,

$$\begin{aligned}\Psi_h(r) &= \frac{h^2}{2} - \int_0^h [(B_r + y)^+ - (B_r - y)^+] dy \\ &\quad - \int_0^r (h - |B_r - B_s|)^+ d\widehat{B}_s.\end{aligned}\quad (3.15)$$

The convergences (3.11) and (3.12) will be proved in the next two steps.

*Step 3* The convergence (3.12) follows from the following lemma.

**Lemma 4.** For any  $t \geq 0$ ,  $\langle M^h, B \rangle_t$  converges to zero in  $L^2(\Omega)$  uniformly in compact sets as  $h$  tends to zero.

*Proof* In view of (3.13) it suffices to show that

$$\sup_{0 \leq t \leq t_1} \left| h^{-3/2} \int_0^t \left( \int_0^r (h - |B_r - B_s|)^+ d\widehat{B}_s \right) dr \right|$$

converges to zero in  $L^2(\Omega)$  as  $h$  tends to zero, for any  $t_1 > 0$ . For any  $p \geq 2$  and any  $0 \leq s < t$  we can write by Fubini's theorem and Burkholder's inequality

$$\begin{aligned}& \mathbb{E} \left( \left| \int_s^t \left( \int_0^r (h - |B_r - B_v|)^+ d\widehat{B}_v \right) dr \right|^p \right) \\ & \leq 2^{p-1} \left\{ \mathbb{E} \left( \left| \int_0^s \left( \int_s^t (h - |B_r - B_v|)^+ dr \right) d\widehat{B}_v \right|^p \right) \right. \\ & \quad \left. + \mathbb{E} \left( \left| \int_s^t \left( \int_v^t (h - |B_r - B_v|)^+ dr \right) d\widehat{B}_v \right|^p \right) \right\} \\ & \leq c_p \left\{ \mathbb{E} \left( \left| \int_0^s \left( \int_s^t (h - |B_r - B_v|)^+ dr \right)^2 dv \right|^{p/2} \right) \right. \\ & \quad \left. + \mathbb{E} \left( \left| \int_s^t \left( \int_v^t (h - |B_r - B_v|)^+ dr \right)^2 dv \right|^{p/2} \right) \right\} \\ & = c_p(I_1 + I_2).\end{aligned}$$

The term  $I_1$  can be expressed using occupation formula as follows

$$\begin{aligned} I_1 &= \mathbb{E} \left( \left| \int_0^s \left( \int_{\mathbb{R}} (h - |x - B_v|)^+ (L_t(x) - L_s(x)) dx \right)^2 dv \right|^{p/2} \right) \\ &\leq s^{p/2} h^{2p} \mathbb{E} \left( \sup_x |L_t(x) - L_s(x)|^p \right). \end{aligned}$$

By the inequalities for local time proved, for instance, in [1] we obtain

$$I_1 \leq c_p h^{2p} |t - s|^{p/2}.$$

Similarly,

$$\begin{aligned} I_2 &= \mathbb{E} \left( \left| \int_s^t \left( \int_{\mathbb{R}} (h - |x - B_v|)^+ (L_t(x) - L_v(x)) dx \right)^2 dv \right|^{p/2} \right) \\ &\leq h^{2p} |t - s|^{p/2} \sup_{s \leq v \leq t} \mathbb{E} \left( \sup_x |L_t(x) - L_v(x)|^p \right) \\ &\leq c_p h^{2p} |t - s|^p. \end{aligned}$$

Finally, a standard application of the Garsia-Rudemich-Rumsey lemma allows us to conclude.

*Step 4* We are going to show that

$$h^{-3} \int_0^t \Psi_h(r)^2 dr \xrightarrow{L^2(\Omega)} \frac{4}{3} \alpha_t, \tag{3.16}$$

as  $h$  tends to zero. Notice that

$$\alpha_t = 2 \int_0^t \int_0^v \delta_0(B_v - B_u) dudv$$

is the self-intersection local time of  $B$ , and Equation (3.16) provides an approximation for this self-intersection local time which has its own interest.

Taking into account (3.13) and (3.14), the convergence (3.16) will follow from

$$h^{-3} \int_0^t \left( \int_0^r (h - |B_r - B_s|)^+ d\widehat{B}_s \right)^2 dr \xrightarrow{L^2(\Omega)} \frac{4}{3} \alpha_t, \tag{3.17}$$

as  $h$  tends to zero. By Itô's formula we can write

$$\begin{aligned} \left( \int_0^r (h - |B_r - B_s|)^+ d\widehat{B}_s \right)^2 &= 2 \int_0^r \left( \int_s^r (h - |B_r - B_u|)^+ d\widehat{B}_u \right) \\ &\quad \times (h - |B_r - B_s|)^+ d\widehat{B}_s + \int_0^r [(h - |B_r - B_s|)^+]^2 ds. \end{aligned} \tag{3.18}$$

Finally, (3.17) follows from (3.18) and the next two lemmas.



**Lemma 5.** *We have*

$$\int_0^t \int_0^r \frac{[(h - |B_r - B_s|)^+]^2}{h^3} ds dr \xrightarrow{L^2(\Omega)} \frac{4}{3} \alpha_t,$$

as  $h$  tends to zero.

*Proof* Notice that

$$\alpha_t = \int_{\mathbb{R}} L_t(x)^2 dx = \int_0^t \int_{\mathbb{R}} L_r(x) L_{dr}(x) dx = \int_0^t L_r(B_r) dr,$$

and

$$\int_0^t \int_0^r \frac{[(h - |B_r - B_s|)^+]^2}{h^3} ds dr = \int_0^t \int_{\mathbb{R}} \frac{[(h - |B_r - x|)^+]^2}{h^3} L_r(x) dx.$$

As a consequence, taking into account that

$$\int_{\mathbb{R}} \frac{[(h - |B_r - x|)^+]^2}{h^3} dx = \int_{\mathbb{R}} \frac{[(h - |x|)^+]^2}{h^3} dx = \frac{4}{3},$$

we obtain

$$\begin{aligned} & \left| \int_0^t \int_0^r \frac{[(h - |B_r - B_s|)^+]^2}{h^3} ds dr - \frac{4}{3} \alpha_t \right| \\ & \leq \int_0^t \int_{\mathbb{R}} \frac{[(h - |B_r - x|)^+]^2}{h^3} |L_r(x) - L_r(B_r)| dx dr \\ & \leq \frac{4}{3} \int_0^t \sup_{|x-y|<h} |L_r(x) - L_r(y)| dr, \end{aligned}$$

which clearly converges to zero in  $L^2$  by the properties of the Brownian local time (see, for instance, [2]).

**Lemma 6.** *We have*

$$\frac{1}{h^6} \mathbb{E} \left[ \left( \int_0^t \left( \int_0^r \left( \int_s^r (h - |B_r - B_u|)^+ d\widehat{B}_u \right) (h - |B_r - B_s|)^+ d\widehat{B}_s \right) dr \right)^2 \right] \rightarrow 0,$$

as  $h$  tends to zero.

*Proof* By the isometry property of the backward Itô integral we can write

$$\begin{aligned} B_h & : = \frac{1}{h^6} \mathbb{E} \left[ \left( \int_0^t \left( \int_0^r \left( \int_s^r (h - |B_r - B_u|)^+ d\widehat{B}_u \right) (h - |B_r - B_s|)^+ d\widehat{B}_s \right) dr \right)^2 \right] \\ & = \frac{1}{h^6} \mathbb{E} \left[ \int_0^t \left( \int_s^t (h - |B_r - B_s|)^+ \left( \int_s^r (h - |B_r - B_u|)^+ d\widehat{B}_u \right) dr \right)^2 ds \right] \\ & \leq \mathbb{E} [B_h^1 B_h^2], \end{aligned}$$

where

$$B_h^1 = \int_0^t \left( \int_s^t \frac{(h - |B_r - B_s|)^+}{h^2} dr \right)^2 ds$$

and

$$B_h^2 = \sup_{0 < s < r < t} \left| \int_s^r \frac{(h - |B_r - B_u|)^+}{h} d\widehat{B}_u \right|^2.$$

As in Lemma 5, we can show that  $B_h^1$  converges to  $\frac{9}{4} \int_0^t (L_t(B_s) - L_s(B_s))^2 ds$ , and the convergence holds in  $L^p(\Omega)$  for any  $p \geq 2$ . Here we use

$$\int_{\mathbb{R}} \frac{(h - |x|)^+}{h^2} dx = \frac{3}{2}.$$

On the other hand,  $\int_s^r \frac{(h - |B_r - B_u|)^+}{h} d\widehat{B}_u$  can be expressed using again Tanaka's formula:

$$\begin{aligned} \frac{1}{h} \int_s^r (h - |B_r - B_u|)^+ d\widehat{B}_u &= \int_0^h \left( \int_s^r I_{y > |B_r - B_u|} d\widehat{B}_u \right) dy \\ &= \frac{1}{h} \int_0^h (L_r(B_r - y) - L_r(B_r + y)) dy \\ &\quad - \frac{1}{h} \int_0^h (L_s(B_r - y) - L_s(B_r + y)) dy + \frac{h}{2} \\ &\quad + \frac{1}{h} \int_0^h [(B_r - B_s + y)^+ - (B_r - B_s - y)^+] dy. \end{aligned}$$

Therefore,

$$\left| \frac{1}{h} \int_s^r (h - |B_r - B_u|)^+ d\widehat{B}_u \right| \leq \sup_{s < r < t} \sup_{|x-y| \leq 2h} |L_r(x) - L_r(y)| + O(h),$$

which also converges to zero in  $L^p(\Omega)$  for any  $p \geq 2$ .

## References

- [1] Barlow, M. T.; Yor, M. (Semi-) martingale inequalities and local times. *Z. Wahrsch. Verw. Gebiete* 55 (1981), no. 3, 237–254. MR0608019
- [2] Borodin, A. N. Brownian local time. *Russian Math. Surveys* 44 (1989), 1–51. MR0998360
- [3] Chen, X., Li, W., Marcus, M. B. and Rosen, J. A CLT for the  $L^2$  modulus of continuity of Brownian local time. Preprint.
- [4] Nualart, D. *The Malliavin Calculus and Related Topics*. Second edition. Springer Verlag, Berlin, 2006. MR2200233
- [5] Ocone, D. Malliavin calculus and stochastic integral representation of diffusion processes. *Stochastics* 12 (1984), 161–185. MR0749372

- 
- [6] Pitman, J.; Yor, M. Asymptotic laws of planar Brownian motion. *Ann. Probab.* **14** (1986), no. 3, 733–779. MR0841582
- [7] Revuz, D.; Yor, M. *Continuous martingales and Brownian motion*. Third edition. Springer-Verlag, Berlin, 1999. MR1725357
- [8] Rosen, J. A stochastic calculus proof of the CLT for the  $L^2$  modulus of continuity of local time. <http://arxiv.org/pdf/0910.2922.pdf>.