

ON THE PRINCIPLE OF SMOOTH FIT FOR KILLED DIFFUSIONS

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Abstract

We explore the principle of smooth fit in the case of the discounted optimal stopping problem

$$V(x) = \sup_{\tau} E_x [e^{-\beta\tau} G(X_{\tau})].$$

We show that there exists a regular diffusion X and differentiable gain function G such that the value function V above fails to satisfy the smooth fit condition $V'(b) = G'(b)$ at the optimal stopping point b . However, if the fundamental solutions ψ and ϕ of the ‘killed’ generator equation $\mathcal{L}_X u(x) - \beta u(x) = 0$ are differentiable at b then the smooth fit condition $V'(b) = G'(b)$ holds (whenever X is regular and G is differentiable at b). We give an example showing that this can happen even when ‘smooth fit through scale’ (in the sense of the discounted problem) fails.

1 Introduction

The *principle of smooth fit* states that the optimal stopping point b which separates the continuation set C from the stopping set D in the (undiscounted) optimal stopping problem

$$V(x) = \sup_{\tau} E_x G(X_{\tau}) \tag{1.1}$$

is characterized by the fact that $V'(b)$ exists and is equal to $G'(b)$. The pivotal role of the scale function in the smooth fit is explored in detail in [6]. Indeed, necessary (and sufficient) conditions for the smooth fit to hold are found to be regularity of the diffusion process X , differentiability of the gain function at the optimal stopping point b and, most notably, differentiability of the scale function at b .

We study the discounted optimal stopping problem

$$V(x) = \sup_{\tau} \mathbb{E}_x [e^{-\beta\tau} G(X_{\tau})] \quad (1.2)$$

with $\beta > 0$ the rate of discounting. It is well known that in this case too the smooth fit principle is preserved in the above intuitive form $V'(b) = G'(b)$. We derive this relationship together with the corresponding necessary (and sufficient) conditions, following the method of [6]. In this way we expose most transparently the role of the fundamental solutions ψ and ϕ of the ‘killed’ generator equation $\mathbb{L}_X u - \beta u = 0$ in the smooth fit, and its analogy with the role of the scale function in the undiscounted case.

Our main findings can be summarized as follows. Firstly, we show that there exists a regular diffusion process X and a differentiable gain function G such that the smooth fit condition $V'(b) = G'(b)$ fails to hold at the optimal stopping point b of the discounted optimal stopping problem (1.2) (Example 2.5). Secondly, we show that if X is regular and each of G , ψ and ϕ are differentiable at b , then V is differentiable at b and $V'(b) = G'(b)$ (Theorem 2.3). Thirdly, we give an example showing that the latter can happen even when ‘smooth fit through scale’ in the sense of the discounted problem (cf. Theorem 2.1) fails at b (Example 2.2).

We remark finally that a more complete study of the smooth fit principle for killed diffusions could aim to address both finite horizon optimal stopping problems as well as higher-dimensional diffusion processes. Whilst the present paper does not look to address these problems directly, we hope the results herein can provide some useful insight in these directions.

2 The smooth fit principle

1. Let $X = (X_t)_{t \geq 0}$ be a diffusion process with values in an interval J of \mathbb{R} . We assume that X can be killed only at the end-points of J which do not belong to J . Thus, if ζ denotes the death time of X , then X is a strong Markov process such that $t \rightarrow X_t$ is continuous on $[0, \zeta)$, and the end-points of J at which X can be killed act as absorbing boundaries (once such a point is reached X stays there forever). We will denote by $I = (l, r)$ the interior of J .

Given $c \in J$ we will let

$$\tau_c = \inf \{ t > 0 \mid X_t = c \} \quad (2.1)$$

denote the hitting time of X to c . We will assume that X is *regular* in the sense that $\mathbb{P}_b(\tau_c < \infty) > 0$ for every $b \in I$ and all $c \in J$. It means that I cannot be decomposed into smaller intervals from which X could not exit.

The infinitesimal generator of X on I is given by the second order differential operator

$$\mathbb{L}_X = \mu(x) \frac{d}{dx} + \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2}. \quad (2.2)$$

Recall that a characteristic of the scale function S of X is that $\mathbb{L}_X S \equiv 0$ everywhere on I .

2. We study the discounted optimal stopping problem

$$V(x) = \sup_{\tau} \mathbb{E}_x [e^{-\beta\tau} G(X_{\tau})] \quad (2.3)$$

for $x \in J$ where the supremum is taken over all stopping times τ of X (i.e. with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \mid 0 \leq s \leq t)$ generated by X for $t \geq 0$). Let the gain function

$G : J \rightarrow \mathcal{R}$ be a measurable function satisfying

$$\mathbb{E} \sup_{0 \leq t < \zeta} |G(X_t)| < \infty. \quad (2.4)$$

It is well known that discounting the gain as in problem (2.3) is equivalent to ‘killing’ the paths of the process X after exponential time T with parameter β . The infinitesimal generator for this killed process, \tilde{X} say, is given by (see e.g. [7] or [10])

$$\mathbb{L}_{\tilde{X}} = \mathbb{L}_X - \beta I \quad (2.5)$$

where I is the identity operator.

The ordinary differential equation

$$\mathbb{L}_X u(x) - \beta u(x) = 0 \quad (2.6)$$

has two linearly independent positive solutions which are uniquely determined up to multiplication if we require one of them to be strictly increasing and the other to be strictly decreasing. We shall denote the increasing solution by ψ and the decreasing solution by ϕ . The general solution of (2.6) is then obtained as a linear combination of ψ and ϕ . Note that one can take $\psi \equiv S$ and $\phi \equiv 1$ if $\beta = 0$. For further characterisation of ψ and ϕ , see e.g [1, Chapter 2] or [5, Section 4.6].

3. Following a similar argument to Dynkin and Yushkevich [4, p. 115], take $c < x < d$ in J and choose stopping times σ_1 and σ_2 such that $\mathbb{E}_c e^{-\beta\sigma_1} G(X_{\sigma_1}) \geq V(c) - \varepsilon$ and $\mathbb{E}_d e^{-\beta\sigma_2} G(X_{\sigma_2}) \geq V(d) - \varepsilon$ where ε is given and fixed. Consider the stopping time $\sigma = (\tau_c + \sigma_1 \circ \theta_{\tau_c}) I(\tau_c < \tau_d) + (\tau_d + \sigma_2 \circ \theta_{\tau_d}) I(\tau_d < \tau_c)$ obtained by applying σ_1 after hitting c (before d) and σ_2 after hitting d (before c). By the strong Markov property of X it follows that

$$\begin{aligned} V(x) &\geq \mathbb{E}_x e^{-\beta\tau_\sigma} G(X_\sigma) \quad (2.7) \\ &= \mathbb{E}_x e^{-\beta(\tau_c + \sigma_1 \circ \theta_{\tau_c})} G(X_{\tau_c + \sigma_1 \circ \theta_{\tau_c}}) I(\tau_c < \tau_d) + \mathbb{E}_x e^{-\beta(\tau_d + \sigma_2 \circ \theta_{\tau_d})} G(X_{\tau_d + \sigma_2 \circ \theta_{\tau_d}}) I(\tau_d < \tau_c) \\ &= \mathbb{E}_x e^{-\beta\tau_c} [e^{-\beta\sigma_1} G(X_{\sigma_1}) \circ \theta_{\tau_c}] I(\tau_c < \tau_d) + \mathbb{E}_x e^{-\beta\tau_d} [e^{-\beta\sigma_2} G(X_{\sigma_2}) \circ \theta_{\tau_d}] I(\tau_d < \tau_c) \\ &= \mathbb{E}_x e^{-\beta\tau_c} \mathbb{E}_{X_{\tau_c}}(e^{-\beta\sigma_1} G(X_{\sigma_1})) I(\tau_c < \tau_d) + \mathbb{E}_x e^{-\beta\tau_d} \mathbb{E}_{X_{\tau_d}}(e^{-\beta\sigma_2} G(X_{\sigma_2})) I(\tau_d < \tau_c) \\ &\geq (V(c) - \varepsilon) \mathbb{E}_x [e^{-\beta\tau_c} I_{\tau_c \leq \tau_d}] + (V(d) - \varepsilon) \mathbb{E}_x [e^{-\beta\tau_d} I_{\tau_c \geq \tau_d}] \end{aligned}$$

where the first inequality follows by definition of V and the second inequality follows by the choice of σ_1 and σ_2 , and the shift operator $\theta_t : \Omega \rightarrow \Omega$ is defined by $\theta_t(\omega)(s) = \omega(t + s)$ for $\omega \in \Omega$ and $t, s \geq 0$. Letting $\varepsilon \downarrow 0$ in (2.7) one concludes that

$$V(x) \geq V(c) \mathbb{E}_x [e^{-\beta\tau_c} I_{\tau_c \leq \tau_d}] + V(d) \mathbb{E}_x [e^{-\beta\tau_d} I_{\tau_c \geq \tau_d}]. \quad (2.8)$$

The function $u_1(x) \equiv \mathbb{E}_x [e^{-\beta\tau_c} I_{\tau_c \leq \tau_d}]$ (respectively $u_2(x) \equiv \mathbb{E}_x [e^{-\beta\tau_d} I_{\tau_c \geq \tau_d}]$) is the unique solution of (2.6) in (c, d) with boundary conditions $u_1(c) = 1$ & $u_1(d) = 0$ (respectively $u_2(c) = 0$ & $u_2(d) = 1$). In terms of the functions ψ and ϕ , using the appropriate boundary conditions, one can calculate

$$u_1(x) = \frac{\psi(x)\phi(d) - \psi(d)\phi(x)}{\psi(c)\phi(d) - \psi(d)\phi(c)} \quad \& \quad u_2(x) = \frac{\psi(c)\phi(x) - \psi(x)\phi(c)}{\psi(c)\phi(d) - \psi(d)\phi(c)} \quad (2.9)$$

for $x \in [c, d]$. Substituting these into inequality (2.8), then dividing both sides of the inequality by $\phi(x)$ (respectively, by $\psi(x)$), one obtains

$$\frac{V(x)}{\phi(x)} \geq \frac{V(c)}{\phi(c)} \frac{F(d) - F(x)}{F(d) - F(c)} + \frac{V(d)}{\phi(d)} \frac{F(x) - F(c)}{F(d) - F(c)} \quad (2.10)$$

$$\frac{V(x)}{\psi(x)} \geq \frac{V(c)}{\phi(c)} \frac{G(d) - G(x)}{G(d) - G(c)} + \frac{V(d)}{\phi(d)} \frac{G(x) - G(c)}{G(d) - G(c)} \quad (2.11)$$

for $x \in [c, d]$ respectively, where the functions

$$F(x) \equiv \frac{\psi(x)}{\phi(x)} \quad \& \quad G(x) \equiv -\frac{1}{F(x)} = -\frac{\phi(x)}{\psi(x)} \quad (2.12)$$

for $x \in [l, r]$ are both well defined and strictly increasing. Observe that inequalities (2.10) and (2.11) imply that V/ϕ is F -concave and V/ψ is G -concave on $[l, r]$ (this result is established in the more general setting of superharmonic functions in [3, Theorem 16.4]). It is easily seen that V/ϕ is F -concave if and only if V/ψ is G -concave. The two are therefore equivalent statements, and henceforth we shall work with the former.

Inequality (2.10) implies that

$$y \mapsto \frac{\frac{V}{\phi}(y) - \frac{V}{\phi}(x)}{F(y) - F(x)} \quad \text{is decreasing} \quad (2.13)$$

on $J \setminus \{x\}$ for every $x \in I$. It follows that

$$-\infty < \frac{d^+ \frac{V}{\phi}}{dF}(x) \leq \frac{d^- \frac{V}{\phi}}{dF}(x) < +\infty \quad (2.14)$$

for every $x \in I$. It also follows that V/ϕ is continuous on I , and hence V is continuous on I , since F is continuous.

4. Let us now assume that $b \in I$ is an optimal stopping point in the problem (2.3). Then $V(b) = G(b)$, and hence by (2.13) we get

$$\frac{\frac{G}{\phi}(b+\varepsilon) - \frac{G}{\phi}(b)}{F(b+\varepsilon) - F(b)} \leq \frac{\frac{V}{\phi}(b+\varepsilon) - \frac{V}{\phi}(b)}{F(b+\varepsilon) - F(b)} \leq \frac{\frac{V}{\phi}(b-\delta) - \frac{V}{\phi}(b)}{F(b-\delta) - F(b)} \leq \frac{\frac{G}{\phi}(b-\delta) - \frac{G}{\phi}(b)}{F(b-\delta) - F(b)} \quad (2.15)$$

for $\varepsilon > 0$ and $\delta > 0$ where the first inequality follows since $G(b+\varepsilon) \leq V(b+\varepsilon)$ and the third inequality follows since $-V(b-\delta) \leq -G(b-\delta)$ (recalling also that F is strictly increasing and that ϕ is strictly positive on $[l, r]$). Passing to the limit for $\varepsilon \downarrow 0$ and $\delta \downarrow 0$ this immediately leads to

$$\frac{d^+ \frac{G}{\phi}}{dF}(b) \leq \frac{d^+ \frac{V}{\phi}}{dF}(b) \leq \frac{d^- \frac{V}{\phi}}{dF}(b) \leq \frac{d^- \frac{G}{\phi}}{dF}(b) \quad (2.16)$$

whenever $d^+(G/\phi)/dF$ and $d^-(G/\phi)/dF$ exist at b . In this way we reach the following result (cf. [9, Theorem 4.7] and [2, Corollary 7.1]):

Theorem 2.1 ('Smooth fit through scale')

If $d(G/\phi)/dF$ exists at b then $d(V/\phi)/dF$ exists at b and

$$\frac{d\frac{V}{\phi}}{dF}(b) = \frac{d\frac{G}{\phi}}{dF}(b) \quad (2.17)$$

whenever $V(b) = G(b)$ for $b \in I$.

5. At an intuitive level, smooth fit means that the graph of the value function leaves that of the gain function smoothly as we move from the stopping region D into the continuation region C . That is, at every point $b \in I$ on the boundary of C and D we have

$$\frac{dV}{dx}(b) = \frac{dG}{dx}(b). \quad (2.18)$$

The example below shows that smooth fit as in (2.17) may fail even though (2.18) holds. We follow the idea of [6, e.g. 2.2] where we saw that smooth fit through scale

$$\frac{dV}{dS}(b) = \frac{dG}{dS}(b) \quad (2.19)$$

could fail to hold even though (2.18) holds. This was seen to be caused by the fact that the derivative of the scale function at the optimal stopping point was zero. The following example asks a similar question of Theorem 2.1 above. If the derivative of F at the optimal stopping point b is zero, could this lead to failure of (2.17) even though (2.18) holds? With this in mind, we construct our diffusion by defining first the functions ψ and ϕ in such a way that $F'(0) = 0$ (we shall see later that zero is indeed the unique optimal stopping point in this problem). Using these functions we shall then identify the generator of our diffusion process.

Example 2.2

Let X be a diffusion process on $[-1, 1]$ such that

$$\psi(x) = \begin{cases} 1 + x^p & \text{if } x \in [0, 1] \\ 1 - |x|^q & \text{if } x \in [-1, 0] \end{cases} \quad (2.20)$$

$$\phi(x) = \begin{cases} 1 - x^q & \text{if } x \in [0, 1] \\ 1 + |x|^p & \text{if } x \in [-1, 0] \end{cases} \quad (2.21)$$

solve the ordinary differential equation (2.6). It is easy to check that for $p > 1$ and $q > 1$ we have $F'(0) = 0$. Since ψ and ϕ are solutions to (2.6), we have

$$\mu\psi' + D\psi'' = \beta\psi \quad (2.22)$$

$$\mu\phi' + D\phi'' = \beta\phi \quad (2.23)$$

with $D = \frac{1}{2}\sigma^2$. For $x \in [0, 1]$ we find that

$$\mu(x) = \frac{\beta}{p-q} \left(\frac{1-q}{p} (1+x^p) x^{1-p} + \frac{1-p}{q} (1-x^q) x^{1-q} \right) \quad (2.24)$$

$$D(x) = \frac{\beta}{p-q} \left(\frac{1}{q} (1-x^q) x^{2-q} + \frac{1}{p} (1+x^p) x^{2-p} \right) \quad (2.25)$$

and for $x \in [-1, 0]$ we find

$$\mu(x) = \frac{\beta}{p-q} \left(\frac{q-1}{p} (1+|x|^p) |x|^{1-p} + \frac{p-1}{q} (1-|x|^q) |x|^{1-q} \right) \quad (2.26)$$

$$D(x) = \frac{\beta}{p-q} \left(\frac{1}{q} (1-|x|^q) |x|^{2-q} + \frac{1}{p} (1+|x|^p) |x|^{2-p} \right). \quad (2.27)$$

From (2.25) and (2.27) we see that we must have $p > q$ to ensure that D is positive. In fact, it is important in the following arguments for us to take $p = 2$ and $q \in (1, 2)$.

Notice that the above expressions (2.24) and (2.26) exhibit terms which are singular at zero. This is also true for a Bessel process Z which has coefficients $\mu_Z(x) = (\alpha-1)/2x$ and $\sigma_Z^2(x) = 1$. We make use of this comparison to justify the existence of our process X , and in particular the regularity of the boundary zero. To this end it is useful for us to examine the squared process X^2 . The existence of the latter process is readily justified by comparison with the existence argument for the squared Bessel process (see e.g. [8, Ch.XI, §1]). Setting $p = 2$ and using Ito's formula we find that the process X^2 has coefficients

$$\mu_{X^2}(x) = \beta(1+x) \quad \& \quad \sigma_{X^2}(x) = \sqrt{\frac{8\beta}{2-q} \left(\frac{1}{2}x + \frac{1}{q}x^{2-\frac{q}{2}} - \left(\frac{q-2}{2q} \right) x^2 \right)} \quad (2.28)$$

for $x \in [0, 1]$. It is easy to check that a process with these coefficients is regular away from zero. To deal with zero boundary behaviour, let us compare X^2 with the squared Bessel process Z^2 which has coefficients

$$\mu_{Z^2}(x) = \alpha \quad \& \quad \sigma_{Z^2}(x) = 2\sqrt{x} \quad (2.29)$$

for $x > 0$. Observe that close to zero the coefficients of our squared process X^2 behave similarly to those of Z^2 . Note here that we may take out a factor of \sqrt{x} in the expression for $\sigma_{X^2}(x)$ of (2.28), which becomes the dominant term close to zero. Furthermore, we know that the squared Bessel process has a regular boundary at zero which is reached almost surely for $0 < \alpha < 2$. Comparing (2.28) and (2.29), removing terms which are negligible close to zero, we can conclude that our squared process X^2 also has a regular boundary at zero which is reached almost surely, for suitable $\beta > 0$.

We now recover our process X from X^2 in the same way as the Bessel process is obtained from the squared Bessel process (see e.g. [8, Ch.XI, §1]). We note that the function $x \rightarrow \sqrt{x}$ is a homeomorphism of \mathbb{R}_+ , therefore if X is a Markov process on \mathbb{R}_+ , then \sqrt{X} is also a Markov process. Thus, taking the square root of X^2 , we obtain X . Since \sqrt{x} is a continuous, increasing function on $[0, 1]$, X will inherit the regularity of X^2 away from zero. Moreover, in the same way as the Bessel process inherits the zero boundary characteristics of the squared Bessel process, so does X inherit those of X^2 , and we conclude that our process X has a regular boundary at zero which is reached almost surely, for suitable β . To extend X from $[0, 1]$ to $[-1, 1]$, notice from (2.24)-(2.27) that X is symmetrical about zero. Following a similar argument to [8, p. 307], for $x \in [0, 1]$ let X have transition density $P_t(x, A)$ for $A \subseteq [0, 1]$. For $x \in [-1, 0]$ we set $\hat{P}_t(x, A) = P_t(-x, \hat{A})$ where $\hat{A} = -A$, and the extension is complete.

Consider now the discounted optimal stopping problem (2.3) with gain function

$$G(x) = 1 - x^{\frac{q+1}{2}} \quad (2.30)$$

for $x \in (-1, 1)$. Using the fact that in the continuation region C our value function is a solution of ordinary differential equation (2.6), and therefore a linear combination of ψ and ϕ , we obtain value function

$$V(x) = \begin{cases} \phi(x) = 1 - x^q & \text{if } x \in [0, 1] \\ \psi(x) = 1 - |x|^q & \text{if } x \in [-1, 0). \end{cases} \quad (2.31)$$

Comparing (2.30) with (2.31) we see that $b = 0$ is an optimal stopping point. This seems to be supported by the fact that our process drifts towards zero from both intervals $(0, 1]$ and $[-1, 0)$ (using (2.24) and (2.26)). Moreover, it is evident that smooth fit (2.18) holds at $b = 0$, both derivatives being zero. However, it is easily verified that

$$\frac{d^+G}{dF} = -\infty < \frac{d^+V}{dF} = 0 < \frac{d^-V}{dF} = 1 < \frac{d^-G}{dF} = +\infty \quad (2.32)$$

at the optimal stopping point $b = 0$. Therefore the smooth fit of Theorem 2.1 fails. This motivates the following extension of Theorem 2.1:

Theorem 2.3 (Smooth fit)

If dG/dx , $d\psi/dx$ and $d\phi/dx$ exist at b , then dV/dx exists at b and

$$\frac{dV}{dx}(b) = \frac{dG}{dx}(b) \quad (2.33)$$

whenever $V(b) = G(b)$ for $b \in I$.

Proof. We proceed similarly to the proof of Theorem 2.3 of [6]. Assume first that $F'(b) \neq 0$. Multiplying by $(F(b+\varepsilon) - F(b))/\varepsilon$ in (2.15) we get

$$\frac{\frac{G}{\phi}(b+\varepsilon) - \frac{G}{\phi}(b)}{\varepsilon} \leq \frac{\frac{V}{\phi}(b+\varepsilon) - \frac{V}{\phi}(b)}{\varepsilon} \leq \frac{F(b+\varepsilon) - F(b)}{\varepsilon} \frac{(\frac{G}{\phi}(b-\delta) - \frac{G}{\phi}(b))/(-\delta)}{(F(b-\delta) - F(b))/(-\delta)}. \quad (2.34)$$

Passing to the limit for $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, and using that $F'(b) \neq 0$, it follows that $d^+(V/\phi)/dx = d(G/\phi)/dx$ at b . (Note that one could take $\varepsilon = \delta$ in this argument.) It follows from this, by a simple application the quotient rule for derivatives, that $d^+V/dx = dG/dx$ at b .

Similarly, multiplying by $(F(b-\delta) - F(b))/(-\delta)$ in (2.15) we get

$$\frac{(\frac{G}{\phi}(b+\varepsilon) - \frac{G}{\phi}(b))/\varepsilon}{(F(b+\varepsilon) - F(b))/\varepsilon} \frac{F(b-\delta) - F(b)}{-\delta} \leq \frac{\frac{V}{\phi}(b-\delta) - \frac{V}{\phi}(b)}{-\delta} \leq \frac{\frac{G}{\phi}(b-\delta) - \frac{G}{\phi}(b)}{-\delta}. \quad (2.35)$$

Passing to the limit for $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, and using that $F'(b) \neq 0$, it follows that $d^-(V/\phi)/dx = d(G/\phi)/dx$ at b . (Note that one could take $\varepsilon = \delta$ in this argument.) As above it follows that $d^-V/dx = dG/dx$ at b .

Combining the two conclusions we see that dV/dx exists at b and (2.33) holds as claimed.

To treat the case $F'(b) = 0$ we need the following facts of real analysis (see [6] for a proof):

Lemma 2.5

Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be two continuous functions satisfying:

$$f(0) = 0 \text{ and } f(\varepsilon) > 0 \text{ for } \varepsilon > 0; \quad (2.36)$$

$$g(0) = 0 \text{ and } g(\delta) > 0 \text{ for } \delta > 0. \quad (2.37)$$

Then for every $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$ there are $\varepsilon_{n_k} \downarrow 0$ and $\delta_k \downarrow 0$ as $k \rightarrow \infty$ such that $f(\varepsilon_{n_k}) = g(\delta_k)$ for all $k \geq 1$. In particular, it follows that

$$\lim_{k \rightarrow \infty} \frac{f(\varepsilon_{n_k})}{g(\delta_k)} = 1. \quad (2.38)$$

Let us continue the proof of Theorem 2.3 in the case when $F'(b) = 0$. Take $\varepsilon_n \downarrow 0$ and by Lemma 2.5 choose $\delta_k \downarrow 0$ such that (2.38) holds with $f(\varepsilon) = (F(b+\varepsilon) - F(b))/\varepsilon$ and $g(\delta) = (F(b) - F(b-\delta))/\delta$. Then (2.34) reads

$$\frac{\frac{G}{\phi}(b+\varepsilon_{n_k}) - \frac{G}{\phi}(b)}{\varepsilon_{n_k}} \leq \frac{\frac{V}{\phi}(b+\varepsilon_{n_k}) - \frac{V}{\phi}(b)}{\varepsilon_{n_k}} \leq \frac{f(\varepsilon_{n_k})}{g(\delta_k)} \frac{\frac{G}{\phi}(b-\delta_k) - \frac{G}{\phi}(b)}{-\delta_k} \quad (2.39)$$

for all $k \geq 1$. Letting $k \rightarrow \infty$ and using (2.38) we see that $(\frac{V}{\phi}(b+\varepsilon_{n_k}) - \frac{V}{\phi}(b))/\varepsilon_{n_k} \rightarrow \frac{G'}{\phi}(b)$. Since this is true for any $\varepsilon_n \downarrow 0$ it follows that $d^+(V/\phi)/dx$ exists and is equal to $d(V/\phi)/dx$ at b , and hence $d^+V/dx = dG/dx$ at b .

Similarly, take $\varepsilon_n \downarrow 0$ and by Lemma 2.5 choose $\delta_k \downarrow 0$ such that (2.38) holds with $f(\varepsilon) = (F(b) - F(b-\varepsilon))/\varepsilon$ and $g(\delta) = (F(b+\delta) - F(b))/\delta$. Then (2.35) (with ε and δ swapped) reads

$$\frac{\frac{G}{\phi}(b+\delta_k) - \frac{G}{\phi}(b)}{\delta_k} \frac{f(\varepsilon_{n_k})}{g(\delta_k)} \leq \frac{\frac{V}{\phi}(b-\varepsilon_{n_k}) - \frac{V}{\phi}(b)}{-\varepsilon_{n_k}} \leq \frac{\frac{G}{\phi}(b-\varepsilon_{n_k}) - \frac{G}{\phi}(b)}{-\varepsilon_{n_k}} \quad (2.40)$$

for all $k \geq 1$. Letting $k \rightarrow \infty$ and using (2.38) we see that $(\frac{V}{\phi}(b-\varepsilon_{n_k}) - \frac{V}{\phi}(b))/(-\varepsilon_{n_k}) \rightarrow \frac{G'}{\phi}(b)$. Since this is true for any $\varepsilon_n \downarrow 0$ it follows that d^-V/dx exists and is equal to dG/dx at b , and hence $d^-V/dx = dG/dx$ at b . Taken together with the previous conclusion on d^+V/dx this establishes (2.33) and the proof of Theorem 2.3 is complete. \square

Remark 2.4

The above proof shows that in fact *minimal* conditions for the smooth fit to hold are differentiability at the optimal stopping point b of the gain function G , of (G/ϕ) , and of $F = (\psi/\phi)$. Of course, the conditions of Theorem 2.3 imply those of this remark.

The following example shows that smooth fit as in (2.33) can fail if ψ and ϕ are not differentiable at the optimal stopping point b , even though we have differentiability of the gain function at b and a regular diffusion.

Example 2.5

Let X be a diffusion process on $[-1, 1]$ such that

$$\psi(x) = \begin{cases} 1 + x^2 & \text{if } x \in [0, 1] \\ 1 - |x| & \text{if } x \in [-1, 0] \end{cases} \quad (2.41)$$

$$\phi(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1] \\ 1 + |x|^2 & \text{if } x \in [-1, 0] \end{cases} \quad (2.42)$$

solve the ordinary differential equation (2.6). Proceeding similarly to Example 2.2, for $x \in [0, 1]$, we find that

$$\mu(x) = -\beta(1-x) \quad \& \quad D(x) = \beta \left((1-x)x + \frac{1}{2}(1+x^2) \right) \quad (2.43)$$

and for $x \in [-1, 0]$ we find

$$\mu(x) = \beta(1 - |x|) \quad \& \quad D(x) = \beta \left((1 - |x|)|x| + \frac{1}{2}(1 + x^2) \right). \quad (2.44)$$

Consider X on $[0, 1]$. One can check that the process is regular away from zero. To deal with boundary behaviour at zero, notice that our process X is exactly that of Example 2.2 if we set $p = 2$ and $q = 1$. Therefore the same arguments for boundary behaviour hold, in particular zero is a regular boundary which is reached almost surely for suitable β . In the same way as Example 2.2 we may define our diffusion on the interval $[-1, 1]$.

Consider now the discounted optimal stopping problem (2.3) with gain function

$$G(x) = \begin{cases} (x - 1)^2 & \text{if } x \in [0, 1] \\ -(|x| - 1)^2 + 2 & \text{if } x \in [-1, 0]. \end{cases} \quad (2.45)$$

Notice in particular that the gain function thus defined is differentiable at zero. Using the fact that in the continuation region C our value function is a solution of ordinary differential equation (2.6), and therefore a linear combination of ψ and ϕ , we obtain value function

$$V(x) = \begin{cases} \phi(x) = 1 - x & \text{if } x \in [0, 1] \\ G(x) = -(|x| - 1)^2 + 2 & \text{if } x \in [-1, 0]. \end{cases} \quad (2.46)$$

Comparing (2.45) with (2.46) we see that $b = 0$ is an optimal stopping point. This seems to be supported by the fact that our process drifts towards zero from both intervals $(0, 1]$ and $[-1, 0)$. However, it is evident that V is not differentiable at $b = 0$ and the smooth fit $V'(b) = G'(b)$ fails.

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