T-MARTIN BOUNDARY OF REFLECTED RANDOM WALKS ON A HALF-SPACE

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Abstract

The *t*-Martin boundary of a random walk on a half-space with reflected boundary conditions is identified. It is shown in particular that the *t*-Martin boundary of such a random walk is not stable in the following sense : for different values of *t*, the *t*-Martin compactifications are not equivalent.

1   Introduction

Before formulating our results we recall the definition and some properties of *t*-Martin compactification. Let $P = (p(x,x'), \; x,x' \in E)$ be a transition kernel of a time-homogeneous, irreducible Markov chains $Z = (Z(t))$ on a countable, discrete state spaces E. Then by irreducibility, for any $t > 0$, the series

$$G_t(z,z') \doteq \sum_{n=0}^{\infty} t^{-n} \mathbb{P}_x(Z(n) = z') \tag{1.1}$$

either converge or diverge simultaneously for all $z, z' \in E$ (see Seneta [18]).

Definition 1.1. *The infimum $\rho(P)$ of the $t > 0$ for which the series* (1.1) *converge is equal to*

$$\rho(P) = \limsup_{n \to \infty} \left(\mathbb{P}_x(Z(n) = x') \right)^{1/n}, \tag{1.2}$$

it is called the convergence norm of the transition kernel P.

Definition 1.2. *For $t > 0$, a positive function $f : E \to \mathbb{R}_+$ is called t-harmonic (resp. t-superharmonic) for P if it satisfies the equality $Pf = tf$ (resp. $Pf \le tf$). A t-harmonic function is therefore an eigenvectors of the transition operator P with respect to the eigenvalue t. For $t = 1$, the t-harmonic functions are called harmonic. A t-harmonic function $f > 0$ is said to be minimal if for any t-harmonic function $\tilde{f} > 0$ the inequality $\tilde{f} \le f$ implies the equality $\tilde{f} = cf$ with some $c > 0$.*

For $t > 0$, the set of *t*-superharmonic functions of an irreducible Markov kernel P on a countable state space E is nonvoid only if $t \ge \rho(P)$, see Pruitt [16] or Seneta [18].

Definition 1.3. *The t-Martin kernel $K_t(x, x')$ of the transition kernel P is defined by*

$$K_t(x, x') = G_t(x, x_n)/G_t(x_0, x_n) \tag{1.3}$$

where x_0 is a reference point in E.
A sequence of points $x_n \in E$ is said to converge to a point of the t-Martin boundary $\partial_{t,M}(E)$ of the set E defined by the transition kernel P if for any finite subset $V \subset E$ there is n_V such that $x_n \notin V$ for all $n > n_V$ and the sequence of functions $K_t(\cdot, x_n)$ converges point-wise on E.

The t-Martin compactification $E_{t,M}$ is therefore the unique smallest compactification of the set E for which the t-Martin kernels $K_t(z, \cdot)$ extend continuously.

Definition 1.4. *The t-Martin compactification is said to be stable if it does not depend on t for $t > \rho(P)$, i.e. if for any sequence of points $x_n \in E$ that leaves the finite subsets of E, the convergence to a point of the t-Martin boundary for some $t > \rho(P)$ implies the convergence to a point of the t-Martin boundary for all $t > \rho(P)$.*

In the case $t = 1$ and with a transient transition kernel P, the t-Martin compactification is the classical Martin compactification, introduced first for Brownian motion by Martin [12]. For countable Markov chains with discrete time, the abstract construction of the Martin compactification was given by Doob [2] and Hunt [6]. The main general results in this domain are the following :
The minimal Martin boundary $\partial_{1,m}(E)$ is the set of all those $\gamma \in \partial_{1,M}(E)$ for which the function $K_1(\cdot, \gamma)$ is minimal harmonic. By the Poisson-Martin representation theorem, for every non-negative 1-harmonic function h there exists a unique positive Borel measure ν on $\partial_{1,m}(E)$ such that

$$h(z) = \int_{\partial_{t,m}E_M} K_1(z, \eta) \, d\nu(\eta)$$

By Convergence theorem, the sequence $(Z(n))$ converges $\mathbb{P}_z$ almost surely for every initial state $z \in E$ to a $\partial_{1,m}(E)$ valued random variable. The Martin boundary provides therefore all non-negative 1-harmonic functions and describes the asymptotic behavior of the transient Markov chain $(Z(n))$. See Woess [19]).
In general it is a non-trivial problem to determine Martin boundary of a given class of Markov chains. The t-Martin boundary plays an important role to determine the Martin boundary of several products of transition kernels.
To identify the Martin boundary of the direct product of two independent transient Markov chains $(X(n))$ and $(Y(n))$, i.e. the Martin boundary of $Z(n) = (X(n), Y(n))$, the determination of the Martin boundary of each of the components $(X(n))$ and $(Y(n))$ is far from being sufficient. Molchanov [13] has shown that for strongly aperiodic irreducible Markov chains $(X(n))$ and $(Y(n))$, every minimal harmonic function h of the couple $Z(n) = (X(n), Y(n))$ is of the form $h(x, y) = f(x)g(y)$ where f is a t-harmonic function of $(X(n))$ and g is a s-harmonic function of $(Y(n))$ with some $t > 0$ and $s > 0$ satisfying the equality $ts = 1$.
In the case of Cartesian product of Markov chains, i.e. by considering a convex combination $Q = aP + (1 - a)P'$, $0 < a < 1$, of the corresponding transition matrices, Picardello and Woess [14] has shown that the minimal harmonic functions of the transition matrix Q have a similar product form but with $t > 0$ and $s > 0$ satisfying the equality $at + (1 - a)s = 1$. In this paper some of the results on the topology of the Martin boundary are obtained under the assumption that the t-Martin boundaries of the components $(X(n))$ and $(Y(n))$ are stable in the above sense.
This stability property is an important ingredient for the identification of the Martin boundary of the product of Markov chains in general. The assumption on stability seems to be non-restrictive in

the case of spatially homogeneous Markov processes, see Woess [19], Picardello and Woess [15]. These previous works suggest in particular the natural conjecture that the *t*-Martin compactification should be stable in general. The purpose of our paper is to show that such a conjecture is not true.

We consider a random walk on a half-space $\mathbb{Z}^{d-1} \times \{0\}$ with reflected conditions on the boundary $\mathbb{Z}^{d-1} \times \{0\}$. For such a random walk, the *t*-Martin compactification was identified in [10] for $t = 1$. In the present paper, we extend these results for an arbitrary $t > \rho(P)$. The convergence norm $\rho(P)$ of the random walk is calculated explicitly and it is shown that under quite general assumptions, the *t*-Martin compactification for a two-dimensional random walk is not stable.

2 Main results

Let $Z(n) = (X(n), Y(n))$ be a random walk on $\mathbb{Z}^{d-1} \times \mathbb{N}$ with transition probabilities

$$p(z, z') = \begin{cases} \mu(z' - z) & \text{for } z = (x, y), z' \in \mathbb{Z}^{d-1} \times \mathbb{N} \text{ with } y > 0, \\ \mu_0(z' - z) & \text{for } z = (x, y), z' \in \mathbb{Z}^{d-1} \times \mathbb{N} \text{ with } y = 0 \end{cases}$$

where μ and μ_0 are two different positive measures on $\mathbb{Z}^d$ with $0 < \mu(\mathbb{Z}^d) \leq 1$ and $0 < \mu_0(\mathbb{Z}^d) \leq 1$. The random walk $Z(n) = (X(n), Y(n))$ can be therefore substochastic if either $\mu(\mathbb{Z}^d) < 1$ or $\mu_0(\mathbb{Z}^d) < 1$.

Throughout this paper we denote by $\mathbb{N}$ the set of all non-negative integers : $\mathbb{N} = \{0, 1, 2, \ldots\}$ and we let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The assumptions we need on the Markov process $(Z(t))$ are the following.

(H0) $\mu(z) = 0$ *for* $z = (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$ *with* $y < -1$ *and* $\mu_0(z) = 0$ *for* $z = (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$ *with* $y < 0$.

(H1) *The Markov process* $Z(t)$ *is irreducible on* $\mathbb{Z}^{d-1} \times \mathbb{N}$.

(H2) *The homogeneous random walk* $S(t)$ *on* $\mathbb{Z}^d$ *having transition probabilities* $p_S(z, z') = \mu(z' - z)$ *is irreducible on* $\mathbb{Z}^d$.

(H3) *The jump generating functions*

$$\varphi(a) = \sum_{z \in \mathbb{Z}^d} \mu(z) e^{a \cdot z} \quad and \quad \varphi_0(a) = \sum_{z \in \mathbb{Z}^d} \mu_0(z) e^{a \cdot z} \tag{2.1}$$

are finite everywhere on $\mathbb{R}^d$.

(H4) *The last coordinate of* $S(t)$ *is an aperiodic random walk on* $\mathbb{Z}$.

Our first result identifies the convergence rate $\rho(P)$ of the transition kernel $P = (p(z, z'), \ z, z' \in \mathbb{Z}^{d-1} \times \mathbb{N})$.

Proposition 2.1. *Under the hypotheses (H0)-(H3),*

$$\rho(P) = \inf_{a \in \mathbb{R}^d} \max\{\varphi(a), \varphi_0(a)\}. \tag{2.2}$$

The proof of this proposition uses the large deviation results obtained in [3, 5, 7, 8] and is given in Section 4.

Remark that under the assumptions (H0)-(H3), for any $t > 0$, the sets

$$D^t \doteq \{a \in \mathbb{R}^d : \varphi(a) \le t\} \quad \text{and} \quad D_0^t \doteq \{a \in \mathbb{R}^d : \varphi_0(a) \le t\} \tag{2.3}$$

are convex and the set D^t is moreover compact. We denote by ∂D^t the boundary of D^t we let

$$\partial_0 D^t \doteq \{a \in \partial D : \nabla \varphi(a) \in \mathbb{R}^{d-1} \times \{0\}\},$$

$$\partial_+ D^t \doteq \{a \in \partial D^t : \nabla \varphi(a) \in \mathbb{R}^{d-1} \times [0, +\infty[\}$$

and

$$\partial_- D^t \doteq \{a \in \partial D^t : \nabla \varphi(a) \in \mathbb{R}^{d-1} \times] - \infty, 0]\}.$$

For $a \in D^t$, the unique point on the boundary $\partial_- D^t$ which has the same first $(d - 1)$ coordinates as the point a is denoted by $\overline{a}^t$,

$$\hat{D}^t \doteq \{a \in D^t : \varphi_0(\overline{a}^t) \le t\} \quad \text{and} \quad \Gamma_+^t \doteq \partial_+ D^t \cap \hat{D}^t. \tag{2.4}$$

Remark that $\partial_0 D^t = \partial_+ D^t \cap \partial_- D^t$ and for $a \in \partial_+ D^t$, one has $a = \overline{a}^t$ if and only if $a \in \partial_0 D^t$. Moreover, under the hypotheses (H0)-(H1), $\varphi_0(\overline{a}^t) \le \varphi_0(a)$ for any $a \in D^t$ because the function $a \to \varphi_0(a)$ is increasing with respect to the last coordinate of $a \in \mathbb{R}^d$. This inequality implies another useful representation of the set $\hat{D}^t$: $a = (\alpha, \beta) \in \hat{D}^t$ if and only if $a \in D^t$ and $a' = (\alpha, \beta') \in D^t \cap D_0^t$ for some $\beta' \in \mathbb{R}$, or equivalently,

$$\hat{D}^t = (\Theta^t \times \mathbb{R}) \cap D^t \tag{2.5}$$

where

$$\Theta^t \doteq \{\alpha \in \mathbb{R}^{d-1} : \inf_{\beta \in \mathbb{R}} \max\{\varphi(\alpha, \beta), \varphi_0(\alpha, \beta)\} \le t\}. \tag{2.6}$$

The set $\Theta^t \times \{0\}$ is therefore the orthogonal projection of the set $D^t \cap D_0^t$ onto the hyper-plane $\mathbb{R}^{d-1} \times \{0\}$ and by Proposition 2.1,

$$\rho(P) = \inf\{t > 0 : D^t \cap D_0^t \ne \emptyset\} = \inf\{t > 0 : \Theta^t \ne \emptyset\}. \tag{2.7}$$

For $t > \rho(P)$ and $a \in \hat{D}^t$, we denote by $V_t(a)$ the normal cone to the set $\hat{D}^t$ at the point a and for $a \in \Gamma_+^t \doteq \hat{D}^t \cap \partial_+ D^t = (\Theta^t \times \mathbb{R}) \cap \partial_+ D^t$ we define the function $h_{a,t}$ on $\mathbb{Z}^{d-1} \times \mathbb{N}$ by letting

$$h_{a,t}(z) = \begin{cases} \exp(a \cdot z) - \dfrac{t - \varphi_0(a)}{t - \varphi_0(\overline{a}^t)} \exp(\overline{a}^t \cdot z) & \text{if } a \notin \partial_0 D^t \text{ and } \varphi_0(\overline{a}^t) < t, \\[3ex] y \exp(a \cdot z) + \dfrac{\frac{\partial}{\partial \beta} \varphi_0(a)}{(t - \varphi_0(a))} \exp(a \cdot z) & \text{if } a = \overline{a}^t \in \partial_0 D^t \text{ and } \varphi_0(a) < t, \\[3ex] \exp(\overline{a}^t \cdot z) & \text{if } \varphi_0(\overline{a}^t) = t \end{cases} \tag{2.8}$$

where $\frac{\partial}{\partial \beta} \varphi_0(a)$ denotes the partial derivative of the function $a \to \varphi_0(a)$ with respect to the last coordinate $\beta \in \mathbb{R}$ of $a = (\alpha, \beta)$.

The following lemma gives an explicit representation of the normal cone $V_t(a)$.

Lemma 2.1. *Under the hypotheses (H0)-(H3), for any $t > \rho(P)$ and $a \in \Gamma_+^t$,*

$$V_t(a) = \begin{cases} \{c\nabla\varphi(a) : c \geq 0\} & \text{if either } \varphi_0(\overline{a}^t) < t \\ & \text{or } a = \overline{a}^t \in \partial_0 D^t, \\ \{c_1\nabla\varphi(a) + c_2(\nabla\varphi_0(\overline{a}^t) + \kappa_a\nabla\varphi(\overline{a}^t)) : c_i \geq 0\} & \text{if } \varphi_0(\overline{a}^t) = t \\ & \text{and } a \notin \partial_0 D^t \end{cases} \tag{2.9}$$

where

$$\kappa_a = -\left. \frac{\partial \varphi_0(\alpha, \beta)}{\partial \beta}\left(\frac{\partial \varphi(\alpha, \beta)}{\partial \beta}\right)^{-1}\right|_{(\alpha,\beta)=\overline{a}^t}$$

Proof. Recall that for any $t > \inf_a \max\{\varphi(a), \varphi_0(a)\}$, the set $\Theta^t \times \{0\}$ is the orthogonal projection of the convex set $D^t \cap D_0^t$ onto the hyperplane $\mathbb{R}^{d-1} \times \{0\}$. This proves that the set Θ^t is convex itself. Moreover, for any $t > \inf_a \max\{\varphi(a), \varphi_0(a)\}$, the set $D^t \cap D_0^t$ has a non-empty interior. Since $D^t \cap D_0^t \subset \hat{D}^t$ from this it follows that for any $t > \inf_a \max\{\varphi(a), \varphi_0(a)\}$, set $\hat{D}^t = (\Theta^t \times \mathbb{R}) \cap D^t$ has also a non-empty interior and consequently, by Corollary 23.8.1 of Rockafellar [17],

$$V_t(a) = V_{\Theta^t \times \mathbb{R}}(a) + V_{D^t}(a), \quad \forall a \in \hat{D}^t, \tag{2.10}$$

where $V_{\Theta^t \times \mathbb{R}}(a)$ denotes the normal cone to the set $\Theta^t \times \mathbb{R}$ at the point a and $V_{D^t}(a)$ is the normal cone to the set D^t at a. Since under the hypotheses of our lemma,

$$V_{D^t}(a) = \{c\nabla\varphi(a) : c \geq 0\}, \quad \forall a \in \partial D^t \tag{2.11}$$

from this it follows that

$$V_t(a) = V_{D^t}(a) = \{c\nabla\varphi(a) : c \geq 0\}$$

whenever the point $a \in \Gamma_+^t$ belongs to the interior of the set $\Theta^t \times \mathbb{R}$, i.e. when $\varphi_0(\overline{a}^t) < t$. The first equality of (2.9) is therefore verified. Suppose now that the point $a \in \Gamma_+^t$ belongs to the boundary of the set $\Theta^t \times \mathbb{R}$, i.e. either $a = \overline{a}^t \in \partial_0 D^t$ or $\varphi_0(\overline{a}^t) = t$. Then

$$V_{\Theta^t \times \mathbb{R}}(a) = V_{D^t \cap D_0^t}(\overline{a}^t) \cap (\mathbb{R}^{d-1} \times \{0\})$$

because the set $\Theta^t \times \{0\}$ is the orthogonal projection of $D^t \cap D_0^t$ onto $\mathbb{R}^{d-1} \times \{0\}$. $V_{D^t \cap D_0^t}(\overline{a}^t)$ denotes here the normal cone to the set $D^t \cap D_0^t$ at the point $\overline{a}^t$. Using therefore again Corollary 23.8.1 of Rockafellar [17], we obtain

$$V_{\Theta^t \times \mathbb{R}}(a) = \left(V_{D^t}(\overline{a}^t) + V_{D_0^t}(\overline{a}^t)\right) \cap (\mathbb{R}^{d-1} \times \{0\})$$

where

$$V_{D_0^t}(\overline{a}^t) = \begin{cases} \{c\nabla\varphi_0(\overline{a}^t) : c \geq 0\} & \text{if } \varphi_0(\overline{a}^t) = t, \\ \{0\} & \text{if } \varphi_0(\overline{a}^t) < t, \end{cases}$$

is the normal cone to the set D_0^t at the point $\overline{a}^t$. Since the function φ_0 is increasing with respect to the last variable, the last coordinate of $\nabla\varphi_0(\overline{a}^t)$ is strictly positive and consequently, the last relations combined with (2.10) and (2.11) prove the second equality of (2.9).

The following result identifies the t-Martin compactification of $\mathbb{Z}^{d-1} \times \mathbb{N}$ defined by the random walk $(Z(n))$. As above, we denote by $K_t(z, z')$ the t-Martin kernel of the Markov process $(Z(n))$ with a reference point $z_0 \in \mathbb{Z}^{d-1} \times \mathbb{N}$.

Proposition 2.2. *Under the hypotheses (H0)-(H4), for any $t > \rho(P)$, the following assertions hold :*

(i) *for any unit vector $q \in \mathbb{R}^{d-1} \times [0, +\infty[$ there exists a unique $a = \hat{a}_t(q) \in \Gamma_+^t$ such that $q \in V_t(\hat{a}_t(q))$,*

(ii) *for any $a \in \hat{D}^t \cap \partial_+ D^t$ and any sequence of points $z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}$,*

$$\lim_{n \to \infty} K_t(z, z_n) = h_{a,t}(z)/h_{a,t}(z_0), \qquad \forall z \in \mathbb{Z}^{d-1} \times \mathbb{N} \qquad (2.12)$$

whenever $\lim_{n \to \infty} |z_n| = \infty$ and $\lim_{n \to \infty} \text{dist}(V_t(a), z_n/|z_n|) = 0$.

Tor $t = 1$, this statement was proved in [10]. For an arbitrary $t > \rho(P)$, we obtain this result as a consequence of the results of the paper [10] by using the exponential change of the measure. The proof of Proposition 2.2 is given in section 5.

Remark that the assertion (ii) of Proposition 2.2 proves that a sequence $z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}$ with $\lim_{n \to \infty} |z_n| = \infty$, converges to a point on the t-Martin boundary if and only if

$$\lim_{n \to \infty} \text{dist}\left(V_t(a), z_n/|z_n|\right) = 0$$

for some $a \in \hat{D} \cap \partial_+ D$. The t-Martin compactification is therefore stable if and only if $V_t(\hat{a}_t(q)) = V_s(\hat{a}_s(q))$ for any unit vector $q \in \mathbb{R}^{d-1} \times [0, +\infty[$ and all $t > s > \rho(P)$.

Now we give an example where the t-Martin compactification is unstable. This is a subject of the following section.

3   Example

Recall that under the hypotheses (H0)-(H3), by Proposition 2.1, the convergence norm of our transition kernel P is given by (2.2). Here, we consider a particular case when $d = 2$ and

$$\inf_{a \in \mathbb{R}^2} \max\{\varphi(a), \varphi_0(a)\} > \inf_{a \in \mathbb{R}^2} \varphi(a). \qquad (3.1)$$

The minimum of the function $\max\{\varphi(a), \varphi_0(a)\}$ over $a \in \mathbb{R}^2$ is then achieved at some point $a^* = (\alpha^*, \beta^*)$ where

$$\left. \frac{\partial}{\partial \beta} \varphi(\alpha, \beta) \right|_{(\alpha,\beta)=a^*} \leq 0 \quad \text{and} \quad \left. \frac{\partial}{\partial \beta} \varphi_0(\alpha, \beta) \right|_{(\alpha,\beta)=a^*} > 0.$$

The second inequality holds because the function $\varphi_0(\alpha, \beta)$ is increasing with respect to the second variable β. To prove the first inequality it is sufficient to notice that otherwise, there is a point $a = (\alpha, \beta)$ with $\alpha = \alpha^*$ and $\beta < \beta^*$ for which $\max\{\varphi(a), \varphi_0(a)\} < \max\{\varphi(a^*), \varphi_0(a^*)\}$. Finally, we will assume that such a point a^* is unique and that

$$\left. \frac{\partial}{\partial \beta} \varphi(\alpha, \beta) \right|_{(\alpha,\beta)=a^*} < 0. \qquad (3.2)$$

Then clearly, $\varphi(a^*) = \varphi_0(a^*)$ and by implicit function theorem, in a neighborhood the point a^*, one can parametrize the intersection of the surfaces $\mathscr{C} = \{(\alpha, \beta, t) \in \mathbb{R}^3 : t = \varphi(\alpha, \beta)\}$ and

$\mathscr{C}_0 = \{(\alpha, \beta, t) \in \mathbb{R}^3 : t = \varphi_0(\alpha, \beta)\}$ as follows : there are $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and a smooth function $\alpha \to \beta(\alpha)$ from $[\alpha^* - \varepsilon_1, \alpha^* + \varepsilon_2]$ to $\mathbb{R}$ such that $\beta(\alpha^*) = \beta^*$ and for any $\alpha^* - \varepsilon_1 \le \alpha \le \alpha^* + \varepsilon_1$,

$$\frac{\partial}{\partial \beta} \varphi(\alpha, \beta)\Big|_{\beta = \beta(\alpha)} < 0, \qquad \frac{\partial}{\partial \beta} \varphi_0(\alpha, \beta)\Big|_{\beta = \beta(\alpha)} > 0, \tag{3.3}$$

and

$$\{(\alpha, \beta, t) \in \mathscr{C} \cap \mathscr{C}_0 : \alpha^* - \varepsilon_1 \le \alpha \le \alpha^* + \varepsilon_2\}$$
$$= \{(\alpha, \beta(\alpha), t(\alpha)), \alpha^* - \varepsilon_1 \le \alpha \le \alpha^* + \varepsilon_2\} \tag{3.4}$$

with

$$t(\alpha) = \varphi(\alpha, \beta(\alpha)) = \varphi_0(\alpha, \beta(\alpha)) \ge t(\alpha^*). \tag{3.5}$$

Moreover, since the point a^*, where the minimum of the function $\max\{\varphi, \varphi_0\}$ is achieved, is assumed to be unique, the last inequality holds with the equality if and only if $\alpha = \alpha^*$ and without any restriction of generality we can assume that $t(\alpha^* - \varepsilon_1) = t(\alpha^* + \varepsilon_2) > t(\alpha^*)$. Then for any $t(\alpha^*) < t \le t(\alpha^* - \varepsilon_1)$, there are exactly two points $\alpha^* - \varepsilon_1 \le \alpha_1(t) < \alpha^*$ and $\alpha^* < \alpha_2(t) \le \alpha^* + \varepsilon_2$ such that for $a_i(t) = (\alpha_i(t), \beta(\alpha_i(t)))$,

$$\varphi(a_i(t)) = \varphi_0(a_i(t)) = t(\alpha_i(t)) = t, \quad \forall i \in \{1, 2\},$$

$$\Theta^t = [\alpha_1(t), \alpha_2(t)], \quad \hat{D}^t = \{(\alpha, \beta) \in \mathbb{R}^2 : \varphi(\alpha, \beta) \le t, \ \alpha_1(t) \le \alpha \le \alpha_2(t)\},$$

and $\Gamma_+^t = \{a = (\alpha, \beta) \in \partial_+ D^t : \alpha_1(t) \le \alpha \le \alpha_2(t)\}$ is the arc on the boundary $\partial_+ D^t$ with the end points in $\tilde{a}_1(t)$ and $\tilde{a}_2(t)$ where $\tilde{a}_i(t) = (\tilde{\alpha}_i(t), \tilde{\beta}_i(t))$ is a unique point on the boundary $\partial_+ D^t$ with $\tilde{\alpha}_i(t) = \alpha_i(t)$ for $i = 1, 2$, (see Figure 1).
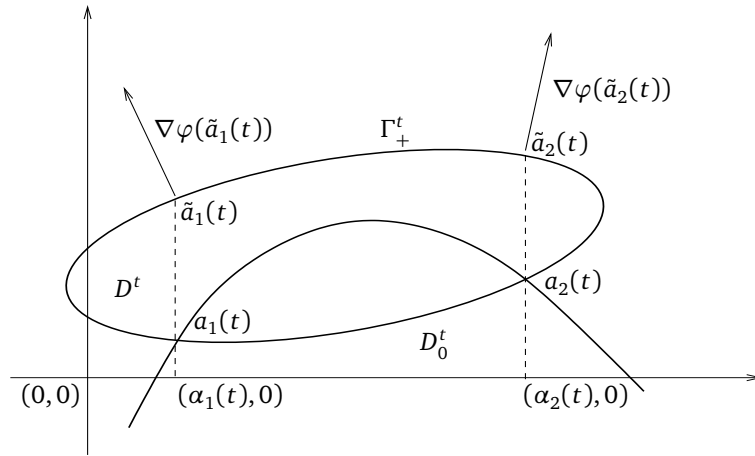

Figure 1:

Furthermore, by Lemma 2.1, for any $t(\alpha^*) < t \le t(\alpha^* - \varepsilon_1)$ and $a \in \Gamma_+^t$,

$$V_t(a) = \begin{cases} \{c_1 e_1 + c_2 \nabla\varphi(\tilde{a}_2(t)) : c_i \ge 0\} & \text{if } a = \tilde{a}_2(t), \\ \{-c_1 e_1 + c_2 \nabla\varphi(\tilde{a}_1(t)) : c_i \ge 0\} & \text{if } a = \tilde{a}_1(t), \\ \{c \nabla\varphi(a) : c \ge 0\} & \text{otherwise.} \end{cases}$$

Hence, by Proposition 2.2 any sequence of points $z_n \in \mathbb{Z} \times \mathbb{N}$ with $\lim_n |z_n| = \infty$ converges in the t-Martin compactification of $\mathbb{Z} \times \mathbb{N}$ if and only if one of the following conditions is satisfied :

  – either $\lim_{n\to\infty} \arg(z_n) = \gamma$ for some $\arg(\nabla\varphi(\tilde{a}_2(t))) < \gamma < \arg(\nabla\varphi(\tilde{a}_2(t)))$,

  – or $\limsup_{n\to\infty} \arg(z_n) \leq \arg(\nabla\varphi(\tilde{a}_2(t)))$,

  – or $\liminf_{n\to\infty} \arg(z_n) \geq \arg(\nabla\varphi(\tilde{a}_2(t)))$.

In particular, any sequence $z_n \in \mathbb{Z} \times \mathbb{N}$ with $\lim_n |z_n| = \infty$ and satisfying the inequality $\arg(z_n) \leq \arg(\nabla\varphi(\tilde{a}_2(t)))$, for all $n \in \mathbb{N}$, converges to a point of the t-Martin boundary of $\mathbb{Z} \times \mathbb{N}$.
Remark finally that $a_i(t) \to a^*$ as $t \to t(\alpha^*)$ for any $i \in \{1,2\}$. From this it follows that $\tilde{a}_i(t) \to \tilde{a}^*$ as $t \to t(\alpha^*)$ for any $i \in \{1,2\}$ where $\tilde{a}^* = (\tilde{\alpha}^*, \tilde{\beta}^*)$ is the unique point on the boundary $\partial_+ D^t$ with $\tilde{\alpha}^* = \alpha^*$, and consequently,

$$\lim_{t\to t(a^*)} \nabla\varphi(\tilde{a}_1(t)) \;=\; \lim_{t\to t(a^*)} \nabla\varphi(\tilde{a}_2(t)) \;=\; \nabla\varphi(\tilde{a}^*).$$

Since clearly, $\nabla\varphi(\tilde{a}_1(t)) \neq \nabla\varphi(\tilde{a}_2(t))$ for $t(\alpha^*) < t \leq t(\alpha^* - \varepsilon_1)$, we conclude that at least one of the function $t \to \nabla\varphi(\tilde{a}_1(t))$ or $t \to \nabla\varphi(\tilde{a}_2(t))$ is not constant on the interval $[t(\alpha^*), t(\alpha^* - \varepsilon_1)]$ and hence, there are $t, t' \in \,]t(\alpha^*), t(\alpha^* - \varepsilon_1)]$ such that $t \neq t'$ and $\nabla\varphi(\tilde{a}_i(t)) \neq \nabla\varphi(\tilde{a}_i(t'))$ either for $i = 1$ or for $i = 2$. Suppose that this relation holds for $i = 2$ (the case when $i = 1$ is quite similar) and let

$$\arg(\nabla\varphi(\tilde{a}_i(t))) \;<\; \arg(\nabla\varphi(\tilde{a}_i(t'))).$$

Then in the t'-Martin compactification, any sequence of points $z_n \in \mathbb{Z} \times \mathbb{N}$ with $\lim_n |z_n| = \infty$ and

$$\arg(\nabla\varphi(\tilde{a}_i(t))) \;\leq\; \arg(z_n) \;\leq\; \arg(\nabla\varphi(\tilde{a}_i(t'))), \quad \forall n \in \mathbb{N},$$

converges to a point of the t'-Martin boundary, while in the t-Martin compactification such a sequence converges to a point of the t-Martin boundary if and only if there exists a limit $\lim_n z_n/|z_n|$. The following proposition is therefore proved.

Proposition 3.1. *Let the conditions (H0)-(H4) be satisfied. Suppose moreover that the minimum of the function $\max\{\varphi, \varphi_0\}$ is attained at a unique point a^* and the inequalities (3.1) and (3.2) hold. Then the t-Martin compactification of the transition kernel P is unstable.*

4   Proof of Proposition 2.1

We prove this proposition by using large deviation principle of the sample paths of scaled processes $Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon])$ with $\varepsilon \to 0$. Before proving this proposition we recall the definition of the sample path large deviation principle.
Throughout this section, for $t \in [0, +\infty[$, we denote by $[t]$ the integer part of t.

Definitions : *1) Let $D([0,T], \mathbb{R}^d)$ denote the set of all right continuous with left limits functions from $[0,T]$ to $\mathbb{R}^d$ endowed with Skorohod metric (see Billingsley [1]). Recall that a mapping $I_{[0,T]} : D([0,T], \mathbb{R}^d) \to [0, +\infty]$ is a good rate function on $D([0,T], \mathbb{R}^d)$ if for any $c \geq 0$ and any compact set $V \subset \mathbb{R}^d$, the set*

$$\{\varphi \in D([0,T], \mathbb{R}^d) : \; \phi(0) \in V \; \text{ and } \; I_{[0,T]}(\varphi) \leq c\}$$

is compact in $D([0,T], \mathbb{R}^d)$. According to this definition, a good rate function is lower semi-continuous.

2) For a Markov chain $(Z(t))$ on $E \subset \mathbb{R}^d$ the family of scaled processes $(Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon]),\ t \in [0,T])$, is said to satisfy sample path large deviation principle in $D([0,T],\mathbb{R}^d)$ with a rate function $I_{[0,T]}$ if for any $z \in \mathbb{R}^d$

$$\lim_{\delta \to 0} \liminf_{\varepsilon \to 0}\ \inf_{z' \in E:|\varepsilon z'-z|<\delta} \varepsilon \log \mathbb{P}_{z'}(Z^\varepsilon(\cdot) \in \mathcal{O}) \geq - \inf_{\phi \in \mathcal{O}:\phi(0)=z} I_{[0,T]}(\phi), \qquad (4.1)$$

for every open set $\mathcal{O} \subset D([0,T],\mathbb{R}^d)$, and

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0}\ \sup_{z' \in E:|\varepsilon z'-z|<\delta} \varepsilon \log \mathbb{P}_{z'}(Z^\varepsilon(\cdot) \in F) \leq - \inf_{\phi \in F:\phi(0)=z} I_{[0,T]}(\phi). \qquad (4.2)$$

for every closed set $F \subset D([0,T],\mathbb{R}^d)$.

We refer to sample path large deviation principle as SPLD principle. Inequalities (4.1) and (4.2) are referred as lower and upper SPLD bounds respectively.

Recall that the convex conjugate f^* of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$f^*(v) = \sup_{a \in \mathbb{R}^d}(a \cdot v - f(a)), \quad v \in \mathbb{R}^d.$$

The following proposition provides the SPLD principle for the scaled processes $Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon])$ for our random walk $(Z(n))$ on $\mathbb{Z} \times \mathbb{N}$.

Proposition 4.1. *Under the hypotheses $(H_0) - (H_4)$, for every $T > 0$, the family of scaled processes $(Z^\varepsilon(t),\ t \in [0,T])$ satisfies SPLD principle in $D([0,T],\mathbb{R}^d)$ with a good rate function*

$$I_{[0,T]}(\phi) = \begin{cases} \int_0^T L(\phi(t),\dot{\phi}(t))\,dt, & \text{if } \phi \text{ is absolutely continuous and} \\ & \phi(t) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ \text{ for all } t \in [0,T], \\ +\infty & \text{otherwise.} \end{cases}$$

where for any $z = (x,y)\mathbb{R}^{d-1} \times [0,+\infty[$ and $v \in \mathbb{R}^d$, the local rate function L is given by

$$L(z,v) = \begin{cases} (\log \varphi)^*(v) & \text{if } y > 0, \\ (\log \max\{\varphi,\varphi_0\})^*(v) & \text{if } y = 0. \end{cases}$$

This proposition is a consequence of the results obtained in [3, 5, 7, 8]. The results of Dupuis, Ellis and Weiss [3] prove that $I_{[0,T]}$ is a good rate function on $D([0,T],\mathbb{R}^d)$ and provide the SPLD upper bound. SPLD lower bound follows from the local estimates obtained in [7], the general SPLD lower bound of Dupuis and Ellis [5] and the integral representation of the corresponding rate function obtained in [8].

We are ready now to complete the proof of Proposition 2.1. The proof of the upper bound

$$\rho(P) \leq \inf_{a \in \mathbb{R}^d} \max\{\varphi(a),\varphi_0(a)\} \qquad (4.3)$$

is quite simple. Recall that $\rho(P)$ is equal to the infimum of all those $t > 0$ for which the inequality $Pf \leq tf$ has a non-zero solution $f > 0$, see Seneta [18]. Since for any $a \in \mathbb{R}^d$, this inequality is satisfied with $t = \max\{\varphi(a),\varphi_0(a)\}$ for an exponential function $f(z) = \exp(a \cdot z)$, one gets therefore $\rho(P) \leq \max\{\varphi(a),\varphi_0(a)\}$ for all $a \in \mathbb{R}^d$, and consequently, (4.3) holds. To prove the lower bound

$$\rho(P) \geq \inf_{a \in \mathbb{R}^d} \max\{\varphi(a),\varphi_0(a)\} \qquad (4.4)$$

we use the results of the paper [9]. Theorem 1 of [9] proves that for a zero constant function $\overline{0}(t) = 0$, $t \in [0, T]$,

$$\log \rho(P) = -I_{[0,T]}(\overline{0})/T$$

whenever the following conditions are satisfied :

(a_1) for every $T > 0$, the family of rescaled processes $(Z_\varepsilon(t), \ t \in [0, T])$ satisfies sample path large deviation principle in $D([0, T], \mathbb{R}^{d-1} \times [0, \infty[)$ with a good rate functions $I_{[0,T]}$;

(a_2) the rate function $I_{[0,T]}$ has an integral form : there is a local rate function $L : (\mathbb{R}^{d-1} \times [0, \infty[) \times \mathbb{R}^d \to \mathbb{R}_+$ such that

$$I_{[0,T]}(\phi) = \int_0^T L(\phi(t), \dot{\phi}(t)) \, dt$$

if the function $\phi : [0, 1] \to \mathbb{R}^{d-1} \times [0, \infty[$ is absolutely continuous, and $I_{[0,1]}(\phi) = +\infty$ otherwise.

(a_3) there are two convex functions l_1 and l_2 on $\mathbb{R}^d$ such that

– $0 \leq l_1(v) \leq L(x, v) \leq l_2(v)$ for all $x \in \mathbb{R}^{d-1} \times [0, \infty[$ and $v \in \mathbb{R}^d$,
– the function l_2 is finite in a neighborhood of zero
– and
$$\lim_{n \to \infty} \inf_{|v| \geq n} l_1(v)/|v| > 0.$$

In our setting, the conditions (a_1) and (a_2) are satisfied by Proposition 4.1 and the condition (a_3) is satisfied with $l_1(v) = (\log(\varphi, \varphi_0))^*(v)$ and $l_2(v) = (\log \varphi)^*(v)$:

– Clearly, $(\log(\varphi, \varphi_0))^*(v) \leq L(x, v) \leq (\log \varphi)^*(v)$ for all $x \in \mathbb{R}^{d-1} \times [0, \infty[$ and $v \in \mathbb{R}^d$.

– Under the hypotheses (H2) and (H3), there is $\delta > 0$ such that

$$\liminf_{|a| \to \infty} \frac{1}{|a|} \log \varphi(a) > \delta,$$

and consequently, for any $v \in \mathbb{R}^d$ with $|v| \leq \delta$ one has

$$(\log \varphi)^*(v) \leq \sup_{a \in \mathbb{R}^d} \sup_{v \in \mathbb{R}^d : |v| \leq \delta} \left(a \cdot v - \log \varphi(a) \right) = \sup_{a \in \mathbb{R}^d} \left(\delta |a| - \log \varphi(a) \right) < +\infty.$$

– For any $r > 0$,

$$(\log(\varphi, \varphi_0))^*(v) \geq \sup_{a \in \mathbb{R}^d : |a| \leq r} \left(a \cdot v - \log(\varphi, \varphi_0)(a) \right)$$
$$\geq \sup_{a \in \mathbb{R}^d : |a| \leq r} a \cdot v - \sup_{a \in \mathbb{R}^d : |a| \leq r} \log(\varphi, \varphi_0)(a)$$
$$\geq r|v| - \sup_{a \in \mathbb{R}^d : |a| \leq r} \log(\varphi, \varphi_0)(a).$$

Since by (H3), the function $\log(\varphi, \varphi_0)$ is finite everywhere on $\mathbb{R}^d$, from this it follows that

$$\lim_{n \to \infty} \inf_{|v| \geq n} \frac{1}{|v|} (\log(\varphi, \varphi_0))^*(v) \geq r > 0.$$

Using Theorem 1 of [9] and the explicit form of the local rate function L one gets

$$\log \rho(P) \ = \ -L(0,0) \ = -(\log \ \max\{\varphi, \varphi_0\})^*(0) \ = \ \log \inf_{a \in \mathbb{R}^d} \max\{\varphi(a), \varphi_0(a)\}.$$

Proposition 2.1 is therefore proved.

5   Proof of Proposition 2.2

For $t = 1$, this result was proved in [10] under the following conditions : in addition to the hypotheses (H0)-(H4), the positive measures μ and μ_0 were assumed to be probability measures and the means

$$m \doteq \sum_{z \in \mathbb{Z}^d} \mu(z) z \quad \text{and} \quad m_0 \ = \ \sum_{z \in \mathbb{Z}^d} \mu_0(z) z$$

were assumed to satisfy the following condition :

$$m/|m| + m_0/|m_0| \ \neq \ 0. \tag{5.1}$$

Remark that under the above assumptions, the set $\partial D^1 \cap \partial D_0^1$ contains the point zero and the set $D^1 \cap D_0^1$ has a non-empty interior. By Proposition 2.1 from this it follows that

$$\rho(P) \ = \ \inf_{a \in \mathbb{R}^d} \max\{\varphi(a), \varphi_0(a)\} \ < \ 1. \tag{5.2}$$

The above additional conditions can be replaced by a weaker one : for $t = 1$, with the same arguments as in [10] one can get Proposition 2.2 when μ is a probability measure on $\mathbb{Z}^d$ and μ_0 is a positive measure on $\mathbb{Z}^d$ satisfying the inequality (5.2) such that $\mu_0(\mathbb{Z}^d) \leq 1$. This result is now combined with the exponential change of the measure in order to prove Proposition 2.2 for $t > \rho(P)$. For any $t > \rho(P)$, there is a point $\tilde{a}_t \in \partial D^t \cap D_0^t$. We consider a twisted random walk $(\tilde{Z}(t))$ on $\mathbb{Z}^{d-1} \times \mathbb{N}$ with transition probabilities

$$\tilde{p}(z, z') \ = \ \begin{cases} \mu(z' - z) \exp(\tilde{a}_t \cdot (z' - z))/t & \text{if } z = (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N} \text{ with } y > 0, \\ \mu_0(z' - z) \exp(\tilde{a}_t \cdot (z' - z))/t & \text{if } z = (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N} \text{ with } y = 0. \end{cases}$$

For such a random walk $(\tilde{Z}(n))$, the jump generating functions are given by

$$\tilde{\varphi}(a) \ = \ \varphi(a + \tilde{a}_t)/t, \quad \text{and} \quad \tilde{\varphi}_0(a) \ = \ \varphi_0(a + \tilde{a}_t)/t.$$

Hence,
$$\tilde{D}^1 \doteq \{a \in \mathbb{R}^d : \ \tilde{\varphi}(a) \leq 1\} \ = \ \{a \in \mathbb{R}^d : \varphi(a + \tilde{a}_t) \leq t\} \ = \ -\tilde{a}_t + D^t,$$

and similarly,
$$\tilde{D}_0^1 \doteq \{a \in \mathbb{R}^d : \ \tilde{\varphi}_0(a) \leq 1\} \ = \ -\tilde{a}_t + D_0^t.$$

Moreover, with the same arguments one gets

$$\tilde{\Theta}^1 \doteq \{\alpha \in \mathbb{R}^{d-1} : \inf_{\beta \in \mathbb{R}} \max\{\tilde{\varphi}(a), \tilde{\varphi}_0(a)\} \leq 1\} \ = \ -\tilde{\alpha}_t + \Theta^t$$

where α_t denotes the vector of $d - 1$ first coordinates of $\tilde{a}_t$,

$$\hat{\tilde{D}}^1 \doteq (\tilde{\Theta}^1 \times \mathbb{R}) \cap \tilde{D}^1 \ = \ -\tilde{a}_t + \hat{D}^t \quad \text{and} \quad \tilde{\Gamma}_+^1 \doteq \hat{\tilde{D}}^1 \cap \partial_+ \tilde{D}^1 \ = \ -\tilde{a}_t + \Gamma_+^t.$$

For any $a \in \Gamma^t_+$ the normal cone $V_t(a)$ to the set $\hat{D}^t$ at the point a is therefore identical to the normal cone $\tilde{V}_1(a - \tilde{a}_t)$ to the set $\hat{\tilde{D}}^1$ at the point $a - \tilde{a}_t \in \tilde{\Gamma}^1_+$. Remark finally that for any $a \in \tilde{\Gamma}^1_+$ the functions $\tilde{h}_{a,1}$ defined by (2.8) with $t = 1$ and the functions $\tilde{\varphi}$ and $\tilde{\varphi}_0$ instead of φ and φ_0, satisfy the equality

$$\tilde{h}_{a,1}(z)(a) \,=\, h_{a+\tilde{a}_t,t}(z)\exp(-\tilde{a}_t \cdot z), \quad \forall z \in \mathbb{Z}^{d-1} \times \mathbb{N}.$$

Since clearly, $\tilde{G}_1(z, z') \,=\, G_t(z, z')\exp(\tilde{a} \cdot (z' - z))$, we conclude that

(i) for any unit vector $q \in \mathbb{R}^{d-1} \times [0, +\infty[$ there exists a unique point $\hat{a}_t(q) \in \Gamma^t_+$ such that $q \in V_t(\hat{a}_t(q))$,

(ii) for any $a \in \hat{D}^t \cap \partial_+ D^t$ and any sequence of points $z_n \in \mathbb{Z}^{d-1} \times \mathbb{N}$,

$$\begin{aligned}
\lim_{n\to\infty} K_t(z, z_n) = \;&\exp(\tilde{a} \cdot (z - z_0)) \lim_{n\to\infty} \tilde{G}_1(z, z_n)/\tilde{G}_1(z_0, z_n) \\
= \;&\exp(\tilde{a} \cdot (z - z_0)) \, \tilde{h}_{a-\tilde{a}_t,t}(z)/\tilde{h}_{a-\tilde{a}_t,t}(z_0) \,=\, h_{a,t}(z)/h_{a,t}(z_0),
\end{aligned}$$

whenever $\lim_{n\to\infty} |z_n| = \infty$ and $\lim_{n\to\infty} \mathrm{dist}(V_t(a), z_n/|z_n|) = 0$.

Proposition 2.2 is therefore proved.

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