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WEAK APPROXIMATION OF FRACTIONAL SDES: THE DONSKER SETTING

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Abstract

In this note, we take up the study of weak convergence for stochastic differential equations driven by a (Liouville) fractional Brownian motion B with Hurst parameter $H \in (1/3, 1/2)$, initiated in [3]. In the current paper, we approximate the d-dimensional fBm by the convolution of a rescaled random walk with Liouville's kernel. We then show that the corresponding differential equation converges in law to a fractional SDE driven by B.

1 Introduction

The current article can be seen as a companion paper to [3], to which we refer for a further introduction. Indeed, in the latter reference, the following equation on the interval $[0,1]$ was considered (the generalization to $[0,T]$ being a matter of trivial considerations):

$$dy_t = \sigma\left(y_t\right) dB_t + b\left(y_t\right) dt, \quad y_0 = a \in \mathbb{R}^n, \tag{1}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $b : \mathbb{R}^n \to \mathbb{R}^n$ are two bounded and smooth enough functions, and B stands for a d-dimensional fBm with Hurst parameter $H > 1/3$.

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Let us be more specific about the driving process for equation (1): we consider in the sequel the so-called d-dimensional Liouville fBm B, with Hurst parameter $H \in (1/3, 1/2)$. Namely, B can be written as $B = (B^1, \dots, B^d)$, where the B^i's are d independent centered Gaussian processes of the form

$$B_t^i = \int_0^t (t-r)^{H-\frac{1}{2}} dW_r^i, \tag{2}$$

for a d-dimensional Wiener process $W = (W^1, \dots, W^d)$. This process is very close to the usual fBm, in the sense that they only differ by a finite variation process (as pointed out in [1]), and we shall see that its simple expression (2) simplifies some of the computations throughout the paper. In any case, B falls into the scope of application of the rough paths theory, which means that equation (1) can be solved thanks to the semi-pathwise techniques contained in [9, 10, 13]. The natural question raised in [3] was then the following: is it possible to approximate equations like (1) in law by ordinary differential equations, thanks to a Wong-Zakai type approximation (see [12, 16, 17] for further references on the topic)?

Some positive answer to this question had already been given in [8], where some Gaussian sequences approximations were considered in a general context. In [3], we focused on a natural and easily implementable (non Gaussian) scheme for B, based on Kac-Stroock's approximation to white noise (see [11, 15]). However, another very natural way to approximate B relies on Donsker's type scheme (see [14] for the case $H > 1/2$ and [4] for the Brownian case), involving a rescaled random walk. We have thus decided to investigate weak approximations to (1) based on this process.

More precisely, as an approximating sequence of B, we shall choose $(X^\varepsilon)_{\varepsilon>0}$, where $X^{\varepsilon,i}$ is defined as follows for $i = 1, \dots, d$: consider a family of independent random variables $\{\eta_k^i;\ k \geq 1,\ 1 \leq i \leq d\}$, satisfying the

Hypothesis 1.1. *The random variables $\{\eta_k^i;\ k \geq 1,\ 1 \leq i \leq d\}$ are independent and share the same law as another random variable η. Furthermore, η is assumed to satisfy $E(\eta) = 0, E(\eta^2) = 1$ and is almost surely bounded by a constant k_η.*

We then define $X^{\varepsilon,i}$ in the following way:

$$X^{i,\varepsilon}(t) = \int_0^t (t+\varepsilon^2-r)^{H-\frac{1}{2}} \theta^{i,\varepsilon}(r) dr, \tag{3}$$

where

$$\theta^{i,\varepsilon}(r) := \frac{1}{\varepsilon} \sum_{k=1}^{+\infty} \eta_k^i I_{[k-1,k)}\left(\frac{r}{\varepsilon^2}\right). \tag{4}$$

Notice that X^ε is really a process given by the convolution of the rescaled random walk θ^ε with Liouville's kernel.

Let us then consider the process y^ε solution to equation (1) driven by X^ε, namely:

$$dy_t^\varepsilon = \sigma\left(y_t^\varepsilon\right) dX_t^\varepsilon + b\left(y_t^\varepsilon\right) dt, \quad y_0^\varepsilon = a \in \mathbb{R}^n, \quad t \in [0,T]. \tag{5}$$

Our main result is as follows:

Theorem 1.2. *Let $(y^\varepsilon)_{\varepsilon>0}$ be the family of processes defined by (5), and let $1/3 < \gamma < H$, where H is the Hurst parameter of B. Then, as $\varepsilon \to 0$, y^ε converges in law to the process y obtained as the solution to (1), where the convergence takes place in the Hölder space $C^\gamma([0,1]; \mathbb{R}^n)$.*

Let us make a few comments about this theorem:

(i) We have presented our results for the Liouville fBm B because of some simplifications, apparent in [3], in the manipulations of some stochastic integrals. However, as mentioned in [1], the usual fractional Brownian motion $\hat{B}$ can be decomposed as $\hat{B} = B + V$, where V is a finite variation process and B is the Liouville fBm. By writing the Lévy area of $\hat{B}$ according to this decomposition, it is certainly possible to extend our results to fBm, which is in a sense a more natural process for its invariance properties. We did not explore this possibility for sake of conciseness.

(ii) We also have chosen the range $(1/3, 1/2)$ for the coefficient H. Indeed, the case $H > 1/2$ can be deduced easily from [5] since in this case, the solution y to (1) is a continuous function of B. The Brownian case $H = 1/2$ is proved easily following the methods in [6]. We could also have dealt with a coefficient H lying in $(1/4, 1/3]$ according to [8], but this case is much harder for two reasons: (1) The approximation of third order integrals is also required in this case. (2) The assumption $H > 1/3$ is needed for the convergence of certain terms in [3, Proof of Proposition 5.1] (see our term $B_{2,1,1}$ there).

(iii) Notice that if $t \in [N\varepsilon^2, (N+1)\varepsilon^2)$ one has

$$X^{i,\varepsilon}(t) = \frac{1}{(H+1/2)\varepsilon}\left[\sum_{k=1}^{N}\eta_k^i\left[(t-(k-2)\varepsilon^2)^{H+1/2} - (t-(k-1)\varepsilon^2)^{H+1/2}\right] + \eta_{N+1}^i\left[(t-(N-1)\varepsilon^2)^{H+1/2} - \varepsilon^{2H+1}\right]\right].$$

Another possibility in order to approximate X^i would have been to define a process $\hat{X}^{i,\varepsilon}$ at any point of the form $N\varepsilon^2$ by

$$\hat{X}^{i,\varepsilon}(N\varepsilon^2) = \frac{1}{(H+1/2)\varepsilon^{-2H}}\sum_{k=1}^{N}\eta_k^i\left[(N-(k-2))^{H+1/2} - (N-(k-1))^{H+1/2}\right],$$

and then compute $\hat{X}^{i,\varepsilon}$ at any other point by linear interpolation. This new approximation might be closer to the spirit of Donsker type results, since the process $\hat{X}^{i,\varepsilon}$ is piecewise linear. In any case, it should be easy (though technical) to show that $\hat{X}^{i,\varepsilon} - X^{i,\varepsilon}$ vanishes when $\varepsilon \to 0$, as L^2-random variables taking values in an appropriate rough path space (including the Lévy area of both processes). Thus Theorem 1.2 certainly holds true when $X^{i,\varepsilon}$ is replaced by $\hat{X}^{i,\varepsilon}$. Here again, we did not explore this possibility for sake of conciseness.

Here is how our paper is structured: as the reader shall see, many of the techniques introduced in [3] are also useful in our context. In the end, as explained at Section 2, most of the technical differences between the two articles arise in the way to evaluate the moments of quantities like $\int_0^1 f(r)\theta^{i,\varepsilon}(r)\,dr$ for a given Hölder function f, and to compare them with the moments of Gaussian random variable. This is thus where we shall concentrate our efforts, and this essential point will be handled at Section 3.

2 Reduction of the problem

We shall recall here briefly some preliminary steps contained in [3], which allow to reduce our problem to the evaluation of the moments of a specific type of Wiener integrals.

First of all, we need to recall the definition of some Hölder spaces, in which our convergences take place. We call for instance $\mathscr{C}_j([0,1];\mathbb{R}^d)$ the space of continuous functions from $[0,1]^j$ to $\mathbb{R}^d$, which will mainly be considered for $j=1$ or 2 variables. The Hölder norms on those spaces are then defined in the following way: for $f\in\mathscr{C}_2([0,1];\mathbb{R}^d)$ let

$$\|f\|_\mu = \sup_{s,t\in[0,T]} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathscr{C}_2^\mu([0,1];\mathbb{R}^d) = \left\{ f\in\mathscr{C}_2([0,1];\mathbb{R}^d);\ \|f\|_\mu<\infty \right\}.$$

The usual Hölder spaces $\mathscr{C}_1^\mu([0,1];\mathbb{R}^d)$ are then determined by setting $\|g\|_\mu = \|\delta g\|_\mu$ for a continuous function $g\in\mathscr{C}_1([0,1];\mathbb{R}^d)$, where $\delta g\in\mathscr{C}_2([0,1];\mathbb{R}^d)$ is defined by $\delta g_{st} = g_t - g_s$. We then say that $g\in\mathscr{C}_1^\mu([0,1];\mathbb{R}^d)$ iff $\|g\|_\mu$ is finite. Note that $\|\cdot\|_\mu$ is only a semi-norm on $\mathscr{C}_1([0,1];\mathbb{R}^d)$, but we will work in general on spaces of the type

$$\mathscr{C}_{1,a}^\mu([0,1];\mathbb{R}^d) = \left\{ g:[0,T]\to V;\ g_0=a,\ \|g\|_\mu<\infty \right\}, \tag{6}$$

for a given $a\in V$, on which $\|g\|_\mu$ is a norm.

The second crucial point one has to recall is the natural definition of a Lévy area for Liouville's fBm. To this purpose, consider $\mathscr{E}$ the set of step-functions on $[0,T]$ with values in $\mathbb{R}^d$. Let $\mathscr{H}$ be the Hilbert space $\mathscr{H}$ defined as the closure of $\mathscr{E}$ with respect to the scalar product induced by

$$\left\langle (\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]}),(\mathbf{1}_{[0,s_1]},\dots,\mathbf{1}_{[0,s_d]}) \right\rangle_{\mathscr{H}} = \sum_{i=1}^d R(t_i,s_i), \quad s_i,t_i\in[0,T],\ i=1,\dots,d,$$

where $R(t,s) := E[B_t^i B_s^i]$. Then a natural representation of the inner product in $\mathscr{H}$ is given via the operator $\mathscr{K}$, defined from $\mathscr{E}$ to $L^2([0,T])$, by:

$$\mathscr{K}\varphi(t) = (T-t)^{H-\frac{1}{2}}\varphi(t) - \left(\frac{1}{2}-H\right)\int_t^T [\varphi(r)-\varphi(t)](r-t)^{H-\frac{3}{2}}\,dr,$$

and it can be checked that $\mathscr{K}$ can be extended as an isometry between $\mathscr{H}$ and the Hilbert space $L^2([0,T];\mathbb{R}^d)$. Thus the inner product in $\mathscr{H}$ can be defined as:

$$\langle\varphi,\psi\rangle_{\mathscr{H}} \triangleq \langle\mathscr{K}\varphi,\mathscr{K}\psi\rangle_{L^2([0,T];\mathbb{R}^d)}.$$

The mapping $(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]})\mapsto\sum_{i=1}^d B_{t_i}^i$ can also be extended into an isometry between $\mathscr{H}$ and the first Gaussian chaos $H_1(B)$ associated with $B=(B^1,\dots,B^d)$. We denote this isometry by $\varphi\mapsto B(\varphi)$, and $B(\varphi)$ is called the Wiener-Itô integral of φ. It is shown in [7, page 284] that $\mathscr{C}_1^\gamma(\mathbb{R}^d)\subset\mathscr{H}$ whenever $\gamma>1/2-H$, which allows to define $B(\varphi)$ for such kind of functions.

Proposition 2.1. *Let B be a d-dimensional Liouville fBm, and suppose that its Hurst parameter satisfies $H\in(1/3,1/2)$. Then*

(1) *B is almost surely a γ-Hölder path for any $1/3<\gamma<H$.*

(2) *A Lévy area based on B can be defined by setting*

$$\mathbf{B}_{st}^2 = \int_s^t dB_u\otimes\int_s^u dB_v, \quad \textit{i. e.} \quad \mathbf{B}_{st}^2(i,j) = \int_s^t dB_u^i\int_s^u dB_v^j, \quad i,j\in\{1,\dots,d\},$$

for $0 \le s < t \le T$. Here, the stochastic integrals are defined as Wiener-Itô integrals when $i \neq j$, while, when $i = j$, they are simply given by

$$\int_s^t dB_u^i \int_s^u dB_v^i = \frac{1}{2}\left(B_t^i - B_s^i\right)^2.$$

(3) *The process $\mathbf{B^2}$ is almost surely an element of $\mathscr{C}_2^{2\gamma}([0,1];\mathbb{R}^{d\times d})$, and satisfies the algebraic relation*

$$\mathbf{B}_{st}^{\mathbf{2}} - \mathbf{B}_{su}^{\mathbf{2}} - \mathbf{B}_{ut}^{\mathbf{2}} = \left(B_u - B_s\right) \otimes \left(B_t - B_u\right),$$

for all $0 \le s \le u \le t \le 1$.

These algebraic and analytic properties of the fBm path allow to invoke the rough path machinery (see [9, 10, 13]) in order to solve equation (1):

Theorem 2.2. *Let B be a Liouville fBm with Hurst parameter $1/3 < H < 1/2$, and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be a C^2 function, which is bounded together with its derivatives. Then*

(1) *Equation (1) admits a unique solution $y \in \mathscr{C}_1^\gamma(\mathbb{R}^n)$ for any $1/3 < \gamma < H$, with the additional structure of weakly controlled process introduced in [10].*

(2) *The mapping $(a, B, \mathbf{B^2}) \mapsto y$ is continuous from $\mathbb{R}^n \times \mathscr{C}_1^\gamma(\mathbb{R}^d) \times \mathscr{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ to $\mathscr{C}_1^\gamma(\mathbb{R}^n)$.*

One of the nice aspects of rough paths theory is precisely the second point in Theorem 2.2, which allows to reduce immediately our weak convergence result for equation (1), namely Theorem 1.2, to the following result on the approximation of $(B, \mathbf{B^2})$:

Theorem 2.3. *Recall that the random variables η_k^i satisfy Hypothesis 1.1, and let X^ε be defined by (3). For any $\varepsilon > 0$, let $\mathbf{X}^{\mathbf{2},\varepsilon} = (\mathbf{X}_{st}^{\mathbf{2},\varepsilon}(i,j))_{s,t\ge 0;\, i,j=1,\dots,d}$ be the natural Lévy's area associated to X^ε, given by*

$$\mathbf{X}_{st}^{\mathbf{2},\varepsilon}(i,j) = \int_s^t (X_u^{j,\varepsilon} - X_s^{j,\varepsilon})\, dX_u^{i,\varepsilon}, \tag{7}$$

where the integral is understood in the usual Lebesgue-Stieltjes sense. Then, as $\varepsilon \to 0$,

$$(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon}) \xrightarrow{\text{Law}} (B, \mathbf{B^2}), \tag{8}$$

where $\mathbf{B^2}$ denotes the Lévy area defined in Proposition 2.1, and where the convergence in law holds in the spaces $\mathscr{C}_1^\mu(\mathbb{R}^d) \times \mathscr{C}_2^{2\mu}(\mathbb{R}^{d\times d})$, for any $\mu < H$.

The remainder of our work is thus devoted to the proof of Theorem 2.3.

As usual in the context of weak convergence of stochastic processes, we divide the proof into weak convergence for finite-dimensional distributions and a tightness type result. Furthermore, the tightness result in our case is easily deduced from the analogous result in [3]:

Proposition 2.4. *The sequence $(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon})_{\varepsilon>0}$ defined at Theorem 2.3 is tight in $\mathscr{C}_1^\mu(\mathbb{R}^d) \times \mathscr{C}_2^{2\mu}(\mathbb{R}^{d\times d})$.*

Proof. The proof follows exactly the steps of [3, Proposition 4.3], the only difference being that our Lemma 3.1 has to be applied here in order to get the equivalent of inequality (28) in [3]. Details are left to the reader.

□

With these preliminaries in hand, we can now turn to the finite dimensional distribution (f.d.d. in the sequel) convergence, which can be stated as:

Proposition 2.5. *Under the assumption 1.1, let* $(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon})$ *be the approximation process defined by (3) and (7). Then*

$$\text{f.d.d.} - \lim_{\varepsilon\to 0}(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon}) = (B, \mathbf{B}^{\mathbf{2}}), \tag{9}$$

where f.d.d. $-\lim$ *stands for the convergence in law of the finite dimensional distributions. Otherwise stated, for any* $k \geq 1$ *and any family* $\{s_i, t_i;\ i \leq k, 0 \leq s_i < t_i \leq T\}$*, we have*

$$\mathscr{L} - \lim_{\varepsilon\to 0}(X^\varepsilon_{t_1}, \mathbf{X}^{\mathbf{2},\varepsilon}_{s_1t_1}, \dots, X^\varepsilon_{t_k}, \mathbf{X}^{\mathbf{2},\varepsilon}_{s_kt_k}) = (B_{t_1}, \mathbf{B}^{\mathbf{2}}_{s_1t_1}, \dots, B_{t_k}, \mathbf{B}^{\mathbf{2}}_{s_kt_k}). \tag{10}$$

Proof. The structure of the proof follows again closely the steps of [3, Proposition 5.1], except that other kind of estimates will be needed in order to handle the Donsker case.

To be more specific, it should be observed that the first series of simplifications in the proof of [3, Proposition 5.1] can be repeated here. They allow to pass from a convergence of double iterated integrals to the convergence of some Wiener type integrals with respect to X^ε. Namely, for $i = 1, 2$ and $0 \leq u < t \leq 1$, set

$$Y^i(u,t) = \int_u^t (B^i_v - B^i_u)(v-u)^{H-\frac{3}{2}}\, dv,$$

and for $0 \leq u < t \leq 1$ and $(u_1, \dots, u_6)$ in a neighborhood of 0 in $\mathbb{R}^6$, set also

$$\begin{aligned} Z_u &= u_1 + u_2 B^2_u + u_3 Y^2(u,t) + u_4 \int_u^t (v-u)^{H-\frac{1}{2}} dW^2_v \\ &\quad + u_5 \int_u^t dw \int_u^w (w-v)^{H-\frac{3}{2}} \left((w-u)^{H-\frac{1}{2}} - (v-u)^{\frac{1}{2}}\right) dW^2_v \\ &\quad + u_6 \int_u^t dw \int_0^u (w-v)^{H-\frac{3}{2}} (w-u)^{H-\frac{1}{2}}\, dW^2_v. \end{aligned}$$

Consider the analogous processes $Y^{i,\varepsilon}, Z^\varepsilon$ defined by the same formulae, except that they are based on the approximations $\theta^{i,\varepsilon}$ of white noise. We still need to recall a little more notation from [3]: for $f \in L^2([0,1])$ and $t \in [0,1]$, we set

$$\Phi_\varepsilon(f) = E\left(e^{i\int_0^t f(u)\theta^{\varepsilon,1}(u)du}\right), \qquad \phi^\varepsilon_f = \int_0^1 \int_0^1 f^2(x) f^2(y) I_{\{|x-y|<\varepsilon^2\}} dx dy, \tag{11}$$

and

$$\Phi(f) = E\left(e^{i\int_0^t f(u)\, dW^1_u}\right) = e^{\frac{1}{2}\int_0^t f^2(u)\, du}.$$

Then it is shown in [3, Proposition 5.1] that one is reduced to prove that $\lim_{\varepsilon\to 0} \nu^a_\varepsilon = 0$, where ν^a_ε is given by

$$\nu^a_\varepsilon = E\left(\Phi_\varepsilon(Z^\varepsilon) e^{iw\int_0^t \theta^{\varepsilon,2}(u)du}\right) - E\left(\Phi(Z^\varepsilon) e^{iw\int_0^t \theta^{\varepsilon,2}(u)du}\right),$$

for an arbitrary real parameter w in a neighborhood of 0. Furthermore, bounding $e^{iw\int_0^t \theta^{\varepsilon,2}(u)du}$ trivially by 1 and conditioning, it is easily shown that ν^a_ε is controlled by the difference $E[|\Phi_\varepsilon(Z^\varepsilon)) -$

$\Phi(Z^\varepsilon)|]$, for which Lemma 3.3 provides the bound

$$
\begin{aligned}
&|\Phi_\varepsilon(Z^\varepsilon)) - \Phi(Z^\varepsilon)| \\
&\leq E\left[4\sqrt{\frac{1}{5}}w^3(\phi^\varepsilon_{Z^\varepsilon})^{\frac{1}{2}}\|Z^\varepsilon\|_{L^2}k_\eta^3\exp(4w^2k_\eta^2\|Z^\varepsilon\|^2_{L^2}) + w^2\varepsilon^{2\alpha}\|Z^\varepsilon\|_\alpha\|Z^\varepsilon\|_{L^2}\exp(w^2\|Z^\varepsilon\|^2_{L^2})\right. \\
&\quad \left. + \frac{8}{\sqrt{5}}w^4(\phi^\varepsilon_{Z^\varepsilon})^{\frac{1}{2}}\|Z^\varepsilon\|^2_{L^2}{k_\eta}^2\exp(4w^2k_\eta^2\|Z^\varepsilon\|^2_{L^2}) + \frac{1}{2}w^4\phi^\varepsilon_{Z^\varepsilon}\exp(w^2\|Z^\varepsilon\|^2_{L^2})\right],
\end{aligned}
$$

for any $\alpha \in (0,1)$. In order to reach our aim, it is thus sufficient to check the following inequalities:

$$
\sup_\varepsilon E\left(\|Z^\varepsilon\|^2_\alpha\right) \leq M, \quad \lim_{\varepsilon\to 0} E\left((\phi^\varepsilon_{Z^\varepsilon})^2\right) = 0
$$

and for $w < w_0$, where w_0 is a small enough constant,

$$
\sup_\varepsilon E\left(\mathrm{e}^{w^2\|Z^\varepsilon\|^2_{L^2}}\right) \leq M.
$$

However, these relations can be deduced, as (39), (40) and (41) in [3], from Lemma 3.1 (it should be noticed however that a one-parameter version of [2, Lemma 5.1] is needed for the adaptation of the latter result to our Donsker setting). The proof is thus finished once the lemmas below are proven.

□

3 Moments estimates in the Donsker setting

In order to deal with our technical estimates, let us first introduce a new notation: set $\rho_1 = (1-5^{1/2})/2$ and $\rho_2 = (1+5^{1/2})/2$. Then the moments of any integral of a deterministic kernel f with respect to $\theta^{i,\varepsilon}$ can be bounded as follows:

Lemma 3.1. *Let $m \in \mathbb{N}$, $f \in L^2([0,1])$, $i \in \{1,2\}$ and $\varepsilon > 0$. Recall that the random variable η is assumed to be almost surely bounded by a constant k_η. Then we have*

$$
\begin{aligned}
&\left|E\left[\left(\int_0^1 f(r)\theta^{i,\varepsilon}(r)dr\right)^{2m}\right]\right| \\
&\leq \frac{(2m)!}{2^m\, m!}\|f\|^{2m}_{L^2} + \frac{(2m)!}{5^{1/2}(m-2)!}k_\eta^{2m}\left(\rho_2^{2m-1} - \rho_1^{2m-1}\right)(\phi^\varepsilon_f)^{\frac{1}{2}}\|f\|^{2m-2}_{L^2},
\end{aligned}
\tag{12}
$$

and

$$
\left|E\left[\left(\int_0^1 f(r)\theta^{i,\varepsilon}(r)dr\right)^{2m+1}\right]\right| \leq \frac{(2m+1)!}{5^{1/2}(m-1)!}k_\eta^{2m+1}\left(\rho_2^{2m} - \rho_1^{2m}\right)(\phi^\varepsilon_f)^{\frac{1}{2}}\|f\|^{2m-1}_{L^2}, \tag{13}
$$

where ϕ^ε_f is the quantity defined at (11).

Proof. We focus first on inequality (12) and divide this proof into several steps.

Step 1: Identification of some key iterated integrals. Notice that

$$\left|E\left[\left(\int_0^1 f(r)\theta^{i,\varepsilon}(r)dr\right)^{2m}\right]\right| \leq \int_{[0,1]^{2m}} |f(r_1)|\cdots|f(r_{2m})||E(\theta^{i,\varepsilon}(r_1)\cdots\theta^{i,\varepsilon}(r_{2m}))|dr_1\cdots dr_{2m}.$$

Transforming the symmetric integral on $[0,1]^{2m}$ into an integral on the simplex, and using expression (4) for θ^ε, we can write the latter expression as:

$$\frac{(2m)!}{\varepsilon^{2m}} \sum_{\substack{k_1,\dots,k_{2m}=1 \\ k_1\geq\cdots\geq k_{2m}}}^{n(\varepsilon)} \int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\cdots\int_{(k_{2m}-1)\varepsilon^2}^{k_{2m}\varepsilon^2} |f(r_1)|\cdots|f(r_{2m})| \times |E(\eta^i_{k_1}\cdots\eta^i_{k_{2m}})|I_{\{r_1\geq r_2\geq\cdots\geq r_{2m}\}}dr_1\cdots dr_{2m}, \quad (14)$$

where $n(\varepsilon)=[\frac{1}{\varepsilon^2}]+1$ and where we understand that $f(x)=0$ whenever $x>1$.

Let us study now the quantities $E(\eta^i_{k_1}\cdots\eta^i_{k_l})$. If there exists l such that $k_l\neq k_j$ for all $j\neq l$ then $E(\eta^i_{k_1}\cdots\eta^i_{k_l})=0$. On the other hand, when $k_{2l-1}=k_{2l}>k_{2l+1}$ for any l, we clearly have $E(\eta^i_{k_1}\cdots\eta^i_{k_l})=1$. Finally, in the general case, for all $l\in\mathbb{N}$, $|E(\eta^i_{k_1}\cdots\eta^i_{k_l})|\leq k_\eta^l$. Separating the cases in this way for $|E(\eta^i_{k_1}\cdots\eta^i_{k_{2m}})|$, we end up with a decomposition of the form $E[(\int_0^1 f(r)\theta^{i,\varepsilon}(r)dr)^{2m}]=T_m^1+T_m^2$, where

$$T_m^1=\frac{(2m)!}{\varepsilon^{2m}} \sum_{\substack{k_1,\dots,k_m=1 \\ k_1>\cdots>k_m}}^{n(\varepsilon)} \int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\cdots\int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2}\int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2} |f(r_1)|\cdots|f(r_{2m})|\times I_{\{r_1\geq r_2\geq\cdots\geq r_{2m}\}}dr_1\cdots dr_{2m},$$

and where the term T_m^2 is defined by:

$$T_m^2=\frac{(2m)!\,k_\eta^{2m}}{\varepsilon^{2m}} \sum_{\substack{n_1,\dots,n_s\geq 2; s\in\{1,\dots,m-1\} \\ n_1+\cdots+n_s=2m}} U_{n_1,\dots,n_s}, \quad (15)$$

with

$$U_{n_1,\dots,n_s}=\sum_{\substack{k_1,\dots,k_s=1 \\ k_1>\cdots>k_s}}^{n(\varepsilon)} \int_{D_{k_1\cdots k_s}} |f(r_1)|\cdots|f(r_{2m})|I_{\{r_1\geq r_2\geq\cdots\geq r_{2m}\}}dr_1\cdots dr_{2m}, \quad (16)$$

and where we have set $D_{k_1\cdots k_s}=\prod_{j=1}^s[(k_j-1)\varepsilon^2,k_j\varepsilon^2]^{n_j}$.

Let us observe at this point that we have split our sum into T_m^1 and T_m^2 because T_m^1 represents the dominant contribution to our moment estimate. This is simply due to the fact that T_m^1 is obtained by assuming some pairwise equalities among the random variables η^i_k, while T_m^2 is based

on a higher number of constraints. In any case, both expressions will be analyzed through the introduction of some iterated integrals of the form

$$K_\nu(k;\nu,w)=\frac{1}{\varepsilon^\nu}\int_{[(k-1)\varepsilon^2,k\varepsilon^2]^\nu}\prod_{j=1}^{\nu}|f(r_j)|\,I_{\{w\geq r_1>\cdots>r_\nu\geq\nu\}}dr_1\cdots dr_\nu,$$

defined for $\nu,k\geq 1$ and $0\leq\nu<w\leq 1$.

Step 2: Analysis of the integrals K_ν. Those iterated integrals are treated in a slightly different way according to the parity of ν. Indeed, for $\nu=2n$, thanks to the elementary inequality $2ab\leq a^2+b^2$, we obtain a bound of the form:

$$\begin{aligned}
&\sum_{k=1}^{n(\varepsilon)}K_{2n}(k;\nu,w) \qquad (17)\\
&\leq\sum_{k=1}^{n(\varepsilon)}\frac{1}{\varepsilon^{2n}}\int_{[(k-1)\varepsilon^2,k\varepsilon^2]^{2n}}\prod_{i=1}^{n}\left(\frac{f^2(x_{2i-1})+f^2(x_{2i})}{2}\right)I_{\{w\geq x_1\geq\cdots\geq x_{2n}\geq\nu\}}dx_1\cdots dx_{2n}\\
&\leq\sum_{k=1}^{n(\varepsilon)}\frac{1}{\varepsilon^{2n}}\int_{[(k-1)\varepsilon^2,k\varepsilon^2]^{2n}}f^2(x_1)\cdots f^2(x_n)I_{\{w\geq x_1\geq\cdots\geq x_n\geq\nu\}}dx_1\cdots dx_{2n}\\
&=\sum_{k=1}^{n(\varepsilon)}\int_{[(k-1)\varepsilon^2,k\varepsilon^2]^{n}}f^2(x_1)\cdots f^2(x_n)I_{\{w\geq x_1\geq\cdots\geq x_n\geq\nu\}}dx_1\cdots dx_{n}\\
&\leq\int_{[0,1]^n}f^2(x_1)\cdots f^2(x_n)I_{\{x_1-x_n<\varepsilon^2\}}I_{\{w\geq x_1\geq\cdots\geq x_n\geq\nu\}}dx_1\cdots dx_n.
\end{aligned}$$

The case $\nu=2n+1$ can be treated along the same lines, except for the fact that one has to cope with some expressions of the form

$$\begin{aligned}
\sum_{k=1}^{n(\varepsilon)}K_3(k;\nu,w) &\leq \sum_{k=1}^{n(\varepsilon)}\frac{1}{\varepsilon}\int_{[(k-1)\varepsilon^2,k\varepsilon^2]^2}|f(x_1)|f^2(x_2)I_{\{w\geq x_1\geq x_2\geq\nu\}}dx_1dx_2\\
&\leq \frac{1}{\varepsilon}\int_{[0,1]^2}|f(x_1)|f^2(x_2)I_{\{x_1-x_2<\varepsilon^2\}}I_{\{w\geq x_1\geq x_2\geq\nu\}}dx_1dx_2. \qquad (18)
\end{aligned}$$

Combining (18) and (17) we can state the following general formula: let $\nu\geq 1$, and define a couple $(\nu^*,\hat{\nu})$ as: (i) $\nu^*=\nu/2$, $\hat{\nu}=0$ if ν is even, (ii) $\nu^*=(\nu+1)/2$, $\hat{\nu}=1$ if ν is odd. With this notation in hand, we have:

$$\begin{aligned}
\sum_{k=1}^{n(\varepsilon)}K_\nu(k;\nu,w)\leq\frac{1}{\varepsilon^{\hat{\nu}}}\int_{[0,1]^{\nu^*}}&|f(x_1)|^{2-\hat{\nu}}f^2(x_2)\cdots f^2(x_{\nu^*})I_{\{x_1-x_{\nu^*}<\varepsilon^2\}}\\
&\times I_{\{w\geq x_1\geq\cdots\geq x_{\nu^*}\geq\nu\}}dx_1\cdots dx_{\nu^*}. \qquad (19)
\end{aligned}$$

Step 3: Bound on T_m^1. It is readily checked that T_m^1 can be decomposed into blocks of the form

$K_2(k;w,v)$, for which one can apply (19). This yields

$$T_m^1 \le \frac{(2m)!}{2^m} \sum_{k_1,\dots,k_m=1}^{n(\varepsilon)} \int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2} \cdots \int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2} f^2(r_1)\cdots f^2(r_m) I_{\{r_1\ge r_2\ge\cdots\ge r_m\}} dr_1\cdots dr_m$$
$$\le \frac{(2m)!}{2^m m!}\|f\|_{L^2}^{2m}.$$

Step 4: Bound on $U_{n_1,\dots,n_s}$. Recall that $U_{n_1,\dots,n_s}$ is defined by (16). We introduce now a recursion procedure in order to control this term. Namely, integrating with respect to the last n_s variables, one obtains that

$$\frac{1}{\varepsilon^{2m}} U_{n_1,\dots,n_s} = \frac{1}{\varepsilon^{2m-n_s}} \sum_{\substack{k_1,\dots,k_{s-1}=1\\ k_1>\cdots>k_{s-1}}}^{n(\varepsilon)} \int_{D_{k_1\cdots k_{s-1}}} \prod_{l=1}^{2m-n_s} |f(r_l)|\, I_{\{r_1\ge r_2\ge\cdots\ge r_{2m-n_s}\}}$$
$$\times K_{n_s}(k_s;0;r_{2m-n_s})\, dr_1\cdots dr_{2m-n_s}$$

Plugging our bound (19) on K_{n_s} into this expression, we get

$$\frac{1}{\varepsilon^{2m}} U_{n_1,\dots,n_s} \le \frac{1}{\varepsilon^{2m-n_s+\hat{n}_s}} \sum_{\substack{k_1,\dots,k_{s-1}=1\\ k_1>\cdots>k_{s-1}}}^{n(\varepsilon)} \int_{D_{k_1\cdots k_{s-1}}} \int_{[0,1]^{n_s^*}} \prod_{l=1}^{2m-n_s} |f(r_l)|\, |f(y_1)|^{2-\hat{n}_s}$$
$$\times \prod_{j=2}^{n_s^*} f^2(y_j) I_{\{y_1-y_{n_s^*}<\varepsilon^2\}} I_{\{r_1\ge r_2\ge\cdots\ge r_{2m-n_s}\ge y_1\ge\cdots\ge y_{n_s^*}\}} dr_1\cdots dr_{2m-n_s} dy_1\cdots dy_{n_s^*}.$$

We can now proceed, and integrate with respect to the variables r_l for $2m-n_s-n_{s-1}<l\le 2m-n_s$. In the end, since $\sum n_s = 2m$, the remaining singularity in ε is of the form $\prod \varepsilon^{-\hat{n}_s}$. However, each of the singularity $\varepsilon^{-\hat{n}_s}$ comes with an integral that compensates the singularity $\varepsilon^{-\hat{n}_s}$ (recall that $\hat{n}_s \le 1$). Hence, iterating the integrations with respect to the variables r, we end up with a bound of the form

$$\frac{1}{\varepsilon^{2m}} U_{n_1,\dots,n_s} \le \frac{1}{(m-2)!}\|f\|_{L^2}^{2(m-2)} \phi_f^\varepsilon \le \frac{1}{(m-2)!}\|f\|_{L^2}^{2m-2} (\phi_f^\varepsilon)^{\frac{1}{2}}. \tag{20}$$

Notice that when one of the terms $n_1, n_2, \dots, n_s$ of the decomposition of $2m$ is even the last bound is easily obtained. On the other hand, we will illustrate with an example how the bound can be obtained when all the terms are odd. Let us consider the case $m = n_1 = n_2 = 3$. Following our procedure we obtain that

$$\begin{aligned}
\frac{1}{\varepsilon^6} U_{3,3} &\le \frac{1}{\varepsilon^2} \int_{[0,1]^4} f^2(y_2)\cdot|f(y_1)|\cdot f^2(y_4)\cdot|f(y_3)| \\
&\quad \times I_{\{y_1-y_2<\varepsilon^2\}} I_{\{y_3-y_4<\varepsilon^2\}} I_{\{y_3\ge y_4\ge y_1\ge y_2\}} dy_1\cdots dy_4 \\
&\le \frac{1}{\varepsilon^2} \int_{[0,1]^4} f^2(y_2) f^2(y_4) \left(\frac{f^2(y_1)+f^2(y_3)}{2}\right) \\
&\quad \times I_{\{y_1-y_2<\varepsilon^2\}} I_{\{y_3-y_4<\varepsilon^2\}} I_{\{y_3\ge y_4\ge y_1\ge y_2\}} dy_1\cdots dy_4 \\
&\le \phi_f^\varepsilon \frac{1}{\varepsilon^2} \int_{[0,1]^2} f^2(y_4) I_{\{0\le y_3-y_4<\varepsilon^2\}} dy_3 dy_4 \\
&= \|f\|_{L^2}^2 \phi_f^\varepsilon.
\end{aligned}$$

Step 5: Bound on T_m^2. Owing to inequality (20), our bound on T_m^2 can be reduced now to an estimate of the number of terms in the sum over $n_1, \dots, n_s$ in formula (15). This boils down to the following question: given a natural number n, how can we write it as a sum of natural numbers (larger than one)?

This is arguably a classical problem, and in order to recall its answer, let us take a simple example: for $n = 6$, the possible decompositions can be written as $\{6; 2+2+2; 2+4; 4+2; 3+3\}$. Furthermore, notice that the decompositions of 6 can be obtained by adding $+2$ to the decompositions of 4 or adding 1 to the last number of the decompositions of 5. Extrapolating to a general integer n, it is easily seen that the number of decompositions can be expressed as u_{n-1}, where $(u_n)_{n\geq 1}$ stands for the Fibonacci sequence. We have thus found a number of decompositions of the form

$$N_n = 5^{-1/2}\left(\rho_2^{n-1} - \rho_1^{n-1}\right),$$

where the quantities ρ_1, ρ_2 appear in formula (12). Moreover, the number of terms in T_2 is given by $N_{2m} - 1$, the -1 part corresponding to the term T_1.

Putting together this expression with (20) and the result of Step 3, our claim (12) is now easily obtained.

Step 6: Proof of (13). The proof of (13) follows the same arguments as for (12). We briefly sketch the main difference between these two proofs, lying in the analysis of the term $U_{n_1,\dots,n_s}$. Indeed, since we are now dealing with an odd power $2m+1$, the equivalent of (20) is an upper bound of the form

$$\frac{1}{\varepsilon}\int_0^1\int_0^1 |f(y_1)|f^2(y_2)I_{\{|y_1-y_2|<\varepsilon^2\}}dy_1dy_2\int_{[0,1]^{m-1}}\prod_{j=1}^{m-1} f^2(x_j)I_{\{x_1\geq\dots\geq x_{m-1}\}}dx_1\cdots dx_{m-1}. \tag{21}$$

Furthermore, applying Hölder's inequality twice, we obtain

$$\frac{1}{\varepsilon}\int_0^1\int_0^1 |f(y_1)|f^2(y_2)I_{\{|y_1-y_2|<\varepsilon^2\}}dy_1dy_2 = \frac{1}{\varepsilon}\int_0^1 f^2(y_2)\int_{0\vee(y_2-\varepsilon^2)}^{1\wedge(y_2+\varepsilon^2)}|f(y_1)|dy_1dy_2$$
$$\leq \left(\int_0^1\int_0^1 f^2(y_1)f^2(y_2)I_{\{|y_1-y_2|<\varepsilon^2\}}dy_1dy_2\right)^{\frac{1}{2}}\|f\|_{L^2} = (\phi_f^\varepsilon)^{\frac{1}{2}}\|f\|_{L^2},$$

and thus we can bound (21) by $\frac{1}{(m-1)!}(\phi_f^\varepsilon)^{\frac{1}{2}}\|f\|_{L^2}^{2m-1}$, which ends the proof.

□

Our next technical lemma compares the moments of a Wiener type integral with respect to θ^ε and with respect to the white noise.

Lemma 3.2. *Let $m \in \mathbb{N}$, $f \in C^\alpha([0,1])$, $i \in \{1,2\}$, $\varepsilon > 0$ and for $m \geq 1$, set*

$$J_m = \left|\frac{1}{(2m)!}E\left[\left(\int_0^1 f(r)\theta^{i,\varepsilon}(r)dr\right)^{2m}\right] - \frac{1}{2^m m!}\int_{[0,1]^m} f^2(s_1)\cdots f^2(s_m)ds_1\cdots ds_m\right|.$$

Then

(1) We have $J_1 \leq \varepsilon^{2\alpha}\|f\|_{L^2}\|f\|_\alpha$.

(2) For any $m > 1$, the following inequality holds true, where we recall that ρ_1, ρ_2 have been defined just before Lemma 3.1:

$$J_m \leq \frac{1}{(m-1)!}\varepsilon^{2\alpha}\|f\|_\alpha \|f\|_{L^2}^{2m-1} + \frac{k_\eta^{2m}}{\sqrt{5}(m-2)!}\left(\rho_1^{2m-1}-\rho_2^{2m-1}\right)(\phi_f^\varepsilon)^{\frac{1}{2}}\|f\|_{L^2}^{2m-2} + \frac{1}{(m-2)!}\|f\|_{L^2}^{2(m-2)}\phi_f^\varepsilon.$$

Proof. We divide again this proof into several steps.

Step 1: Variance estimates. We prove here the first of our assertions: Notice that

$$\frac{1}{2}\int_0^1 f^2(s_1)ds_1 = \frac{1}{2\varepsilon^2}\sum_{k=1}^{n(\varepsilon)}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2} f^2(s_1)ds_2 ds_1.$$

On the other hand

$$\frac{1}{2}E\left[\left(\int_0^1 f(r)\theta^{i,\varepsilon}(r)dr\right)^2\right] = \frac{1}{2\varepsilon^2}\sum_{k=1}^{n(\varepsilon)}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2} f(r_1)f(r_2)dr_2 dr_1.$$

We thus get

$$\begin{aligned} J_1 &= \left|\frac{1}{2\varepsilon^2}\sum_{k=1}^{n(\varepsilon)}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2} f(r_1)(f(r_2)-f(r_1))dr_2 dr_1\right| \\ &= \left|\frac{1}{2\varepsilon^2}\int_0^1\int_0^1 f(r_1)(f(r_2)-f(r_1))\left(\sum_{k=1}^{+\infty} I_{[k-1,k)^2}\left(\frac{r_1}{\varepsilon^2},\frac{r_2}{\varepsilon^2}\right)\right)dr_2 dr_1\right|, \end{aligned}$$

and hence this quantity can be bounded as follows:

$$\begin{aligned} J_1 &\leq \frac{1}{2\varepsilon^2}\int_0^1\int_0^1 |f(r_1)||(f(r_2)-f(r_1))|I_{\{|r_1-r_2|<\varepsilon^2\}}dr_2 dr_1 \\ &\leq \frac{1}{2\varepsilon^2}\int_0^1 |f(r_1)|\|f\|_\alpha \int_0^1 |r_2-r_1|^\alpha I_{\{|r_1-r_2|<\varepsilon^2\}}dr_2 dr_1 \\ &\leq \varepsilon^{2\alpha}\|f\|_{L^2}\|f\|_\alpha, \end{aligned}$$

which is the first claim of our lemma.

Step 2: decomposition for higher moments: We can follow exactly the computations of Lemma 3.1, Step 1, in order to get

$$\frac{1}{(2m)!}E\left[\left(\int_0^1 f(r)\theta^{i,\varepsilon}(r)dr\right)^{2m}\right] = \tilde{T}_m^1 + \tilde{T}_m^2,$$

with $\tilde{T}_m^j = \frac{T_m^j}{(2m)!}$ for $j = 1, 2$. Furthermore, the term $\tilde{T}_m^2$ can be bounded as in Lemma 3.1, and we obtain

$$|\tilde{T}_m^2| \leq \frac{k_\eta^{2m}}{\sqrt{5}(m-2)!}\left(\rho_2^{2m-1}-\rho_1^{2m-1}\right)(\phi_f^\varepsilon)^{\frac{1}{2}}\|f\|_{L^2}^{2m-2}. \tag{22}$$

Step 3: Study of $\tilde{T}_m^1$: We analyze $\tilde{T}_m^1$ in a slightly different way as in Lemma 3.1. Namely, we first write

$$\tilde{T}_m^1 = \frac{1}{2^m \varepsilon^{2m}} \sum_{\substack{k_1,\dots,k_m=1 \\ k_1>\cdots>k_m}}^{n(\varepsilon)} \int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2} \cdots \int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2}\int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2} f(r_1)\cdots f(r_{2m}) \times I_{\{\{r_1,r_2\}\geq\cdots\geq\{r_{2m-1},r_{2m}\}\}} dr_1\cdots dr_{2m}, \quad (23)$$

where we have written $\{a,b\}\geq\{c,d\}$ for $a\wedge b\geq c\vee d$. We will now compare this quantity with another expression of the same type, called $\hat{T}_m^1$ and defined by

$$\hat{T}_m^1 = \frac{1}{2^m m!}\int_{[0,1]^m} f^2(s_1)\cdots f^2(s_m) ds_1\cdots ds_m.$$

Let us thus write $\hat{T}_m^1$ as

$$\begin{aligned}
\hat{T}_m^1 &= \frac{1}{2^m}\int_{[0,1]^m}\prod_{j=1}^m f^2(s_j)\, I_{\{s_1\geq\cdots\geq s_m\}}\, ds_1\cdots ds_m \\
&= \frac{1}{2^m}\sum_{\substack{k_1,\dots,k_m=1 \\ k_1\geq\cdots\geq k_m}}^{n(\varepsilon)} \int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\cdots\int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2}\prod_{j=1}^m f^2(s_j)\, I_{\{s_1\geq\cdots\geq s_m\}}\, ds_1\cdots ds_m \\
&= \frac{1}{2^m}\sum_{\substack{k_1,\dots,k_m=1 \\ k_1>\cdots>k_m}}^{n(\varepsilon)} \int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\cdots\int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2}\prod_{j=1}^m f^2(s_j)\, I_{\{s_1\geq\cdots\geq s_m\}}\, ds_1\cdots ds_m + \tilde{T}_m^3,
\end{aligned}$$

where $\tilde{T}_m^3$ represents the part of the sum taken over the indices $k_1,\dots,k_m$ such that there exist l satisfying $k_l = k_{l+1}$. However, this latter term can be bounded as in (20), yielding

$$|\tilde{T}_m^3| \leq \frac{m-1}{2^m}\frac{1}{(m-2)!}\|f\|_{L^2}^{2(m-2)}\phi_f^\varepsilon \leq \frac{1}{2}\frac{1}{(m-2)!}\|f\|_{L^2}^{2(m-2)}\phi_f^\varepsilon. \quad (24)$$

Step 4: Conclusion. Putting together the decompositions we have obtained so far, we end up with

$$\begin{aligned}
J_m &\leq |\tilde{T}_m^2| + |\tilde{T}_m^3| + \Bigg| \frac{1}{2^m\varepsilon^{2m}} \sum_{\substack{k_1,\dots,k_m=1 \\ k_1>\cdots>k_m}}^{n(\varepsilon)} \int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\int_{(k_1-1)\varepsilon^2}^{k_1\varepsilon^2}\cdots\int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2}\int_{(k_m-1)\varepsilon^2}^{k_m\varepsilon^2} \\
&\qquad \Big[f(r_1)\cdots f(r_{2m}) - f^2(r_1)f^2(r_3)\cdots f^2(r_{2m-1})\Big] I_{\{\{r_1,r_2\}\geq\cdots\geq\{r_{2m-1},r_{2m}\}\}} dr_1\cdots dr_{2m}\Bigg| \\
&\leq |\tilde{T}_m^2| + |\tilde{T}_m^3| + \frac{1}{2^m m!\varepsilon^{2m}}\int_{[0,1]^{2m}} \Big|f(r_1)\cdots f(r_{2m}) - f^2(r_1)f^2(r_3)\cdots f^2(r_{2m-1})\Big| \\
&\qquad \times I_{\{|r_1-r_2|<\varepsilon^2\}}\cdots I_{\{|r_{2m-1}-r_{2m}|<\varepsilon^2\}}\, dr_1\cdots dr_{2m}.
\end{aligned}$$

Invoking our estimates (22) and (24) on $\tilde{T}_m^2$ and $\tilde{T}_m^3$, our bound on J_m easily reduced to check that

$$\frac{1}{2^m m! \varepsilon^{2m}} \int_{[0,1]^{2m}} \left| f(r_1)\cdots f(r_{2m}) - f^2(r_1) f^2(r_3)\cdots f^2(r_{2m-1}) \right| \\ \times I_{\{|r_1-r_2|<\varepsilon^2\}} \cdots I_{\{|r_{2m-1}-r_{2m}|<\varepsilon^2\}} dr_1 \cdots dr_{2m} \leq \frac{1}{(m-1)!} \varepsilon^{2\alpha} \|f\|_\alpha \|f\|_{L^2}^{2m-1}.$$

The latter inequality can now be obtained from the decomposition

$$\begin{aligned} &\left| f(r_1)\cdots f(r_{2m}) - f^2(r_1) f^2(r_3)\cdots f^2(r_{2m-1}) \right| \\ &= \Big| f(r_1)(f(r_2)-f(r_1)) f(r_3) f(r_4)\cdots f(r_{2m}) \\ &\quad + f^2(r_1) f(r_3)(f(r_4)-f(r_3)) f(r_5) f(r_6)\cdots f(r_{2m}) \\ &\quad + \cdots \\ &\quad + f^2(r_1) f^2(r_3) \ldots f^2(r_{2m-3}) f(r_{2m-1})(f(r_{2m})-f(r_{2m-1})) \Big|, \end{aligned}$$

the inequalities

$$\begin{aligned} \frac{1}{2\varepsilon^2} \int_0^1 \int_0^1 f^2(r_1) I_{|r_1-r_2|<\varepsilon^2} dr_2 dr_1 &\leq \|f\|_{L^2}^2, \\ \frac{1}{2\varepsilon^2} \int_0^1 \int_0^1 f(r_1) f(r_2) I_{|r_1-r_2|<\varepsilon^2} dr_2 dr_1 &\leq \|f\|_{L^2}^2, \end{aligned}$$

and from the estimate we have already obtained for J_1. This finishes the proof.

□

Finally, the characteristic function of a Wiener type integral of the form $\int_0^1 f(r)\theta^{k,\varepsilon}(r)dr$ can be compared to its expected limit $\int_0^1 f(r)dW_r^k$ in the following way:

Lemma 3.3. *Let $f \in \mathscr{C}^\alpha([0,1])$ for a certain $\alpha \in (0,1)$, $k \in \{1,\ldots,d\}$ and $\varepsilon > 0$. For any $u \in \mathbb{R}$, we have:*

$$\begin{aligned} &\left| E\left[e^{iu\int_0^1 f(r)\theta^{k,\varepsilon}(r)dr} \right] - E\left[e^{iu\int_0^1 f(r)dW_r^k} \right] \right| \\ &\leq 4(1/5)^{1/2} u^3 (\phi_f^\varepsilon)^{\frac{1}{2}} \|f\|_{L^2} k_\eta^3 \exp(4u^2 k_\eta^2 \|f\|_{L^2}^2) + u^2 \varepsilon^{2\alpha} \|f\|_\alpha \|f\|_{L^2} \exp(u^2 \|f\|_{L^2}^2) \\ &\quad + 8\,5^{-1/2} u^4 (\phi_f^\varepsilon)^{\frac{1}{2}} \|f\|_{L^2}^2 k_\eta^2 \exp(4u^2 k_\eta^2 \|f\|_{L^2}^2) + (1/2) u^4 \phi_f^\varepsilon \exp(u^2 \|f\|_{L^2}^2). \end{aligned}$$

Proof. Let us control first the imaginary part of the difference. Using lemma 3.1, and invoking the

fact that the odd moments of a Gaussian random variable are null, we get

$$\begin{aligned}
&\left|\mathrm{Im}\left(E\left[\mathrm{e}^{iu\int_0^1 f(r)\theta^{k,\varepsilon}(r)dr}\right]-E\left[\mathrm{e}^{iu\int_0^1 f(r)dW_r^k}\right]\right)\right| \\
&\quad\le\sum_{m=1}^{\infty}\frac{|u|^{2m+1}}{(2m+1)!}\left|E\left[\left(\int_0^1 f(r)\theta^{k,\varepsilon}(r)dr\right)^{2m+1}\right]\right| \\
&\quad\le\sum_{m=1}^{\infty}\frac{(1/5)^{1/2}|u|^{2m+1}}{(m-1)!}k_\eta^{2m+1}\left(\rho_2^{2m}-\rho_1^{2m}\right)(\phi_f^\varepsilon)^{\frac12}\|f\|_{L^2}^{2m-1} \\
&\quad\le 4(1/5)^{1/2}u^3(\phi_f^\varepsilon)^{\frac12}\|f\|_{L^2}k_\eta^3\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\left(4k_\eta^2\|f\|_{L^2}^2u^2\right)^{m-1} \\
&\quad\le 4(1/5)^{1/2}u^3(\phi_f^\varepsilon)^{\frac12}\|f\|_{L^2}k_\eta^3\exp(4u^2k_\eta^2\|f\|_{L^2}^2).
\end{aligned}$$

In order to control the real part of the difference, we will use Lemma 3.2. This yields:

$$\begin{aligned}
&\left|\mathrm{Re}\left(E\left[\mathrm{e}^{iu\int_0^1 f(r)\theta^{k,\varepsilon}(r)dr}\right]-E\left[\mathrm{e}^{iu\int_0^1 f(r)dW_r^k}\right]\right)\right| \\
&\le\sum_{m=1}^{+\infty}u^{2m}\Bigg[\frac{1}{(m-1)!}\varepsilon^{2\alpha}\|f\|_\alpha\|f\|_{L^2}^{2m-1} \\
&\qquad\qquad+u^4(\phi_f^\varepsilon)^{\frac12}\frac{8}{\sqrt5}\|f\|_{L^2}^2k_\eta^2\sum_{m=2}^{+\infty}\frac{1}{(m-2)!}\left(4k_\eta^2u^2\|f\|_{L^2}^2\right)^{m-2} \\
&\qquad\qquad+\frac12u^4\phi_f^\varepsilon\sum_{m=2}^{+\infty}\frac{1}{(m-2)!}\left(u^2\|f\|_{L^2}^2\right)^{m-2}\Bigg].
\end{aligned}$$

The latter quantity can be bounded by

$$\begin{aligned}
u^2\varepsilon^{2\alpha}\|f\|_\alpha\|f\|_{L^2}\exp(u^2\|f\|_{L^2}^2)+\frac{8}{\sqrt5}u^4(\phi_f^\varepsilon)^{\frac12}\|f\|_{L^2}^2k_\eta^2\exp(4u^2k_\eta^2\|f\|_{L^2}^2) \\
+\frac12u^4\phi_f^\varepsilon\exp(u^2\|f\|_{L^2}^2),
\end{aligned}$$

which ends the proof.

□

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