Elect. Comm. in Probab. **16** (2011), 9–21

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON RELAXING THE ASSUMPTION OF DIFFERENTIAL SUBORDINATION IN SOME MARTINGALE INEQUALITIES

ADAM OSĘKOWSKI [1]
Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
email: ados@mimuw.edu.pl

Submitted March 30, 2010, *accepted in final form* December 20, 2010

AMS 2000 Subject classification: Primary: 60G44. Secondary: 60G42.
Keywords: Martingale, differential subordination, moment inequality, weak-type inequality

Abstract

Let X, Y be continuous-time martingales taking values in a separable Hilbert space $\mathscr{H}$.
(i) Assume that X, Y satisfy the condition $[X,X]_t \geq [Y,Y]_t$ for all $t \geq 0$. We prove the sharp inequalities

$$\sup_t ||Y_t||_p \leq (p-1)^{-1} \sup_t ||X_t||_p, \qquad 1 < p \leq 2,$$

$$\mathbb{P}(\sup_t |Y_t| \geq 1) \leq \frac{2}{\Gamma(p+1)} \sup_t ||X_t||_p^p, \qquad 1 \leq p \leq 2,$$

and for any $K > 0$ we determine the optimal constant $L = L(K)$ depending only on K such that

$$\sup_t ||Y_t||_1 \leq K \sup_t \mathbb{E}|X_t| \log|X_t| + L(K).$$

(ii) Assume that X, Y satisfy the condition $[X,X]_\infty - [X,X]_{t-} \geq [Y,Y]_\infty - [Y,Y]_{t-}$ for all $t \geq 0$. We establish the sharp bounds

$$\sup_t ||Y_t||_p \leq (p-1) \sup_t ||X_t||_p, \qquad 2 \leq p < \infty$$

and

$$\mathbb{P}(\sup_t |Y_t| \geq 1) \leq \frac{p^{p-1}}{2} \sup_t ||X_t||_p^p, \qquad 2 \leq p < \infty.$$

This generalizes the previous results of Burkholder, Suh and the author, who showed the above estimates under the more restrictive assumption of differential subordination. The proof is based on Burkholder's technique and integration method.

[1] RESEARCH PARTIALLY SUPPORTED BY MNISW GRANT N N201 397437

1 Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, filtered by a nondecreasing family $(\mathscr{F}_n)_{n\geq 0}$ of sub-σ-fields of $\mathscr{F}$. Let $f=(f_n)$, $g=(g_n)$ be adapted martingales taking values in a separable Hilbert space $\mathscr{H}$ (which may and will be assumed to be equal to ℓ^2), with a norm $|\cdot|$ and a scalar product denoted by $\cdot$. The difference sequences $df=(df_n)$, $dg=(dg_n)$ of f and g are given by the equalities

$$f_n=\sum_{k=0}^n df_k, \qquad g_n=\sum_{k=0}^n dg_k, \qquad n=0,\,1,\,2,\,\ldots.$$

The following notion of *differential subordination* is due to Burkholder: we say that g is differentially subordinate to f, if for any n we have $|dg_n|\leq |df_n|$. This condition implies many interesting estimates which have numerous applications in many areas of mathematics, see the surveys [4], [6] by Burkholder and references therein. Consult also Bañuelos and Wang [1], Wang [13], Bañuelos and Méndez-Hernández [2], Geiss, Montgomery-Smith and Saksman [8], Suh [12] and the papers [10], [11] by the author for some more recent results in this direction. To begin, let us recall the following classical moment inequality, due to Burkholder [3]. We use the notation $||f||_p=\sup_n ||f_n||_p$ for $1\leq p\leq\infty$.

Theorem 1.1. *If g is differentially subordinate to f, then for $1<p<\infty$,*

$$||g||_p \leq (p^*-1)||f||_p. \tag{1.1}$$

Here $p^*=\max\{p-1,(p-1)^{-1}\}$ and the constant is the best possible.

Moreover, we have the weak-type bounds, proved by Burkholder [3] for $1\leq p\leq 2$, and Suh [12] for $2\leq p<\infty$. Let $g^*=\sup_n|g_n|$ denote the maximal function of g.

Theorem 1.2. *Assume that g is differentially subordinate to f. Then*

$$\mathbb{P}(g^*\geq 1)\leq \frac{2}{\Gamma(p+1)}||f||_p^p, \qquad 1\leq p\leq 2, \tag{1.2}$$

and, if f and g are real-valued,

$$\mathbb{P}(g^*\geq 1)\leq \frac{p^{p-1}}{2}||f||_p^p, \qquad 2\leq p<\infty. \tag{1.3}$$

Both inequalities are sharp.

In the case $p=1$ the moment inequality does not hold with any finite constant. The author established in [10] the following substitute.

Theorem 1.3. *Assume that g is differentially subordinate to f. Then for $K>1$,*

$$||g||_1\leq K\sup_n \mathbb{E}|f_n|\log|f_n|+L(K), \tag{1.4}$$

where

$$L(K)=\begin{cases} \frac{K^2}{2(K-1)}\exp(-K^{-1}) & \text{if } K<2,\\ K\exp(K^{-1}-1) & \text{if } K\geq 2.\end{cases} \tag{1.5}$$

The constant is the best possible. Furthermore, for $K\leq 1$ the inequality does not hold in general with any universal $L(K)<\infty$.

Let us now turn to the continuous-time setting. Assume that the probability space is complete and is equipped with a filtration $(\mathscr{F}_t)_{t\in[0,\infty)}$ such that $\mathscr{F}_0$ contains all the events of probability 0. Let $X=(X_t)_{t\geq 0}$, $Y=(Y_t)_{t\geq 0}$ be $\mathscr{H}$-valued martingales, which have right-continuous trajectories with limits from the left. The generalization of the differential subordination is as follows (see [1] and [13]): Y is differentially subordinate to X, if the process $([X,X]_t-[Y,Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. Here $[X,X]$ denotes the square bracket (quadratic variation) of X: that is, if $X_t=(X_t^1,X_t^2,\ldots)\in\mathscr{H}$, then $[X,X]=\sum_{k=1}^{\infty}[X^k,X^k]$, where $[X^k,X^k]$ is the usual square bracket of a real-valued martingale X^k, see e.g. Dellacherie and Meyer [7] for details. We use the notation $||X||_p=\sup_t||X_t||_p$ and $X^*=\sup_t|X_t|$, analogous to the one in the discrete-time case. Furthermore, throughout the paper, we set $X_{0-}=Y_{0-}=0$ and $[X,X]_{0-}=[Y,Y]_{0-}=0$.
The inequalities (1.1), (1.2), (1.3) and (1.4) can be successfully extended to the continuous-time setting (this will be clear from our results below, see also the papers by Wang [13] and Suh [12] for the proofs of (1.1) and (1.3)). The motivation in the present paper comes from the interesting and challenging question raised in [4]: Burkholder asked whether the moment inequality

$$||Y||_p\leq(p^*-1)||X||_p,\qquad 1<p<\infty \tag{1.6}$$

remains valid under a weaker assumption that the process $([X,X]_t-[Y,Y]_t)_{t\geq 0}$ is nonnegative (and not necessarily nondecreasing). We will prove that this is true for $p\leq 2$, and introduce a dual condition, weaker than the differential subordination, which implies the validity of (1.6) for $p\geq 2$. Furthermore, we will show that under these relaxed conditions the corresponding weak-type and logarithmic bounds hold.
The main results of the paper are stated in the two theorems below.

Theorem 1.4. *Suppose that X, Y are $\mathscr{H}$-valued martingales such that*

$$[X,X]_t\geq[Y,Y]_t\qquad \text{for any } t\geq 0. \tag{1.7}$$

Then

$$||Y||_p\leq(p-1)^{-1}||X||_p,\qquad 1<p\leq 2, \tag{1.8}$$

and

$$\mathbb{P}(Y^*\geq 1)\leq\frac{2}{\Gamma(p+1)}||X||_p^p,\quad 1\leq p\leq 2. \tag{1.9}$$

Furthermore, if $K>1$, then

$$||Y||_1\leq K\sup_{t\geq 0}\mathbb{E}|X_t|\log|X_t|+L(K), \tag{1.10}$$

where $L(K)$ is given by (1.5). For $K\leq 1$ the inequality does not hold in general with any universal $L(K)<\infty$. All the inequalities above are sharp.

Theorem 1.5. *Suppose that X, Y are $\mathscr{H}$-valued martingales such that*

$$[X,X]_\infty-[X,X]_{t-}\geq[Y,Y]_\infty-[Y,Y]_{t-}\qquad \text{for any } t\geq 0. \tag{1.11}$$

Then

$$||Y||_p\leq(p-1)||X||_p,\qquad 2\leq p<\infty, \tag{1.12}$$

and

$$\mathbb{P}(Y^*\geq 1)\leq\frac{p^{p-1}}{2}||X||_p^p,\qquad 2\leq p<\infty. \tag{1.13}$$

The inequalities are sharp.

Obviously, the condition (1.11) is weaker than the differential subordination. It concerns the quadratic variations of X and Y on the intervals $[t,\infty)$, $t \geq 0$, and hence it can be seen as a dual to (1.7), which compares the square brackets on the intervals $[0,t]$, $t \geq 0$. It should also be stressed that Suh's result (the weak-type inequality for $p \geq 2$) concerned only real-valued martingales. Our approach is not only much simpler, but it also enables us to obtain the bound for processes taking values in a Hilbert space.

A few words about our approach and the organization of the paper. The original proofs of (1.1), (1.2), (1.3) and (1.4) are based on Burkholder's method, which reduces the problem of showing a given martingale inequality to the problem of finding a certain biconcave function. This approach has also been been successful in a number of other estimates, see [3], [5] for the detailed description of the method and related remarks, and [13] for the extension of the technique to the continuous-time setting. Our approach is slightly different and exploits an integration argument developed by the author in [9]. In Section 2 we introduce two „basis" functions which are used in two simple inequalities (2.7) and (2.8) for martingales satisfying (1.7) or (1.11). Then we complicate these inequalities by integrating the basis functions against various kernels; this yields the desired estimates. Section 3 contains the description of the integration argument, and in Section 4 we present the detailed calculations leading to (1.8), (1.9), (1.10), (1.12) and (1.13).

2 Two basis functions

Let D_1, D_∞ be subsets of $\mathscr{H} \times \mathscr{H} \times [0,\infty)$, defined as follows:

$$D_1 = \{(x,y,s) : |x| + \sqrt{|y|^2 + s} \leq 1\},$$

$$D_\infty = \{(x,y,t) : \sqrt{|x|^2 + t} + |y| \leq 1\}.$$

Consider $u_1 : \mathscr{H} \times \mathscr{H} \times [0,\infty) \to \mathbb{R}$, given by

$$u_1(x,y,s) = \begin{cases} |y|^2 - |x|^2 + s & \text{if } (x,y,s) \in D_1, \\ 1 - 2|x| & \text{if } (x,y,s) \notin D_1. \end{cases} \tag{2.1}$$

Furthermore, introduce the functions ϕ_1, $\psi_1 : \mathscr{H} \times \mathscr{H} \times [0,\infty) \to \mathscr{H}$ by

$$(\phi_1(x,y,s), \psi_1(x,y,s)) = \begin{cases} (-2x, 2y) & \text{if } (x,y,s) \in D_1, \\ (-2x', 0) & \text{if } (x,y,s) \notin D_1. \end{cases}$$

Here $x' = x/|x|$ if $x \neq 0$ and $x' = 0$ if $x = 0$. The dual to u_1 is the function $u_\infty : \mathscr{H} \times \mathscr{H} \times [0,\infty) \to \mathbb{R}$, defined by the formula

$$u_\infty(x,y,t) = \begin{cases} 0 & \text{if } (x,y,t) \in D_\infty, \\ (|y| - 1)^2 - |x|^2 - t & \text{if } (x,y,t) \notin D_\infty. \end{cases} \tag{2.2}$$

In addition, let ϕ_∞, $\psi_\infty : \mathscr{H} \times \mathscr{H} \times [0,\infty) \to \mathscr{H}$ be the functions given by

$$(\phi_\infty(x,y,t), \psi_\infty(x,y,t)) = \begin{cases} (0,0) & \text{if } (x,y,t) \in D_\infty, \\ (-2x, 2y - 2y') & \text{if } (x,y,t) \notin D_\infty. \end{cases}$$

The key property of the functions u_1 and u_∞ is described in the lemma below.

Lemma 2.1. *Let x, y, h, $k \in \mathscr{H}$ and s, $t \geq 0$.*
(i) If $s+|h|^2-|k|^2 \geq 0$, then

$$u_1(x+h,y+k,s+|h|^2-|k|^2) \leq u_1(x,y,s)+\phi_1(x,y,s)\cdot h+\psi_1(x,y,s)\cdot k. \tag{2.3}$$

(ii) If $t-|h|^2+|k|^2 \geq 0$, then

$$u_\infty(x+h,y+k,t-|h|^2+|k|^2) \leq u_\infty(x,y,t)+\phi_\infty(x,y,t)\cdot h+\psi_\infty(x,y,t)\cdot k. \tag{2.4}$$

Proof. (i) We start from the observation that

$$u_1(x,y,s) \leq 1-2|x| \quad \text{for all } (x,y,s)\in \mathscr{H}\times\mathscr{H}\times[0,\infty). \tag{2.5}$$

Indeed, both sides are equal on D_1^c, while on D_1 we have $u_1(x,y,s)-(1-2|x|)=(\sqrt{|y|^2+s})^2-(1-|x|)^2 \leq 0$, by the definition of D_1. Now if $(x,y,s)\notin D_1$, then

$$\begin{aligned} u_1(x,y,s)+\phi_1(x,y,s)\cdot h+\psi_1(x,y,s)\cdot k &= 1-2|x|-2x'\cdot h\\ &\geq 1-2|x+h|\\ &\geq u_1(x+h,y+k,s+|h|^2-|k|^2). \end{aligned}$$

Suppose then, that (x,y,s) lies in D_1. If $(x+h,y+k,s+|h|^2-|k|^2)$ also belongs to D_1, then both sides of (2.3) are equal. If $(x+h,y+k,s+|h|^2-|k|^2)\notin D_1$ and $|x+h|\leq 1$, then (2.3) can be rewritten in the form

$$(1-|x+h|)^2 \leq |y+k|^2+s+|h|^2-|k|^2,$$

which holds due to the definition of D_1. Finally, if $(x+h,y+k,s+|h|^2-|k|^2)\notin D_1$ and $|x+h|>1$, we have, using the bound $|k|^2\leq s+|h|^2$,

$$\begin{aligned} &u_1(x,y,s)+\phi_1(x,y,s)\cdot h+\psi_1(x,y,s)\cdot k-u_1(x+h,y+k,s+|h|^2-|k|^2)\\ &\quad=|y|^2-|x|^2+s-2x\cdot h+2y\cdot k-1+2|x+h|\\ &\quad\geq |y|^2-|x|^2+s-2x\cdot h-2|y||k|-1+2|x+h|\\ &\quad\geq |y|^2-|x|^2+s-2x\cdot h-2|y|\sqrt{s+|h|^2}-1+2|x+h|\\ &\quad=(|y|-\sqrt{s+|h|^2})^2-(|x+h|-1)^2. \end{aligned} \tag{2.6}$$

It suffices to note that $|x|+|y|\leq |x|+\sqrt{|y|^2+s}\leq 1$, so

$$||x+h|-1| = |x+h|-1 \leq |x|+|h|-1 \leq -|y|+|h| \leq \sqrt{s+|h|^2}-|y|$$

and we are done.
(ii) This follows immediately from (i) and the identities

$$\begin{aligned} u_\infty(x,y,t) &= u_1(y,x,t)+|y|^2-|x|^2-t,\\ \phi_\infty(x,y,t) &= \psi_1(y,x,t)-2x,\\ \psi_\infty(x,y,t) &= \phi_1(y,x,t)+2y \end{aligned}$$

valid for all x, $y\in\mathscr{H}$ and $t\geq 0$.
The lemma above leads to the following martingale inequalities. Let $u_1^0(x,y)=u_1(x,y,0)$ and $u_\infty^0(x,y)=u_\infty(x,y,0)$.

Lemma 2.2. *Suppose that X, Y are $\mathcal{H}$-valued martingales.*
(i) Suppose that (1.7) *holds. Then for all $t \geq 0$,*

$$\mathbb{E}u_1^0(X_t, Y_t) \leq 0. \tag{2.7}$$

(ii) Suppose that (1.11) *holds and $||X||_2 < \infty$. Then*

$$\mathbb{E}u_\infty^0(X_\infty, Y_\infty) \leq 0. \tag{2.8}$$

Proof. Using standard approximation (cf. [13]), it suffices to show the inequalities for finite dimensional case: $\mathcal{H} = \mathbb{R}^d$ for some positive integer d.
(i) Let $t \geq 0$ be fixed. We will prove that

$$\mathbb{E}u_1(X_t, Y_t, [X,X]_t - [Y,Y]_t) \leq 0,$$

a stronger statement, since $u_1(x,y,s) \geq u_1^0(x,y)$ for any x, $y \in \mathcal{H}$ and $s \geq 0$. Let $Z_s = (X_s, Y_s, [X,X]_s - [Y,Y]_s)$ and consider a stopping time $\tau = \inf\{s : Z_s \notin D_1\}$. By (2.5),

$$\begin{aligned}\mathbb{E}u_1(Z_t)I_{\{\tau\leq t\}} &= \mathbb{E}\big[I_{\{\tau\leq t\}}\mathbb{E}(u_1(Z_t)|\mathcal{F}_\tau)\big] \leq \mathbb{E}\big[I_{\{\tau\leq t\}}\mathbb{E}(1-2|X_t||\mathcal{F}_\tau)\big]\\ &\leq \mathbb{E}((1-2|X_\tau|)I_{\{\tau\leq t\}} = \mathbb{E}u_1(Z_{\tau\wedge t})I_{\{\tau\leq t\}},\end{aligned}$$

which gives

$$\mathbb{E}u_1(Z_t) \leq \mathbb{E}u_1(Z_{\tau\wedge t}). \tag{2.9}$$

Since u_1 is of class C^2 on D_1 and the range of the process $(Z_{\tau\wedge s-})_{s\leq t}$ is contained in D_1, we may apply Itô's formula to obtain

$$u_1(Z_{\tau\wedge t}) = I_0 + I_1 + I_2/2 + I_3 + I_4 + I_5, \tag{2.10}$$

where

$$\begin{aligned}
I_0 &= u_1(Z_0),\\
I_1 &= \int_{0+}^{\tau\wedge t} u_{1x}(Z_{s-})dX_s + \int_{0+}^{\tau\wedge t} u_{1y}(Z_{s-})dY_s,\\
I_2 &= \sum_{i,j\leq d}\int_{0+}^{\tau\wedge t} u_{1x_ix_j}(Z_{s-})d[X^{ic},X^{jc}]_{s-} + \sum_{i,j\leq d}\int_{0+}^{\tau\wedge t} u_{1y_iy_j}(Z_{s-})d[Y^{ic},Y^{jc}]_{s-}\\
&\qquad + 2\sum_{i,j\leq d}\int_{0+}^{\tau\wedge t} u_{1x_iy_j}(Z_{s-})d[X^{ic},Y^{jc}]_{s-},\\
I_3 &= \sum_{i=1}^{d}\int_{0+}^{\tau\wedge t} u_{1s}(Z_{s-})d[X^{ic},X^{ic}]_{s-} - \sum_{i=1}^{d}\int_{0+}^{\tau\wedge t} u_{1s}(Z_{s-})d[Y^{ic},Y^{ic}]_{s-}\\
I_4 &= \sum_{0<s\leq\tau\wedge t} u_{1s}(Z_{s-})(|\Delta X_s|^2 - |\Delta Y_s|^2),\\
I_5 &= \sum_{0<s\leq\tau\wedge t}\Big[u_1(Z_s) - u_1(Z_{s-}) - u_{1x}(Z_{s-})\cdot\Delta X_s - u_{1y}(Z_{s-})\cdot\Delta Y_s\\
&\qquad\qquad - u_{1s}(Z_{s-})(|\Delta X_s|^2 - |\Delta Y_s|^2)\Big].
\end{aligned}$$

As one easily verifies, $I_2/2 + I_3 = 0$; this is due to the fact that $u_{1x_ix_j}(Z_{s-}) = -u_{1y_iy_j}(Z_{s-}) = -2$ if $i = j$; $u_{1x_ix_j}(Z_{s-}) = -u_{1y_iy_j}(Z_{s-}) = 0$ if $i \neq j$; $u_{1x_iy_j}(Z_{s-}) = 0$ for all i, j; and $u_{1s}(Z_{s-}) = 1$. Furthermore, $I_4 + I_5 \leq 0$, in view of (1.7) and (2.3). By the properties of stochastic integrals, I_1 has mean 0. Combine the above facts about the terms I_k with (2.9) and (2.10) to get $\mathbb{E}u_1(Z_t) \leq \mathbb{E}u_1(Z_0)$. However, the latter expression is nonpositive; this is due to $u_1(x, y, |x|^2 - |y|^2) = 0$ for $|x| \leq 1/2$, and $u_1(x, y, |x|^2 - |y|^2) = 1 - 2|x| \leq 0$ for remaining x.
(ii) By (1.7) and the condition $||X||_2 < \infty$ we have that Y is also bounded in L^2 and $[X,X]_\infty$, $[Y,Y]_\infty$ are finite almost surely. We will show a stronger statement: for any stopping time η,

$$\mathbb{E}u_\infty(X_\eta, Y_\eta, [X,X]_\infty - [X,X]_\eta - [Y,Y]_\infty + [Y,Y]_\eta) \leq 0 \tag{2.11}$$

(then the claim follows by taking $\eta \equiv \infty$). Let $\xi = [X,X]_\infty - [Y,Y]_\infty$,

$$Z_s = (X_s, Y_s, [X,X]_\infty - [X,X]_s - [Y,Y]_\infty + [Y,Y]_s)_{s\geq 0} \tag{2.12}$$

and $\tau = \inf\{s : Z_s \notin D_\infty\}$. By Doob's optional sampling theorem,

$$\begin{aligned}\mathbb{E}u_\infty(Z_\eta)I_{\{\tau\leq\eta\}} &= \mathbb{E}\big[(|Y_\eta|^2 - [Y,Y]_\eta) - (|X_\eta|^2 - [X,X]_\eta) - 2|Y_\eta| + 1 - \xi\big]I_{\{\tau\leq\eta\}}\\ &\leq \mathbb{E}\big[(|Y_\tau|^2 - [Y,Y]_\tau) - (|X_\tau|^2 - [X,X]_\tau) - 2|Y_\tau| + 1 - \xi\big]I_{\{\tau\leq\eta\}},\end{aligned}$$

which gives $\mathbb{E}u_\infty(Z_\eta) \leq \mathbb{E}u_\infty(Z_{\tau\wedge\eta})$. Applying Itô's formula for $u_\infty(Z_{\tau\wedge\eta})$, we get the analogue of (2.10), with similar terms I_0–I_5 (simply replace u_1 by u_∞). We have that $I_1 = I_2 = I_3 = 0$ and $I_4 + I_5 \leq 0$ due to (1.11) and (2.4); this implies $\mathbb{E}u_\infty(Z_\eta) \leq \mathbb{E}u_\infty(Z_0)$. The proof is completed by the observation that $u(Z_0) \leq 0$ almost surely, which can be verified readily. □

3 An integration method

In the proof of the announced inequalities we will use the following procedure. Suppose that $V : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is a given Borel function and assume that we are interested in proving the estimate

$$\mathbb{E}V(X_t, Y_t) \leq 0, \qquad t \geq 0 \tag{3.1}$$

for a pair (X, Y) of martingales satisfying the condition (1.7). Let $k : [0,\infty) \to [0,\infty)$ be a Borel function such that

$$\mathbb{E}\int_0^\infty k(r)|u_1^0(X_t/r, Y_t/r)|dr < \infty, \qquad t \geq 0, \tag{3.2}$$

and take

$$U(x, y) = \int_0^\infty k(r)u_1^0(x/r, y/r)dr.$$

If the kernel k is chosen in such a way that

$$U(x, y) \geq V(x, y) \qquad \text{for all } x,\ y \in \mathscr{H}, \tag{3.3}$$

then (3.1) holds. Indeed, for any $r > 0$ and $t \geq 0$ we have $[X/r,X/r]_t \geq [Y/r,Y/r]_t$, so $\mathbb{E}u_1^0(X_t/r, Y_t/r) \leq 0$ by Lemma 2.2. Thus $\mathbb{E}V(X_t, Y_t) \leq \mathbb{E}U(X_t, Y_t) \leq 0$, by (3.3) and Fubini's theorem, permitted due to (3.2).
Similarly, suppose we are interested in the bound

$$\mathbb{E}V(X_\infty, Y_\infty) \leq 0,$$

for a pair (X,Y) satisfying (1.11) and $||X||_2 < \infty$. Arguing as previously, we see that it suffices to find a function $k : [0,\infty) \to [0,\infty)$ which enjoys the condition

$$\mathbb{E}\int_0^\infty k(r)|u_\infty^0(X_\infty/r, Y_\infty/r)|dr < \infty, \qquad t \geq 0 \tag{3.4}$$

and the majorization property (3.3), where $U : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is given by

$$U(x,y) = \int_0^\infty k(r)u_\infty^0(x/r, y/r)dr.$$

This approach will be successful in proving (1.8), (1.9), (1.10) and (1.12). In the case of (1.13) we will need a slight modification of this method. The details are presented in the next section.

4 The proof of Theorem 1.4 and Theorem 1.5

We start from the observation that the constants appearing in (1.8), (1.9), (1.10), (1.12) and (1.13) are optimal: indeed, they are already the best possible under the differential subordination. See pages 13–14 in [5] for the sharpness of the moment estimates (1.8) and (1.12), pages 752–755 in [10] for the logarithmic estimate (1.10), pages 684–685 in [3] and pages 1548–1553 in [12] for the weak-type inequalities (1.9) and (1.13). Furthermore, the moment and weak-type inequalities are trivial for $p = 2$, so in what follows, we assume that $p \neq 2$.

4.1 The proof of (1.8)

Clearly, we may assume that $||X||_p < \infty$, since otherwise there is nothing to prove. We will be done if we show that $\mathbb{E}|Y_t|^p \leq (p-1)^{-p}\mathbb{E}|X_t|^p$ for any $t \geq 0$. Let V_p, $U_p : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ be given by

$$V_p(x,y) = |y|^p - (p-1)^{-p}|x|^p,$$
$$U_p(x,y) = \alpha_p \int_0^\infty r^{p-1}u_1^0(x/r, y/r)\mathrm{d}r,$$

where $\alpha_p = p^{3-p}(2-p)/2$. A direct computation gives

$$U_p(x,y) = \frac{p^{2-p}}{p-1}((p-1)|y| - |x|)(|x| + |y|)^{p-1}.$$

Now we show (3.2) and (3.3). The second estimate was shown by Burkholder, see page 17 in [5]. To establish (3.2), note that by Burkholder-Davis-Gundy inequality, for some constant $c_p > 0$, $||Y_t||_p \leq c_p||[Y,Y]_t^{1/2}||_p \leq c_p||[X,X]_t^{1/2}||_p \leq c_p^2||X_t||_p$, so Y is bounded in L^p. Since

$$|u_1^0(x,y)| \leq \begin{cases} |x|^2 + |y|^2 & \text{if } |x| + |y| \leq 1, \\ 1 + 2|x| & \text{if } |x| + |y| > 1, \end{cases} \tag{4.1}$$

we obtain

$$\mathbb{E}\int_0^\infty r^{p-1}|u_1^0(X_t/r, Y_t/r)|\mathrm{d}r \leq J_1 + J_2 + J_3,$$

where

$$J_1 = p^{-1}\mathbb{E}(|X_t| + |Y_t|)^p,$$
$$J_2 = 2(p-1)^{-1}\mathbb{E}|X_t|(|X_t| + |Y_t|)^{p-1},$$
$$J_3 = (2-p)^{-1}\mathbb{E}(|X_t|^2 + |Y_t|^2)(|X_t| + |Y_t|)^{p-2}.$$

It is straightforward to verify that J_1, J_2, J_3 are finite, and (3.2) follows. This completes the proof of (1.8).

4.2 The proof of (1.9)

This is a bit more technical. We start with the following auxiliary fact. Let

$$A(r) = e^r \int_r^\infty e^{-u}u^p \mathrm{d}u, \qquad r \geq 0. \tag{4.2}$$

Lemma 4.1. *(i) For $r \geq 0$ we have*

$$\int_r^\infty e^{-u}u^p du - (r+1)^p e^{-r} - \frac{\Gamma(p+1)}{2}e^{-r} \leq 0. \tag{4.3}$$

(ii) If $|x| + |y| > 1$, then

$$|y|(|x| + |y| - 1)^p + (1 - |y|)A(|x| + |y| - 1) \leq \frac{\Gamma(p+1)}{2} I_{\{|y|<1\}} + |x|^p. \tag{4.4}$$

Proof. (i) Denote the left-hand side by $F(r)$. We have

$$F'(r) = e^{-r}\left((r+1)^p - r^p - p(r+1)^{p-1} + \frac{\Gamma(p+1)}{2}\right) = e^{-r}G(r) \geq 0,$$

since $G(0) = 1 - p + \Gamma(p+1)/2 \geq 0$ and

$$G'(r) = p((r+1)^{p-1} - r^{p-1} - (p-1)(r+1)^{p-2}) \geq 0,$$

in virtue of the concavity of the function $r \mapsto r^{p-1}$. Hence $F(r) \leq \lim_{s\to\infty} F(s) = 0$.
(ii) A standard analysis shows that for fixed $|x|$, the left-hand side, as a function of $|y|$, is nonincreasing. Therefore it suffices to verify the inequality (4.4) for $|y| = 1$ and $|y| = \max\{1 - |x|, 0\}$. If $|y| = 1$, both sides are equal. If $|x| \leq 1$ and $|y| = 1 - |x| < 1$, then the left hand side of (4.4) equals $\Gamma(p+1)|x|$ and

$$\Gamma(p+1)|x| \leq \frac{\Gamma(p+1)}{2}(1 + |x|^2) \leq \frac{\Gamma(p+1)}{2} + |x|^p.$$

Finally, for $|x| > 1$ and $|y| = 0$ the inequality reduces to (4.3). □
We turn to the proof of (1.9). It suffices to show that

$$\mathbb{P}(|Y_t| \geq 1) \leq \frac{2}{\Gamma(p+1)}||X_t||_p^p \qquad \text{for all } t \geq 0. \tag{4.5}$$

Indeed, let $\varepsilon > 0$ and consider a stopping time $\tau = \inf\{s : |Y_s| \geq 1-\varepsilon\}$. Apply the above estimate to the martingale pair $(X_{\tau\wedge s}/(1-\varepsilon), Y_{\tau\wedge s}/(1-\varepsilon))_{s\geq 0}$ (for which (1.7) is still valid). Since $\{Y^* \geq 1\} \subseteq \bigcup_{t\geq 0}\{|Y_{\tau\wedge t}| \geq 1-\varepsilon\}$, we get

$$\mathbb{P}(Y^* \geq 1) \leq \lim_{t\to\infty} \mathbb{P}(|Y_{\tau\wedge t}| \geq 1-\varepsilon) \leq \frac{2}{\Gamma(p+1)(1-\varepsilon)^p}||X||_p^p \tag{4.6}$$

and letting $\varepsilon \to 0$ yields (1.9).
To prove (4.5), note that for $p = 1$ it follows directly from (2.7), since

$$u_1^0(x, y) \geq 1_{\{|y|\geq 1\}} - 2|x| \qquad \text{for all } x,\, y \in \mathscr{H}$$

(see e.g. pages 20–21 in [5]). For $1 < p < 2$, introduce $V_{p,\infty}$, $U_{p,\infty} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ by

$$V_{p,\infty}(x, y) = I_{\{|y|\geq 1\}} - \frac{2}{\Gamma(p+1)}|x|^p,$$

$$U_{p,\infty}(x, y) = \int_0^\infty k(r)u_1^0(x/r, y/r)dr,$$

where

$$k(r) = I_{[1,\infty)}(r) \cdot \frac{p(p-1)(2-p)}{\Gamma(p+1)} r^2 e^{r-1} \int_{r-1}^\infty e^{-u}u^{p-3}\mathrm{d}u.$$

It can be verified that if $|x| + |y| \leq 1$, then

$$U_{p,\infty}(x, y) = |y|^2 - |x|^2,$$

and if $|x| + |y| > 1$, then

$$U_{p,\infty}(x, y) = 1 - \frac{2}{\Gamma(p+1)}\big[|y|(|x| + |y| - 1)^p + (1-|y|)A(|x| + |y| - 1)\big],$$

where A is given by (4.2). Now the condition (3.2) can be easily verified using (4.1). To prove (3.3), note that if $|x| + |y| \leq 1$, then

$$U_{p,\infty}(x, y) \geq -|x|^2 \geq -|x|^p \geq -\frac{2}{\Gamma(p+1)}|x|^p = V_{p,\infty}(x, y),$$

while for $|x| + |y| > 1$, the majorization reduces to (4.4). This completes the proof.

4.3 The proof of (1.10)

We will prove that $\mathbb{E}|Y_t| \leq K\mathbb{E}|X_t|\log|X_t| + L(K)$ for all $t \geq 0$. Clearly, we may consider only those martingales X, which satisfy the condition $\sup_t \mathbb{E}|X_t|\log|X_t| < \infty$. Let $V_{\log}$, $U_{\log} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ be given by

$$\begin{aligned} V_{\log}(x, y) &= |y| - K|x|\log|x| - L(K), \\ U_{\log}(x, y) &= \alpha \int_1^\infty u_1^0(x/r, y/r)dr \\ &= \begin{cases} \alpha(|y|^2 - |x|^2) & \text{if } |x| + |y| \leq 1, \\ \alpha(2|y| - 2|x|\log(|x| + |y|) - 1) & \text{if } |x| + |y| > 1, \end{cases} \end{aligned}$$

where $\alpha > 0$ will be chosen later. The condition (3.2) is verified easily using (4.1) and the observation that for some positive constants c, d,

$$||Y_t||_1 \le c||[Y,Y]_t^{1/2}||_1 \le c||[X,X]_t^{1/2}||_1 \le c^2\mathbb{E}|X_t|\log|X_t| + d < \infty.$$

The inequality (3.3), for a proper choice of α, was shown in Lemma 3.3 in [10].

4.4 The proof of (1.12)

As previously, we restrict ourselves to the case $||X||_p < \infty$ and the inequality takes the form $\mathbb{E}|Y_\infty|^p \le (p-1)^p\mathbb{E}|X_\infty|^p$. Consider the functions V_p, $U_p : \mathscr{H}\times\mathscr{H}\to\mathbb{R}$ given by

$$V_p(x,y) = |y|^p - (p-1)^p|x|^p,$$
$$U_p(x,y) = \alpha_p\int_0^\infty r^{p-1}u_\infty^0(x/r,y/r)\mathrm{d}r,$$

where $\alpha_p = p^{3-p}(p-1)^p(p-2)/2$. It can be verified that

$$U_p(x,y) = p^{2-p}(p-1)^{p-1}(|y| - (p-1)^{-1}|x|)(|x|+|y|)^{p-1}.$$

The inequality (3.2) can be proved in the same manner as in the case $1 < p \le 2$; the majorization (3.3) was established by Burkholder on page 17 in [5].

4.5 The proof of (1.13)

We assume that $||X||_p < \infty$. We cannot proceed as in the proof of (1.9): the inequality $\mathbb{P}(|Y_\infty| \ge 1) \le \frac{p^{p-1}}{2}\mathbb{E}|X_\infty|^p$, the analogue of (4.5), is of no value to us. The problem is that if τ is a stopping time, then $(X_{\tau\wedge t}, Y_{\tau\wedge t})$ may no longer satisfy (1.11). To overcome this difficulty, we consider a stopping time $\tau = \inf\{t : |Y_t| \ge 1\}$ and show that

$$\mathbb{P}(|Y_\tau| \ge 1) \le \frac{p^{p-1}}{2}||X_\infty||_p^p.$$

This yields the claim: see the argumentation leading to (4.6) above. Introduce $V_{p,\infty} : \mathscr{H}\times\mathscr{H}\to\mathbb{R}$ and $U_{p,\infty} : \mathscr{H}\times\mathscr{H}\times[0,\infty)\to\mathbb{R}$ by

$$V_{p,\infty}(x,y) = I_{\{|y|\ge 1\}} - \frac{p^{p-1}}{2}|x|^p,$$
$$U_{p,\infty}(x,y,t) = \alpha_p\int_0^{1-p^{-1}} r^{p-1}u_\infty(x/r,y/r,t/r^2)\mathrm{d}r,$$

where $\alpha_p = p^p(p-1)^{2-p}(p-2)/4$. A little calculation shows that

$$U_{p,\infty}(x,y,t) = \frac{1}{2}\left(\frac{p}{p-1}\right)^{p-1}(|y| - (p-1)\sqrt{|x|^2+t})(\sqrt{|x|^2+t}+|y|)^{p-1},$$

if $\sqrt{|x|^2+t}+|y| \le 1-p^{-1}$, while for remaining (x,y,t),

$$U_{p,\infty}(x,y,t) = \frac{p^2}{4}\left[|y|^2 - |x|^2 - t - \frac{2(p-2)|y|}{p} + \frac{(p-1)^2(p-2)}{p^3}\right].$$

It is easy to check the analogue of (3.2), that is,

$$\mathbb{E}\int_0^{1-p^{-1}} r^{p-1}|u_\infty(X_t/r, Y_t/r, ([X,X]_\infty - [X,X]_t - [Y,Y]_\infty + [Y,Y]_t)/r^2)|\mathrm{d}r < \infty.$$

Use (2.11) with $\eta = \tau$ to get $\mathbb{E}U_{p,\infty}(Z_\tau) \leq 0$, where Z is given by (2.12). We have

$$\mathbb{E}(|X_\tau|^2 - [X,X]_\tau + [X,X]_\infty)I_{\{\tau<\infty\}} = \mathbb{E}|X_\infty|^2 I_{\{\tau<\infty\}} \quad \text{and} \quad [Y,Y]_\infty \geq [Y,Y]_\tau.$$

Therefore, since $|Y_\tau| \geq 1$ on $\{\tau < \infty\}$, we obtain

$$\begin{aligned}\mathbb{E}U_{p,\infty}(Z_\tau)I_{\{\tau<\infty\}} &\geq \frac{p^2}{4}\mathbb{E}\left[|Y_\tau|^2 - |X_\infty|^2 - \frac{2(p-2)|Y_\tau|}{p} + \frac{(p-1)^2(p-2)}{p^3}\right]I_{\{\tau<\infty\}} \\ &= \mathbb{E}U_{p,\infty}(X_\infty, Y_\tau, 0)I_{\{\tau<\infty\}}.\end{aligned}$$

Combining this with $\mathbb{E}U_{p,\infty}(Z_\tau)I_{\{\tau=\infty\}} = \mathbb{E}U_{p,\infty}(X_\infty, Y_\infty, 0)I_{\{\tau=\infty\}}$, we obtain

$$\mathbb{E}U_{p,\infty}(X_\infty, Y_\tau, 0) \leq \mathbb{E}U_{p,\infty}(Z_\tau) \leq 0$$

and the proof will be complete if we show that $U_{p,\infty}(x,y,0) \geq V_{p,\infty}(x,y)$. To this end, note that the function F given by

$$F(s) = \frac{1}{2}\left(\frac{p}{p-1}\right)^{p-1}(1-ps) + \frac{p^{p-1}}{2}s^p, \qquad s \in [0,1],$$

is nonnegative: indeed, it is convex and satisfies $F((p-1)^{-1}) = F'((p-1)^{-1}) = 0$. This gives the majorization for $|x| + |y| \leq 1 - p^{-1}$, since then it is equivalent to $F(|x|/(|x|+|y|)) \geq 0$. The next step is to show that $U_{p,\infty}(x,y,0) \geq V_{p,\infty}(x,y)$ for $|y| \geq 1$. For fixed x, the function $y \mapsto U_{p,\infty}(x,y,0)$ increases as $|y|$ increases, so it suffices to establish the bound for $|y| = 1$. After some manipulations, it reads

$$(p|x|)^p - 1 \geq \frac{p}{2}((p|x|)^2 - 1), \tag{4.7}$$

and follows from the mean value property of the convex function $t \mapsto t^{p/2}$. It remains to show the majorization for $|x| + |y| > 1 - p^{-1}$ and $|y| < 1$: it takes the form

$$\frac{p^2}{4}\left[|y|^2 - |x|^2 - \frac{2(p-2)|y|}{p} + \frac{(p-1)^2(p-2)}{p^3}\right] \geq -\frac{p^{p-1}|x|^p}{2}.$$

But this bound is valid for all x, $y \in \mathscr{H}$. Indeed, observe that as a function of $|y|$, the left-hand side attains its minimum for $|y| = 1 - 2/p$, and one easily checks that the inequality again reduces to (4.7). This completes the proof.

Acknowledgement

The author would like to express his gratitude to an anonymous referee, whose comments and suggestions greatly improved the paper.

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