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ON THE SPECTRUM OF SUM AND PRODUCT OF NON-HERMITIAN RANDOM MATRICES

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Abstract
In this note, we revisit the work of T. Tao and V. Vu on large non-hermitian random matrices with independent and identically distributed (i.i.d.) entries with mean zero and unit variance. We prove under weaker assumptions that the limit spectral distribution of sum and product of non-hermitian random matrices is universal. As a byproduct, we show that the generalized eigenvalues distribution of two independent matrices converges almost surely to the uniform measure on the Riemann sphere

1 Introduction

We start with some usual definitions. We endow the space of probability measures on $\mathbb{C}$ with the topology of weak convergence: a sequence of probability measures $(\mu_n)_{n\geq 1}$ *converges weakly* to μ is for any bounded continuous function $f : \mathbb{C} \to \mathbb{R}$,

$$\int f \, d\mu_n - \int f \, d\mu$$

converges to 0 as n goes to infinity. In this note, we shall denote this convergence by $\mu_n \underset{n\to\infty}{\rightsquigarrow} \mu$. Similarly, for two sequences of probability measures $(\mu_n)_{n\geq 1}, (\mu'_n)_{n\geq 1}$, we will use $\mu_n - \mu'_n \underset{n\to\infty}{\rightsquigarrow} 0$, or say that $\mu_n - \mu'_n$ tends weakly to 0, if

$$\int f \, d\mu_n - \int f \, d\mu'_n$$

converges to 0 for any bounded continuous function f. We will say that a measurable function $f : \mathbb{C} \to \mathbb{R}$ is *uniformly bounded* for $(\mu_n)_{n\geq 1}$ if

$$\limsup_{n\to\infty} \int |f| d\mu_n < \infty.$$

Finally, recall that a function f is *uniformly integrable* for $(\mu_n)_{n\geq 1}$ if

$$\lim_{t\to+\infty}\limsup_{n\to\infty}\int_{|f|\geq t}|f|d\mu_n=0.$$

The above definitions will also be used for probability measures on $\mathbb{R}_+=[0,\infty)$ and functions $f:\mathbb{R}_+\to\mathbb{R}$.
The *eigenvalues* of an $n\times n$ complex matrix M are the roots in $\mathbb{C}$ of its characteristic polynomial. We label them $\lambda_1(M),\dots,\lambda_n(M)$ so that $|\lambda_1(M)|\geq\cdots\geq|\lambda_n(M)|\geq 0$. We also denote by $s_1(M)\geq\cdots\geq s_n(M)\geq 0$ the *singular values* of M, defined for every $1\leq k\leq n$ by $s_k(M):=\lambda_k(\sqrt{MM^*})$ where M^* is the conjugate transpose of M. We define the empirical spectral measure and the empirical singular values measure as

$$\mu_M=\frac{1}{n}\sum_{k=1}^n\delta_{\lambda_k(M)}\quad\text{and}\quad\nu_M=\frac{1}{n}\sum_{k=1}^n\delta_{s_k(M)}.$$

Note that μ_M is a probability measure on $\mathbb{C}$ while ν_M is a probability measure on $\mathbb{R}_+$. The *generalized eigenvalues* of (M,N), two $n\times n$ complex matrices, are the zeros of the polynomial $\det(M-zN)$. If N is invertible, it is simply the eigenvalues of $N^{-1}M$.
Let $(X_{ij})_{i,j\geq 1}$ and $(Y_{ij})_{i,j\geq 1}$ be independent i.i.d. complex random variables with mean 0 and variance 1. Similarly, let $(G_{ij})_{i,j\geq 1}$ and $(H_{ij})_{i,j\geq 1}$ be independent complex centered gaussian variables with variance 1, independent of (X_{ij},Y_{ij}). We consider the random matrices $X_n=(X_{ij})_{1\leq i,j\leq n}$, $Y_n=(Y_{ij})_{1\leq i,j\leq n}$, $G_n=(G_{ij})_{1\leq i,j\leq n}$ and $H_n=(H_{ij})_{1\leq i,j\leq n}$. For ease of notation, we will sometimes drop the subscript n. It is known that almost surely (a.s.) for n large enough, X is invertible (see the forthcoming Theorem 11) and then $\mu_{X^{-1}Y}$ is a well defined random probability measure on $\mathbb{C}$.
Now, let μ be the probability measure whose density with respect to the Lebesgue measure on $\mathbb{C}\simeq\mathbb{R}^2$ is

$$\frac{1}{\pi(1+|z|^2)^2}.$$

Through stereographic projection, μ is easily seen to be the uniform measure on the Riemann sphere. Haagerup and several authors afterwards have independently observed the following beautiful identity (see Krishnapur [17], Rogers [21] and Forrester and Mays [7]).

Lemma 1 (Spherical ensemble). *For each integer $n\geq 1$,*

$$\mathbb{E}\mu_{G^{-1}H}=\mu.$$

By reorganizing the results of Tao and Vu [23, 24], we will prove a universality result.

Theorem 2 (Universality of generalized eigenvalues). *Almost surely,*

$$\mu_{X^{-1}Y}-\mu_{G^{-1}H}\underset{n\to\infty}{\rightsquigarrow}0.$$

Applying once Lemma 1 and twice Theorem 2, we get

Corollary 3 (Spherical law). *Almost surely*

$$\mu_{X^{-1}Y}\underset{n\to\infty}{\rightsquigarrow}\mu.$$

This statement was recently conjectured in [21, 7]. More generally, our argument also leads to the following universality result for sums and products of random matrices.

Theorem 4 (Universality of sum and product of random matrices). *For every integer n, let M_n, K_n, L_n be $n \times n$ complex matrices such that, for some $\alpha > 0$,*

(i) $x \mapsto x^{-\alpha}$ is uniformly bounded for $(\nu_{K_n})_{n\geq 1}$, and $(\nu_{L_n})_{n\geq 1}$ and $x \mapsto x^{\alpha}$ is uniformly bounded for $(\nu_{M_n})_{n\geq 1}$,

(ii) for almost all (a.a.) $z \in \mathbb{C}$, $\nu_{K_n^{-1}M_nL_n^{-1}-K_n^{-1}L_n^{-1}z}$ converges weakly to a probability measure ν_z.

Then, almost surely,

$$\mu_{M+KXL/\sqrt{n}} \underset{n\to\infty}{\rightsquigarrow} \mu,$$

where μ depends only $(\nu_z)_{z\in\mathbb{C}}$.

For $M = K = L = I$, the identity matrix, this statement gives the famous circular law theorem, that was established through a long sequence of partial results [19, 8, 10, 16, 6, 9, 1, 11, 2, 20, 12, 23, 24]. In this note, the steps of proof are elementary and they borrow all difficult technical statements from previously known results. Nevertheless, this theorem generalizes Theorem 1.18 in [24] in two directions. First, we have removed the assumption of uniformly bounded second moment for $\nu_{M+KXL/\sqrt{n}}$, $\nu_{K^{-1}ML^{-1}}$ and $\nu_{K^{-1}L^{-1}}$. Secondly, it proves the convergence of the spectral measure. The explicit form of μ in terms of ν_z is quite complicated. It is given by the forthcoming equations (2-3). This expression is not very easy to handle. However, following ideas developed in [21] or using tools from free probability as in [22, 14], it should be possible to find more elegant formulas. For nice examples of limit spectral distributions, see e.g. [21]. It is interesting to notice that we may deal with non-centered variables (X_{ij}) by including the average matrix of $X/\sqrt{n}$ into M, and recover [4]. Finally, as in [13], it is also possible by induction to apply Theorem 4 to product of independent copies of X (with the use of the forthcoming Theorem 8).

2 Proof of Theorem 2

2.1 Replacement Principle

The following is an extension of Theorem 2.1 in [24]. The idea goes back essentially to Girko.

Lemma 5 (Replacement principle). *Let A_n, B_n be $n \times n$ complex random matrices. Suppose that for a.a. $z \in \mathbb{C}$, a.s.*

(i) $\nu_{A_n-z} - \nu_{B_n-z}$ tends weakly to 0,

(ii) $\ln(\cdot)$ is uniformly integrable for $(\nu_{A_n-z})_{n\geq 1}$ and $(\nu_{B_n-z})_{n\geq 1}$.

Then a.s. $\mu_{A_n} - \mu_{B_n}$ tends weakly to 0. Moreover the same holds if we replace (i) by

(i') $\int \ln(x) d\nu_{A_n-z} - \int \ln(x) d\nu_{B_n-z}$ tends to 0.

Proof. It is a straightforward adaptation of [3, Lemma A.2]. □

Corollary 6. *Let A_n, B_n, M_n be $n \times n$ complex random matrices. Suppose that a.s. M_n is invertible and for a.a. $z \in \mathbb{C}$, a.s.*

(i) $\nu_{A_n-zM_n^{-1}} - \nu_{B_n-zM_n^{-1}}$ tends weakly to 0,

(ii) $\ln(\cdot)$ is uniformly integrable for $(\nu_{A_n-zM_n^{-1}})_{n\geq 1}$ and $(\nu_{B_n-zM_n^{-1}})_{n\geq 1}$.

Then a.s. $\mu_{M_nA_n} - \mu_{M_nB_n}$ tends weakly to 0.

Proof. If M_n is invertible, note that

$$\int \ln(x) d\nu_{M_nA_n - z} = \frac{1}{n} \ln|\det(M_nA_n - z)| = \int \ln(x) d\nu_{A_n - zM_n^{-1}} + \frac{1}{n} \ln|\det M_n|.$$

We may thus apply Lemma 5(i')-(ii). Indeed, in the expression $\int \ln(x) d\nu_{A_n-z} - \int \ln(x) d\nu_{B_n-z}$, the term $\frac{1}{n} \ln|\det M_n|$ cancels. □

2.2 Convergence of singular values

The following result is due to Dozier and Silverstein.

Theorem 7 (Convergence of singular values, [5]). *Let $(M_n)_{n\geq 1}$ be a sequence of $n \times n$ complex matrices such that ν_{M_n} converges weakly to a probability measure ν. Then a.s. $\nu_{X_n/\sqrt{n}+M_n}$ converges weakly to a probability measure ρ which depends only on ν.*

The measure ρ has an explicit characterization in terms of ν. Its exact form is not relevant here.

2.3 Uniform integrability

In order to use the replacement principle, it is necessary to prove the uniform integrability of $\ln(\cdot)$ for some empirical singular values measures. This is achieved by proving that, for some $\beta > 0$, $x \mapsto x^{-\beta} + x^{\beta}$ is uniformly bounded.

Theorem 8 (Uniform integrability). *Let $(M_n)_{n\geq 1}$ be a sequence of $n \times n$ complex matrices, and assume that $x \mapsto x^{\alpha}$ is uniformly bounded for $(\nu_{M_n})_{n\geq 1}$ for some $\alpha > 0$. Then there exists $\beta > 0$ such that a.s. $x \mapsto x^{-\beta} + x^{\beta}$ is uniformly bounded for $(\nu_{X_n/\sqrt{n}+M_n})_{n\geq 1}$.*

In the remainder of the paper, the notation $n \gg 1$ means *large enough* n. We start with an elementary lemma.

Lemma 9 (Large singular values). *Almost surely, for $n \gg 1$,*

$$\int x^2 d\nu_{X/\sqrt{n}} \leq 2.$$

Proof. We have $\frac{1}{n}\sum_{i=1}^{n} s_i^2(X/\sqrt{n}) = \frac{1}{n^2}\mathrm{tr} X^*X = \frac{1}{n^2}\sum_{1\leq i,j\leq n} |X_{ij}|^2$, and the latter converges a.s. to 1 by the law of large number. □

Corollary 10. *Let $0 < \alpha \leq 2$ and let $(M_n)_{n\geq 1}$ be a sequence of $n \times n$ complex matrices such that $x \mapsto x^{\alpha}$ is uniformly bounded for $(\nu_{M_n})_{n\geq 1}$. Then, a.s. $x \mapsto x^{\alpha}$ is uniformly bounded for $(\nu_{X_n/\sqrt{n}+M_n})_{n\geq 1}$.*

Proof. If M, N are $n \times n$ complex matrices, from [15, Theorem 3.3.16], for all $1 \leq i, j \leq n$ with $1 \leq i + j \leq n + 1$,

$$s_{i+j-1}(M+N) \leq s_i(M) + s_j(N).$$

Hence,

$$s_{2i}(M+N) \leq s_{2i-1}(M+N) \leq s_i(M) + s_i(N).$$

We deduce that for any non-decreasing function, $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $t > 0$,

$$\int f(x)d\nu_{M+N} \leq 2\int f(2x)d\nu_M + 2\int f(2x)d\nu_N,$$

where we have used the inequality

$$f(x+y) \leq f(2x) + f(2y).$$

Now, in view of Lemma 9, we may apply the above inequality to $f(x) = x^\alpha$ and deduce the statement. $\square$

The above corollary settles the problem of the uniform integrability of $\ln(\cdot)$ at $+\infty$ for $\nu_{X/\sqrt{n}+M}$. The uniform integrability at $0+$ is a much more delicate matter. The next theorem is a deep result of Tao and Vu.

Theorem 11 (Small singular values, [23, 24]). *Let $(M_n)_{n\geq 1}$ be a sequence of $n\times n$ complex invertible matrices such that $x \mapsto x^\alpha$ is uniformly bounded for $(\nu_{M_n})_{n\geq 1}$ for some $\alpha > 0$. There exist $c_1, c_0 > 0$ such that a.s. for $n \gg 1$,*

$$s_n(X_n/\sqrt{n} + M_n) \geq n^{-c_1}.$$

Moreover for $i \geq n^{1-\gamma}$ with $\gamma = 0.01$, a.s. for $n \gg 1$,

$$s_{n-i}(X_n/\sqrt{n} + M_n) \geq c_0 \frac{i}{n}.$$

Proof. The first statement is Theorem 2.1 in [23] and the second is contained in [24] (see the proof of Theorem 1.20 and observe that the statement of Proposition 5.1 remains unchanged if we consider a row of the matrix $X_n + \sqrt{n}M_n$). $\square$

Proof of Theorem 8.
By Corollary 10, it is sufficient to prove that $x \mapsto x^{-\beta}$ is uniformly bounded for $(\nu_{X/\sqrt{n}+M})$ and some $\beta > 0$. We have

$$\limsup_n \frac{1}{n}\sum_{i=1}^n s_i^{-\beta}(X/\sqrt{n}+M) < \infty,$$

By Theorem 11, we may a.s. write for $n \gg 1$,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n s_i^{-\beta}(X/\sqrt{n}+M) &\leq \frac{1}{n}\sum_{i=1}^{\lfloor n^{1-\gamma}\rfloor} n^{\beta c_1} + \frac{1}{n}\sum_{i=\lfloor n^{1-\gamma}\rfloor+1}^{n} c_2\left(\frac{n}{i}\right)^\beta \\ &\leq n^{\beta c_1 - \gamma} + \frac{1}{n}\sum_{i=1}^n c_2\left(\frac{n}{i}\right)^\beta .\end{aligned}$$

This last expression is uniformly bounded if $0 < \beta < \min(\gamma/c_1, 1)$. $\square$

2.4 End of proof of Theorem 2

If ρ is a probability measure on $\mathbb{C}\backslash\{0\}$, we define $\check{\rho}$ as the pull-back measure of ρ under $\phi : z \mapsto 1/z$, for any Borel E in $\mathbb{C}\backslash\{0\}$, $\check{\rho}(E) = \rho(\phi^{-1}(E))$. Obviously, if $(\rho_n)_{n\geq 1}$ is a sequence of probability measures on $\mathbb{C}\backslash\{0\}$, then ρ_n converges weakly to ρ is equivalent to $\check{\rho}_n$ converges weakly to $\check{\rho}$.

Note that by Theorem 8, a.s. for $n \gg 1$, X_n is invertible and $x \mapsto x^{-\beta} + x^{\beta}$ is uniformly bounded for $(\nu_{X_n/\sqrt{n}})_{n\geq 1}$. Also, from the quarter circular law theorem, $\nu_{X_n/\sqrt{n}}$ converges a.s. to a probability distribution with density

$$\frac{1}{\pi}\sqrt{4-x^2}\mathbb{1}_{[0,2]}(x),$$

(see Marchenko-Pastur theorem [18, 25, 26]). From the independence of $(X_{ij}),(Y_{ij}),(G_{ij}),(H_{ij})$, we may apply Corollary 6, Theorem 7 and Theorem 8 conditioned on (X_{ij}) to $M_n = zX_n/\sqrt{n}$. We get a.s.

$$\mu_{X^{-1}Y} - \mu_{X^{-1}H} \underset{n\to\infty}{\rightsquigarrow} 0.$$

By Theorem 11, a.s. for $n \gg 1$, $X^{-1}H$ and $G^{-1}H$ are invertible, it follows that

$$\mu_{X^{-1}H} - \mu_{G^{-1}H} \underset{n\to\infty}{\rightsquigarrow} 0 \quad \text{is equivalent to} \quad \check{\mu}_{X^{-1}H} - \check{\mu}_{G^{-1}H} \underset{n\to\infty}{\rightsquigarrow} 0.$$

However since $\mu_{MN} = \mu_{NM}$ and $\check{\mu}_M = \mu_{M^{-1}}$, we get

$$\mu_{X^{-1}H} - \mu_{G^{-1}H} \underset{n\to\infty}{\rightsquigarrow} 0 \quad \text{is equivalent to} \quad \mu_{H^{-1}X} - \mu_{H^{-1}G} \underset{n\to\infty}{\rightsquigarrow} 0.$$

The right hand side holds by applying again, Corollary 6, Theorem 7 and Theorem 8.

3 Proof of Theorem 4

3.1 Bounds on singular values

Lemma 12 (Singular values of sum and product). *If M,N are $n \times n$ complex matrices, for any $\alpha > 0$,*

$$\begin{aligned}\int x^{\alpha} d\nu_{M+N} &\leq 2^{1+\alpha}\left(\int x^{\alpha} d\nu_M + \int x^{\alpha} d\nu_N\right),\\ \int x^{\alpha} d\nu_{MN} &\leq 2\left(\int x^{2\alpha} d\nu_M\right)^{1/2}\left(\int x^{2\alpha} d\nu_N\right)^{1/2}.\end{aligned}$$

Proof. The first statement was already treated in the proof of Corollary 10. Also, from [15, Theorem 3.3.16], for all $1 \leq i,j \leq n$ with $1 \leq i+j \leq n+1$,

$$s_{i+j-1}(MN) \leq s_i(M)s_j(N).$$

Hence,

$$s_{2i}(MN) \leq s_{2i-1}(MN) \leq s_i(M)s_i(N).$$

We deduce

$$\int x^{\alpha} d\nu_{MN} \leq \frac{2}{n}\sum_{i=1}^{n} s_i^{\alpha}(M)s_i^{\alpha}(N).$$

We conclude by applying the Cauchy-Schwarz inequality. □

3.2 Logarithmic potential and Girko's hermitization method

We denote by $\mathscr{D}'(\mathbb{C})$ the set of Schwartz distributions endowed with its usual convergence with respect to all infinitely differentiable functions with bounded support. Let $\mathscr{P}(\mathbb{C})$ be the set of probability measures on $\mathbb{C}$ which integrate $\ln|\cdot|$ in a neighborhood of infinity. For every $\mu \in \mathscr{P}(\mathbb{C})$, the *logarithmic potential* U_μ of μ on $\mathbb{C}$ is the function $U_\mu : \mathbb{C} \to [-\infty, +\infty)$ defined for every $z \in \mathbb{C}$ by

$$U_\mu(z) = \int_{\mathbb{C}} \ln|z - z'|\,\mu(dz'),$$

(in classical potential theory, the definition is opposite in sign). Since $\ln|\cdot|$ is Lebesgue locally integrable on $\mathbb{C}$, one can check by using the Fubini theorem that U_μ is Lebesgue locally integrable on $\mathbb{C}$. In particular, $U_\mu < \infty$ a.e. (Lebesgue almost everywhere) and $U_\mu \in \mathscr{D}'(\mathbb{C})$. Since $\ln|\cdot|$ is the fundamental solution of the Laplace equation in $\mathbb{C}$, we have, in $\mathscr{D}'(\mathbb{C})$,

$$\Delta U_\mu = \pi \mu, \tag{1}$$

where Δ is the Laplace differential operator on $\mathbb{C}$ is given by $\Delta = \frac{1}{4}(\partial_x^2 + \partial_y^2)$.
We now state an alternative statement of Lemma 5 which is closer to Girko's original method, for a proof see [3, Lemma A.2].

Lemma 13 (Girko's hermitization method)**.** *Let A_n be a $n \times n$ complex random matrix. Suppose that for a.a. $z \in \mathbb{C}$, a.s.*

(i) $\nu_{A_n - z}$ *tends weakly to a probability measure ν_z on $\mathbb{R}_+$,*

(ii) $\ln(\cdot)$ *is uniformly integrable for $(\nu_{A_n-z})_{n\geq 1}$.*

Then there exists a probability measure $\mu \in \mathscr{P}(\mathbb{C})$ such that a.s.

(j) μ_{A_n} *converges weakly to μ*

(jj) for a.a. $z \in \mathbb{C}$,

$$U_\mu(z) = \int \ln(x)\, d\nu_z.$$

Moreover the same holds if we replace (i) by

(i') $\int \ln(x) d\nu_{A_n - z}$ *tends to* $\int \ln(x)\, d\nu_z$.

Corollary 14. *Let A_n, K_n, M_n be $n \times n$ complex random matrices. Suppose that a.s. K_n is invertible and* $\ln(\cdot)$ *is uniformly bounded for $(\nu_{K_n})_{n\geq 1}$, and for a.a. $z \in \mathbb{C}$, a.s.*

(i) $\nu_{A_n + K_n^{-1}(M_n - z)}$ *tends weakly to a probability measure ν_z,*

(ii) $\ln(\cdot)$ *is uniformly integrable for $(\nu_{A_n + K_n^{-1}(M_n - z)})_{n\geq 1}$.*

Then there exists a probability measure $\mu \in \mathscr{P}(\mathbb{C})$ such that a.s.

(j) $\mu_{M_n + K_n A_n}$ *converges weakly to μ,*

(jj) in $\mathscr{D}'(\mathbb{C})$,

$$\mu = \frac{1}{\pi} \Delta \int \ln(x)\, d\nu_z.$$

Proof. If K_n is invertible, we write

$$\begin{aligned}\int \ln(x) d\nu_{M_n+K_nA_n-z} &= \frac{1}{n}\ln|\det(A_n+K_n^{-1}(M_n-z))|+\frac{1}{n}\ln|\det K_n| \\ &= \int \ln(x) d\nu_{A_n+K_n^{-1}(M_n-z)}+\frac{1}{n}\ln|\det K_n|.\end{aligned}$$

By assumption, $\frac{1}{n}\ln|\det K_n| = \int \ln(x) d\nu_{K_n}$ is a.s. bounded. We may thus consider any converging subsequence and apply Lemma 5(i')-(ii) together with (1). □

3.3 End of proof of Theorem 4

We first notice that

$$\mu_{M+KXL/\sqrt{n}} = \mu_{LML^{-1}+LKX/\sqrt{n}}.$$

It is thus sufficient to prove that the right hand side converges. We set $\widetilde{M} = LML^{-1}$ and $\widetilde{K} = LK$. Since $\widetilde{K}^{-1}(\widetilde{M}-z) = K^{-1}ML^{-1} - K^{-1}L^{-1}z$, we may apply Lemma 12 and deduce that $x \mapsto x^{\alpha/4}$ is uniformly bounded for $(\nu_{\widetilde{K}_n(\widetilde{M}_n-z)})_{n\geq 1}$. It only remains to invoke Theorem 8 and Theorem 7 applied to $\widetilde{K}^{-1}(\widetilde{M}-z)$, and use Corollary 14 for $\widetilde{M}+\widetilde{K}X/\sqrt{n}$.

3.4 Explicit expression of the limit spectral measure

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, for a probability measure ρ on $\mathbb{R}$, its *Cauchy-Stieltjes transform* is defined as, for all $z \in \mathbb{C}_+$,

$$m_\rho(z) = \int \frac{1}{x-z} d\rho(x).$$

By Corollary 14 and Theorem 7, in $\mathscr{D}'(\mathbb{C})$,

$$\mu = \frac{1}{2\pi}\Delta \int \ln(x)\, d\rho_z(x), \tag{2}$$

where for $z \in \mathbb{C}$, ρ_z is a probability distribution on $\mathbb{R}_+$. From [5], for a.a. $z \in \mathbb{C}$, ρ_z has a Cauchy-Stieltjes transform that satisfies the integral equation: for all $w \in \mathbb{C}_+$,

$$m_{\rho_z}(w) = \int \frac{2x(1+m_{\rho_z}(w))}{x^2-(1+m_{\rho_z}(w))^2 w} d\nu_z(x), \tag{3}$$

where ν_z is as in Theorem 4.

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