A FAMILY OF EXCEPTIONAL PARAMETERS FOR NON-UNIFORM SELF-SIMILAR MEASURES

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Abstract

We present plane algebraic curves that have segments of points for which non uniform self-similar measures get singular. We calculate appropriate points on the curves using Mathematica. These points are in the parameter domain where we generically have absolute continuity of the measures, see [9, 11].

1 Introduction

The study of self-similar Borel probability measures on the real line is a classical subject in geometric measure theory. It is known that uniform self-similar, usually called Bernoulli convolutions, are generically absolutely continues with density in L^2, see [17, 14, 13], and are even equivalent to the Lebesgue measure [7]. On the other hand there is a result of Paul Erdös [3], that Bernoulli convolutions get singular for certain algebraic numbers (the reciprocals of Pisot numbers) and have Hausdorff dimension less than one [6].
We consider non-uniform self-similar measures on the real line, given by similitudes with different contraction rates. In [9, 11] we proved results on absolute continuity and density of these measures which hold generically in appropriate parameter domains. Moreover we proved the existence of certain plane algebraic curves with points such that the corresponding self-similar measures get singular and have Hausdorff dimension less than one in the domain of generic absolute continuity [10]. This result on singularity of non-uniform self similar measures is merely an existence theorem without examples of such curves and points on them.
Here we present a sequence of plane algebraic curves and prove that each of them has a segment of points for which non uniform self-similar measures get singular with respect to Lebesgue measure and have Hausdorff dimension less than one. In addition we use Mathematica to calculate points on algebraic curves of degree four, five and six. For these points in the domain of generic

absolute continuity self-similar measures are singular. We are able to compute upper estimates of the Hausdorff in this exceptional case. **Acknowledgment** Special thanks to Antonios Bisbas for valuable discussion and his help.

2   Rigorous results on non-uniform self-similar measures

For $\beta_1, \beta_2 \in (0,1)$ consider the contractions $T_1, T_2 : \mathbb{R} \longmapsto \mathbb{R}$ given by

$$T_1 x = \beta_1 x \qquad \text{and} \qquad T_2 x = \beta_2 x + 1.$$

These maps induce a contracting operated **T** with

$$\mathbf{T}(\mu) = 1/2(T_1\mu + T_2\mu) = 1/2 \cdot \mu \circ T_1^{-1} + 1/2 \cdot \mu \circ T_2^{-1}$$

on the compact metric space of Borel probability measures on an interval with the Prokhorov metric. By Banach's fixed point theorem there is an unique non-uniform self-similar Borel probability measure μ_{β_1,β_2} on the real line fulfilling

$$\mathbf{T}(\mu_{\beta_1,\beta_2}) = \mu_{\beta_1,\beta_2},$$

compare [5]. By decomposing this measure into singular and absolute continuous part and using uniqueness we immediately have that μ_{β_1,β_2} is of pure type, either singular ($\mu_{\beta_1,\beta_2} \perp \ell$) or absolutely continues ($\mu_{\beta_1,\beta_2} \ll \ell$) with respect to the Lebesgue measure ℓ. Moreover it is possible to show that absolute continuity of self-similar measures implies equivalence to Lebesgue measure ($\mu_{\beta_1,\beta_2} \sim \ell$), see [13]. To formulate the main result on properties of μ_{β_1,β_2} and for further use we remind the reader on the definition of Hausdorff dimension of a Borel measure μ,

$$\dim_H \mu = \inf\{\dim_H B | \mu(B) = 1\}$$

where the Hausdorff dimension of a set B is given by

$$\dim_H B = \inf\{d | H^d(B) = 0\} = \sup\{d | H^d(B) = \infty\}$$

and H^d is the d-dimensional Hausdorff measure.

$$H^d(B) = \lim_{\epsilon \longrightarrow \infty} \inf\{\sum_{U_i \in \mathfrak{U}} |U_i|^d \mid B \subseteq \bigcup_{U_i \in \mathfrak{U}} U_i, \ |U_i| \le \epsilon\}$$

For an introduction to dimension theory we recommend [4] and [15]. If $\dim_H \mu < 1$ we have that μ is singular with respect to Lebesgue measure. Dimension is hence a useful tool to prove singularity of measures. With these notations we state:

Theorem 2.1. *For $\beta_1, \beta_2 \in (0,1)$ with $\beta_1\beta_2 < 1/4$ the measure μ_{β_1,β_2} is singular with*

$$\dim_H \mu_{\beta_1,\beta_2} \le \frac{2\log 2}{\log \beta_1^{-1} + \log \beta_1^{-2}}.$$

It is absolutely continues with respect to Lebesgue measure for almost all $\beta_1, \beta_2 \in (0, 0.649)$ with $\beta_1\beta_2 \ge 1/4$.

We proved this result in [9], and independently it was proved a bit later by Ngau and Wang [8]. The upper bound 0.649 is due to the techniques of the proof. Computational results show that the bound can be extended to 0.66, see [16]. We conjecture that it can be replaced by one, but this conjecture seems to be difficult to prove.

Also theorem 2.1 leaves the challenging question open, if there are exceptional pairs $(\beta_1, \beta_2) \in (0,1)$ with $\beta_1 \beta_2 > 1/4$ such the μ_{β_1, β_2} is singular. In fact a classical result of Paul Erdös [3] shows that the Bernoulli convolutions, which are uniform self-similar measures $\mu_{\beta,\beta}$, are singular if $\beta \in (0.5, 1)$ is the reciprocal of a Pisot number (an algebraic integer with all its Galois conjugates inside the untie circle). Examples of such numbers are the real solutions of the algebraic equations

$$f_n(x) = -1 + \sum_{i=1}^{n} x^i = 0$$

for $n \geq 2$. We consider here similar algebraic curves given by

$$F_n(x, y) = x\left(-1 + \sum_{i=1}^{n} y^i\right) + y\left(-1 + \sum_{i=1}^{n} x^i\right) = 0$$

for $n \geq 2$. Note that F is symmetric $F_n(x, y) = F_n(y, x)$ and $F_n(x, y) = x f_n(y) + y f_n(x)$ especially $F_n(x, x) = 2x f_n(x)$. We prove the following analogue of Erdös result in the last section of this paper

Theorem 2.2. *For all $n \geq 4$ there are segments of the algebraic curve given by $F_n(x, y) = 0$ in $(0, 1)^2$ with $xy > 1/4$ such that the non uniform self-similar measure $\mu_{x,y}$ are singular with $\dim_H \mu_{x,y} \leq D_{x,y}^n < 1$ where*

$$D_{x,y}^n := \frac{2 \log \sqrt[n]{2^n - 1}}{\log x^{-1} + \log y^{-1}}$$

We conjecture that the result on singularity is as well true for $n = 2$ and $n = 3$. In this case the curve $F_n(x, y) = 0$ in the square $(0, 1)^2$ is far away form the critical curve $xy = 1/4$. The dimension estimate we use to prove singularity is not strong enough in this situation.

In the next section we use Mathemmatica to compute appropriate points on the curves for $n = 4, 5, 6$ and estimate the Hausdorff dimension of self-similar measures for these points.

3   Computational results

We use Mathematica to compute solutions $(x, y) \in (0, 1)^2$ of the equations $F_n(x, y) = 0$ for $x \in \{0.001t \mid t = 0 \ldots 500\}$ and checked computational whether $xy > 1/4$ and $D_{x,y}^n < 1$. As mentioned above for $n = 2$ and $n = 3$ there are no solutions with this condition. Our computational result on appropriate solutions of $F_4(x, y) = 0$ are summarized in the following table:

x	y	xy	$D_{x,y}^4$
0.298	0.839132	0.250061	0.991015
0.299	0.837024	0.25027	0.991612
0.300	0.834924	0.250477	0992204
0.301	0.832833	0.250683	0.992793
0.302	0.830751	0.250887	0.993377
0.303	0.828678	0.251089	0.993957
0.304	0.826613	0.251290	0.994533
0.305	0.824557	0.251490	0.995104
0.306	0.822509	0.251688	0.995672
0.307	0.82047	0.251884	0.996236
0.308	0.818439	0.252079	0.996795
0.309	0.818439	0.252273	0.997350
0.310	0.816417	0.252465	0.997902
0.311	0.814403	0.252655	0.998449
0.312	0.812397	0.252845	0.998992
0.313	0.810399	0.253071	0.999532

Table I

Note that the points given here are in the domain where we conjecture generic absolute continuity of self-similar measures. Points on the curve $F_5(x, y) = 0$ satisfying $xy > 1/4$ and $D_{x,y}^n < 1$ are in the domain where we have generic absolute continuity by theorem 2.1.

x	y	xy	$D_{x,y}^5$
0.385	0.649720	0.250142	0.991248
0.386	0.648418	0.2502893	0.991666
0.387	0.647108	0.250430	0.992071
0.388	0.645809	0.250573	0.992480
0.389	0.644526	0.250720	0.992906
0.390	0.643318	0.250894	0.993397
0.391	0.641943	0.250999	0.993699
0.392	0.640649	0.251134	0.994085
0.393	0.639365	0.251270	0.994475
0.394	0.638090	0.251407	0.994868
0.395	0.636806	0.251538	0.995243

Table II

Our results on appropriate points on the curve $F_6(x, y) = 0$ are contained in the following table:

x	y	xy	$D_{x,y}^6$
0.427	0.585485	0.250002	0.998390
0.428	0.584372	0.250111	0.998704
0.429	0.583260	0.250219	0.999014
0.430	0.582151	0.250325	0.999319
0.431	0.581042	0.250429	0.999612
0.432	0.579936	0.250526	0.999918

Table III

Again the points are in the domain of generic absolute continuity of self-similar measures.

4   Proof of the result

Define a map π from $\{1,2\}^{\mathbb{N}}$ to $\mathbb{R}$ by

$$\pi(s) := \sum_{k=1}^{\infty} (s_k - 1) \beta_1^{1_{k-1}(s)} \beta_2^{2_{k-1}(s)}$$

where

$$1_k(s) = Card\{i | s_i = 1 \text{ for } i = 1 \ldots k\},$$
$$2_k(s) = Card\{i | s_i = 2 \text{ for } i = 1 \ldots k\}.$$

Let b be the Bernoulli measure $(1/2, 1/2)$ on $\{1,2\}^{\mathbb{N}}$. We have $\mu_{\beta_1,\beta_2} = \pi(b)$ since $\pi(b)$ fulfillers the self-similarity relation defining μ_{β_1,β_2}, see [9]. Furthermore define π_n by the finite sums

$$\pi_n(s) := \sum_{k=1}^{n} (s_k - 1) \beta_1^{1_{k-1}(s)} \beta_2^{2_{k-1}(s)}.$$

For our purpose here we need the following proposition very similar to theorem 3.1 of [10]

Proposition 4.1. *If $\pi_n(s) = \pi_n(t)$ for finite sequences $s, t \in \{1,2\}^n$ with $s \neq t$ we have*

$$\dim_H \mu_{\beta_1,\beta_2} \leq \frac{2 \log \sqrt[n]{2^n - 1}}{\log \beta_1^{-1} + \log \beta_2^{-1}}$$

Proof. Define a sequence of partitions P_m of $\{1,2\}^{\mathbb{N}}$ by $\pi_m(s) = \pi_m(t)$. Let $P_m(s)$ be the partition element that contains the sequence s and let $H(P_m)$ be the entropy of this partition. Using lemma 4 of [6] based on Shannon's local entropy theorem we get

$$\liminf_{k \longrightarrow \infty} -\frac{1}{k} \log b(P_{nk}(s)) \leq H(P_n)$$

for almost all sequences s with respect to the Bernoulli measure b. Note that by the assumption $\pi_n(s) = \pi_n(t)$ we have $H(P_n) \leq \log(2^n - 1)$.
Now define intervals by $I_m(s) := \pi(P_m(s))$ and let $|I_m(s)| = C\beta_1^{1_m(s)} \beta_2^{2_m(s)}$ be their length. Since $P_m(s) \subseteq \pi^{-1}(I_m(s))$ we have $\mu_{\beta_1,\beta_2}(I_m(s)) \geq b(P_m(s))$ and get

$$\liminf_{k \longrightarrow \infty} -\frac{1}{nk} \log \mu_{\beta_1,\beta_2}(I_{nk}(s)) < \log \sqrt[n]{2^n - 1}$$

for almost all s. On the other hand using Birkhoff's ergodic theorem we get

$$\lim_{k \longrightarrow \infty} -\frac{1}{nk} |I_{nk}(s)| = \frac{1}{2}(\log \beta_1^{-1} + \log \beta_2^{-1})$$

for almost all s. Hence

$$\liminf_{k \longrightarrow \infty} \frac{\mu_{\beta_1,\beta_2}(I_{nk}(s))}{|I_{nk}(s)|} \leq \frac{2 \log \sqrt[n]{2^n - 1}}{\log \beta_1^{-1} + \log \beta_2^{-1}}$$

for almost all s. Now the result follows by Frostman's lemma, see lemma 1 of [6]. $\square$

Now we are prepared to proof the main result of this paper:

Proof of theorem 2.2. Applying proposition 4.1 for sequences $s = (1, 2\ldots, 2)$ and $t = (2, 1, \ldots 1)$ of length n we conclude that the relation

$$F_n(x, y) = x(-1 + \sum_{i=1}^{n} y^i) + y(-1 + \sum_{i=1}^{n} x^i) = 0$$

implies

$$\dim_H \mu_{x,y} \leq \frac{2 \log \sqrt[n]{2^n - 1}}{\log x^{-1} + \log y^{-1}}.$$

Hence to prove theorem 2.2 we have to show that there is a segment of the curve given by $F_n(x, y) = 0$ in $(0, 1)^2$ such that

$$\frac{1}{4} < xy < \frac{1}{2\sqrt[n]{2^n - 1}} \tag{1}$$

for the points (x, y) of this segment. Consider

$$C_n = \{(x, y) \in (0, 1)^2 | F_n(x, y) = 0\}$$

We first show that C_n is a smooth curve in $(0, 1)^2$. If (x, y) is a singular point of $F_n(x, y)$ we have

$$\frac{\partial F_n}{\partial x}(x, y) = \frac{\partial F_n}{\partial y}(y, x) = y f_n'(x) + f_n(y) = 0$$

with

$$f_n(x) = -1 + \sum_{i=1}^{n} x^i.$$

Since $F_n(x, y) = 0$ this implies $xy f_n'(x) + xy f_n'(y) = 0$ and if $xy \neq 0$ we get $f'(x) = -f'(y)$. This gives

$$-x f_n'(x) + f_n(x) = -1 + \sum_{i=1}^{n} (1 - i)x^i = 0$$

which obviously does not hold for $x > 0$. Hence there are no singular points in $(0, 1)^2$, which implies that C_n is smooth and especially continuous, see [2]. Now we show that C_n intersects the graphs of

$$\underline{f}(x) = \frac{1}{4x} \quad \text{and} \quad \overline{f}(x) = \frac{1}{2\sqrt[n]{2^n - 1}x}.$$

To this end let

$$\underline{s}_n(x) = F_n(x, \frac{1}{4x}) \quad \text{and} \quad \overline{s}_n(x) = F_n(x, \frac{1}{2\sqrt[n]{2^n - 1}x})$$

Than for $n \geq 4$

$$\underline{s}_n(1) = \frac{n}{4} - 1 + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} > 0$$

and

$$\underline{s}_n(\frac{1}{2}) = -1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} < 0$$

By intermediate value theorem the curve C_n intersects the graph of $\underline{f}_n$. Furthermore

$$\bar{s}_n(1) = \frac{n}{2\sqrt[n]{2^n - 1}} - 1 + \frac{1}{(2\sqrt[n]{2^n - 1})^2} + \cdots + \frac{1}{(2\sqrt[n]{2^n - 1})^n} > 0$$

and using the geometric sum formula we get

$$\bar{s}_n(\frac{1}{2}) = -\frac{1}{2^n} \frac{1}{\sqrt[n]{2^n - 1}} + \frac{1}{2} \frac{2 - \sqrt[n]{2^n - 1} - \frac{1}{2^n - 1}}{\sqrt[n]{2^n - 1} - 1}.$$

Now a simple estimate gives $\bar{s}_n(\frac{1}{2}) < 0$. Again by intermediate value theorem the curve C_n intersects the graph of $\overline{f}_n$. Since $\overline{f}(x) > \underline{f}(x)$ for $x \in (0,1)$ and C_n is continuous in $(0,1)^2$, our results on the intersections imply that there exists curve segments $S_n \subseteq C_n$ such that for all $(x, y) \in S_n$ relation (1) is satisfied. This completes the proof. □

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