

PARAMETRIX TECHNIQUES AND MARTINGALE PROBLEMS FOR SOME DEGENERATE KOLMOGOROV EQUATIONS

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Abstract

We prove the uniqueness of the martingale problem associated to some degenerate operators. The key point is to exploit the strong parallel between the new technique introduced by Bass and Perkins [2] to prove uniqueness of the martingale problem in the framework of non-degenerate elliptic operators and the Mc Kean and Singer [13] parametrix approach to the density expansion that has previously been extended to the degenerate setting that we consider (see Delarue and Menozzi [3]).

1 Introduction

1.1 Martingale problem and parametrix techniques

The martingale approach turns out to be particularly useful when trying to get uniqueness results for the stochastic process corresponding to an operator. In a recent work, R. Bass and E. Perkins [2] introduced in the framework of non-degenerate, non-divergence, time-homogeneous operators a new technique to prove uniqueness of the associated martingale problem. Precisely, for an operator of the form

$$Lf(x) = \frac{1}{2} \text{Tr}(a(x)D_x^2 f(x)), \quad f \in C_0^2(\mathbb{R}^d, \mathbb{R}), \quad x \in \mathbb{R}^d, \quad (1.1)$$

the authors prove uniqueness provided a is uniformly elliptic, bounded and uniformly η -Hölder continuous in space ($\eta \in (0, 1]$), i.e. there exists $C > 0$ s.t. for all $(x, y) \in \mathbb{R}^d$, $|a(x) - a(y)| \leq C(1 \wedge |x - y|^\eta)$. That is, for a given starting point $x \in \mathbb{R}^d$, there exists a unique probability measure \mathbb{P} on $C(\mathbb{R}^+, \mathbb{R}^d)$ s.t. denoting by $(X_t)_{t \geq 0}$ the canonical process, $\mathbb{P}[X_0 = x] = 1$ and for every $f \in C_0^2(\mathbb{R}^d, \mathbb{R})$, $f(X_t) - f(x) - \int_0^t Lf(X_s)ds$ is a \mathbb{P} -martingale.

In the indicated framework, this result can be derived from the more involved Calderón-Zygmund like L^p estimates established by Stroock and Varadhan [17], that only require continuity of the

diffusion matrix a , or from a more analytical viewpoint from some appropriate Schauder estimates, see e.g. Friedman [5].

Anyhow, the technique introduced in [2] can be related with the first step of Gaussian approximation of the parametrix expansion of the fundamental solution of (1.1) developed by McKean and Singer [13] that we now shortly describe. Suppose first that, additionally to the previous assumptions of ellipticity, boundedness and uniform Hölder continuity, the diffusion coefficient a is smooth (say $C^\infty(\mathbb{R}^d, \mathbb{R})$). Thus, the fundamental solution $p(s, t, x, y)$ of (1.1) exists and is smooth for $t > s$, see e.g [5]. Precisely, we have:

$$\partial_t p(s, t, x, y) = L^* p(s, t, x, y), \quad t > s, \quad (x, y) \in (\mathbb{R}^d)^2, \quad p(s, t, x, \cdot) \xrightarrow[t \downarrow s]{} \delta_x(\cdot),$$

where L^* stands for the adjoint of L and acts on the y variable. For fixed starting and final points $x, y \in \mathbb{R}^d$ and a given final time $t > 0$, in order to estimate $p(0, t, x, y)$, one introduces the Gaussian process $\tilde{X}_u^y = x + \sigma(y)W_{u-s}$, $u \in [s, t], s \leq t$, where $(W_u)_{u \in [0, t-s]}$ is a standard d -dimensional Brownian motion and $\sigma\sigma^*(y) = a(y)$. Observe that the coefficient of \tilde{X}^y is frozen here at the point where we consider the density. Denote by $\tilde{p}^y(s, t, x, \cdot)$ the density of \tilde{X}^y at time t starting from x at time s , and for $\varphi \in C_0^2(\mathbb{R}^d, \mathbb{R})$, define by $\tilde{L}^y \varphi(x) = \frac{1}{2} \text{Tr}(a(y)D_x^2 \varphi(x))$ its generator. The density $\tilde{p}^y(s, t, x, \cdot)$ satisfies the Kolmogorov equation:

$$\partial_s \tilde{p}^y(s, t, x, z) = -\tilde{L}^y \tilde{p}^y(s, t, x, z), \quad s < t, \quad (x, z) \in (\mathbb{R}^d)^2, \quad \tilde{p}^y(s, t, \cdot, z) \xrightarrow[s \uparrow t]{} \delta_z(\cdot),$$

where \tilde{L}^y acts here on the x variable. Take now $z = y$ in the above equation. By formal derivation and the previous Kolmogorov equations we obtain:

$$\begin{aligned} p(0, t, x, y) &- \tilde{p}^y(0, t, x, y) = \int_0^t ds \partial_s \int_{\mathbb{R}^d} p(0, s, x, w) \tilde{p}^y(s, t, w, y) dw \\ &= \int_0^t ds \int_{\mathbb{R}^d} (L^* p(0, s, x, w) \tilde{p}^y(s, t, w, y) - p(0, s, x, w) \tilde{L}^y \tilde{p}^y(s, t, w, y)) dw \\ &= \int_0^t ds \int_{\mathbb{R}^d} p(0, s, x, w) (L - \tilde{L}^y) \tilde{p}^y(s, t, w, y) dw \\ &= \int_0^t ds \int_{\mathbb{R}^d} p(0, s, x, w) H(s, t, w, y) dw := p \otimes H(0, t, x, y), \end{aligned} \tag{1.2}$$

where \otimes denotes a time-space convolution. Observe that $H(s, t, w, y) := (a(w) - a(y)) \times D_w^2 \tilde{p}^y(s, t, w, y)$. From direct computations, there exist $c, C > 0$ (depending on d , the uniform ellipticity constant and the L_∞ bound of a) s.t. $|D_w^2 \tilde{p}^y(s, t, w, y)| \leq \frac{C}{(t-s)^{d/2+1}} \exp(-c \frac{|w-y|^2}{t-s})$. The previous uniform Hölder continuity assumption on a is therefore a sufficient (and quite sharp) condition to remove the time-singularity in H . The idea of the parametrix expansion is then to proceed in (1.2) by applying the same freezing technique to $p(0, s, x, w)$ introducing the density $\tilde{p}^w(0, s, x, \cdot)$ of the process with coefficients frozen at point w . One eventually gets the formal expansion

$$p(0, t, x, y) = \tilde{p}(0, t, x, y) + \sum_{k \geq 1} \tilde{p} \otimes H^{\otimes k}(0, t, x, y), \tag{1.3}$$

where $H^{\otimes k}$, $k \geq 1$, stands for the iterated convolutions of H , and $\forall (s, t, z, y) \in (\mathbb{R}^+)^2 \times \mathbb{R}^d$, $\tilde{p}(s, t, z, y) := \tilde{p}^y(s, t, z, y)$. The Hölder continuity gives that H is a "smoothing" kernel in the sense

that there exist $c, C > 0$ (with the same previous dependence) s.t.

$$|\tilde{p} \otimes H^{\otimes k}(s, t, z, y)| \leq C^{k+1}(t-s)^{k\eta/2} \prod_{i=1}^{k+1} B\left(1 + \frac{(i-1)\eta}{2}, \frac{\eta}{2}\right) (t-s)^{-d/2} \exp(-c \frac{|y-z|^2}{t-s}),$$

$B(m, n) = \int_0^1 s^{m-1}(1-s)^{n-1} ds$ standing for the β function. From this estimate, equation (1.3) and the asymptotics of the β function, one directly gets the Gaussian upper bound over a compact time interval. Namely for all $T > 0$, there exist constants $c, C > 0$ s.t.

$$\forall 0 \leq s < t \leq T, \forall (x, y) \in (\mathbb{R}^d)^2, p(s, t, x, y) \leq \frac{C}{(t-s)^{d/2}} \exp(-c \frac{|y-x|^2}{t-s}), \tag{1.4}$$

with c, C depending on d , the uniform ellipticity constant and L_∞ bound of a and C depending on T as well. We refer to Konakov and Mammen [9] for details in this framework.

Up to now we supposed a was smooth in order to guarantee the existence of the density and justify the formal derivation in (1.2). On the other hand, the r.h.s. of (1.3) can be defined without additional smoothness on a than uniform η -Hölder continuity. The Gaussian upper bound (1.4) also only depends on the Hölder regularity of a . A natural question is to know whether the r.h.s. of (1.3) corresponds to the density of some stochastic differential equation under the sole assumptions of uniform ellipticity, boundedness and Hölder continuity on a . A positive answer is given by the uniqueness of the martingale problem associated to (1.1). Indeed, considering a sequence of equations with mollified coefficients, we derive from convergence in law, the Radon-Nikodym theorem and (1.4) that the unique weak solution of $dX_t = \sigma(X_t)dW_t$ associated to L admits a density that satisfies the previous Gaussian bound. It is actually remarkable that the uniqueness of the martingale problem can be proved using exactly the smoothing properties of the previous kernel H . That is what was achieved by Bass and Perkins [2] in the framework we described and it is the main purpose of this note in a degenerate setting.

To conclude this paragraph, let us emphasize that the previous parametrix approach has been used in various contexts. It turns out to be particularly well suited to the approximation of the underlying processes by Markov chains, see Konakov and Mammen [9, 10] for the non-degenerate continuous case or [11] for the approximation of stable driven SDEs. On the other hand, recently, we used this technique to give a local limit theorem for the Markov chain approximation of a Langevin process [12] or two-sided bounds of some more general degenerate hypoelliptic operators [3]. In particular, in both works, we have an unbounded drift term. The unboundedness of the first order term imposes a more subtle strategy than the previous one for the choice of the frozen Gaussian density. Namely, one has to take into consideration in the frozen process the “geometry” of the deterministic differential equation associated to the first order terms of the operator. This will be thoroughly explained in the next section. Anyhow, the strategy of the previous articles allows to extend the technique of Bass and Perkins to prove uniqueness of the martingale problem for some degenerate operators with unbounded coefficients.

1.2 Statement of the Problem and Main Results

Consider the following system of Stochastic Differential Equations (SDEs in short)

$$\begin{aligned} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + \sigma(t, X_t^1, \dots, X_t^n)dW_t, \\ dX_t^2 &= F_2(t, X_t^1, \dots, X_t^n)dt, \\ dX_t^3 &= F_3(t, X_t^2, \dots, X_t^n)dt, \\ &\dots \\ dX_t^n &= F_n(t, X_t^{n-1}, X_t^n)dt, \end{aligned} \tag{1.5} \quad t \geq 0,$$

$(W_t)_{t \geq 0}$ standing for a d -dimensional Brownian motion, and each $(X_t^i)_{t \geq 0}$, $1 \leq i \leq n$, being \mathbb{R}^d -valued as well.

From the applicative viewpoint, systems of type (1.5) appear in many fields. Let us for instance mention for $n = 2$ stochastic Hamiltonian systems (see e.g. Soize [16] for a general overview or Talay [18] and Hérau and Nier [6] for convergence to equilibrium). Again for $n = 2$, the above dynamics is used in mathematical finance to price Asian options (see for example [1]). For $n \geq 2$, it appears in heat conduction models (see e.g. Eckmann et al. [4] and Rey-Bellet and Thomas [15] when the chain is forced by two heat baths).

In what follows, we denote a quantity in \mathbb{R}^{nd} by a bold letter: i.e. $\mathbf{0}$, stands for zero in \mathbb{R}^{nd} and the solution $(X_t^1, \dots, X_t^n)_{t \geq 0}$ to (1.5) is denoted by $(\mathbf{X}_t)_{t \geq 0}$. Introducing the embedding matrix B from \mathbb{R}^d into \mathbb{R}^{nd} , i.e. $B = (I_d, 0, \dots, 0)^*$, where “ $*$ ” stands for the transpose, we rewrite (1.5) in the shortened form

$$d\mathbf{X}_t = \mathbf{F}(t, \mathbf{X}_t) + B\sigma(t, \mathbf{X}_t)dW_t,$$

where $\mathbf{F} = (F_1, \dots, F_n)$ is an \mathbb{R}^{nd} -valued function. Moreover, for $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, we set $\mathbf{x}^{i,n} = (x_i, \dots, x_n) \in (\mathbb{R}^d)^{n-i+1}$.

We introduce the following assumptions:

(R- η) The functions $(F_i)_{i \in \llbracket 1, n \rrbracket}$ are uniformly Lipschitz continuous with constant $\kappa > 0$ (alternatively we can suppose for $i = 1$ that the drift of the non degenerated component F_1 is measurable and bounded by κ). The diffusion matrix $(a(t, \cdot))_{t \geq 0}$ is uniformly η -Hölder continuous in space with constant κ , i.e.

$$\forall t \geq 0, \sup_{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{nd}, \mathbf{x} \neq \mathbf{y}} \frac{|a(t, \mathbf{x}) - a(t, \mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\eta} \leq \kappa.$$

(UE) There exists $\Lambda \geq 1$, $\forall t \geq 0, \mathbf{x} \in \mathbb{R}^{nd}$, $\xi \in \mathbb{R}^d$, $\Lambda^{-1}|\xi|^2 \leq \langle a(t, \mathbf{x})\xi, \xi \rangle \leq \Lambda|\xi|^2$.

(ND- η) For each integer $2 \leq i \leq n$, $(t, (x_i, \dots, x_n)) \in \mathbb{R}_+ \times \mathbb{R}^{(n-i+1)d}$, the function $x_{i-1} \in \mathbb{R}^d \mapsto F_i(t, \mathbf{x}^{i-1, n})$ is continuously differentiable. Also, the derivative $(t, \mathbf{x}^{i-1, n}) \in \mathbb{R}_+ \times \mathbb{R}^{(n-i+2)d} \mapsto D_{x_{i-1}} F_i(t, \mathbf{x}^{i-1, n})$, is η -Hölder continuous with constant κ . There exists a closed convex subset $\mathcal{E}_{i-1} \subset GL_d(\mathbb{R})$ (set of invertible $d \times d$ matrices) s.t., for all $t \geq 0$ and $(x_{i-1}, \dots, x_n) \in \mathbb{R}^{(n-i+2)d}$, the matrix $D_{x_{i-1}} F_i(t, \mathbf{x}^{i-1, n})$ belongs to \mathcal{E}_{i-1} . For example, \mathcal{E}_i , $1 \leq i \leq n - 1$, may be a closed ball included in $GL_d(\mathbb{R})$, which is an open set.

Assumptions **(UE)**, **(ND- η)** can be seen as a kind of (weak) Hörmander condition. They allow to transmit the non degenerate noise of the first component to the other ones. Also, the particular structure of $F(t, \cdot) = (F_1(t, \cdot), \dots, F_n(t, \cdot))$ yields that the i^{th} component has intrinsic time scale $(2i - 1)/2$, $i \in \llbracket 1, n \rrbracket$. We notice that the coefficients may be irregular in time. The last part of Assumption **(ND- η)** will be explained in Section 2.1. We say that assumption **(A- η)** is satisfied if **(R- η)**, **(UE)**, **(ND- η)** hold.

Under **(A- η)**, we established in [3] Gaussian Aronson like estimates for the density of (1.5) over compact time interval $[0, T]$, for $\eta > 1/2$. Precisely, we proved that the unique weak solution of (1.5) admits a density that satisfies that for all $T > 0, \exists C := C(T, \mathbf{A-}\eta)$ s.t. $\forall (t, \mathbf{x}, \mathbf{y}) \in (0, T] \times \mathbb{R}^{nd} \times \mathbb{R}^{nd}$:

$$C^{-1} t^{-n^2 d/2} \exp\left(-C t |\mathbb{T}_t^{-1}(\boldsymbol{\theta}_t(\mathbf{x}) - \mathbf{y})|^2\right) \leq p(t, \mathbf{x}, \mathbf{y}) \leq C t^{-n^2 d/2} \exp\left(-C^{-1} t |\mathbb{T}_t^{-1}(\boldsymbol{\theta}_t(\mathbf{x}) - \mathbf{y})|^2\right), \tag{1.6}$$

where $\mathbb{T}_t := \text{diag}((t^i I_d)_{i \in \llbracket 1, n \rrbracket})$ is a scale matrix and $\dot{\boldsymbol{\theta}}_t(x) = \mathbf{F}(t, \boldsymbol{\theta}_t(\mathbf{x})), \boldsymbol{\theta}_0(\mathbf{x}) = \mathbf{x}$.

To derive (1.6), we proceeded using a “formal” parametrix expansion considering a sequence of equations with smooth coefficients for which Hörmander’s theorem guaranteed the existence of the density, see. e.g. Hörmander [7] or Norris [14]. Anyhow, as in the previous paragraph, our estimates did not depend on the derivatives of the mollified coefficients but only on the η -Hölder continuity assumed in $(\mathbf{A}\text{-}\eta)$. Anyhow, to pass to the limit following the previously described procedure, some uniqueness in law is needed. Using the comparison principle for viscosity solutions of fully non-linear PDEs, see Ishii and Lions [8], we managed to obtain the bounds under $(\mathbf{A}\text{-}\eta)$, $\eta > 1/2$. However, the viscosity approach totally ignores the smoothing effects of the heat kernel and is not a “natural technique” to derive uniqueness in law.

Introduce the generator of (1.5): $\forall \varphi \in C_0^2(\mathbb{R}^+ \times \mathbb{R}^{nd}, \mathbb{R}), \forall (t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^{nd}$,

$$L_t \varphi(t, \mathbf{x}) = \langle \mathbf{F}(t, \mathbf{x}), \mathbf{D}_x \varphi(t, \mathbf{x}) \rangle + \frac{1}{2} \text{Tr}(a(t, \mathbf{x}) D_{x_1}^2 \varphi(t, \mathbf{x})). \quad (1.7)$$

Adapting the technique of Bass and Perkins [2] we obtain the following results.

Theorem 1.1. *For $\eta \in (0, 1]$, under $(\mathbf{A}\text{-}\eta)$ the martingale problem associated to L in (1.7) is well-posed. In particular, weak uniqueness in law holds for the SDE (1.5).*

As a by-product we derive from [3] the following:

Corollary 1.1. *For η in $(0, 1]$, under $(\mathbf{A}\text{-}\eta)$, the unique weak solution of (1.5) admits for all $t > 0$ a density that satisfies the Aronson like bounds of equation (1.6).*

Remark 1.1. *Let us mention that Theorem 1.1 remains valid if the coefficients are Dini continuous (as in [2]).*

2 Choice of the Gaussian process for the parametrix

In this section we describe the Gaussian processes that will be involved in the study of the martingale problem and that have been previously involved in the parametrix expansions of [3]. We first introduce in Section 2.1 a class of degenerate linear stochastic differential equations that admit a density satisfying bounds similar to those of equation (1.6). We then specify, how to properly linearize the dynamics of (1.5) so that the linearized equations belong to the class considered in Section 2.1.

2.1 Some estimates on degenerate Gaussian processes with linear drift

Introduce the stochastic differential equation:

$$d\mathbf{G}_t = \mathbf{L}_t \mathbf{G}_t + B \Sigma_t dW_t \quad (2.1)$$

where $\mathbf{L}_t \mathbf{x} = (0, \alpha_t^1 x_1, \dots, \alpha_t^{n-1} x_{n-1})^* + \mathbf{U}_t \mathbf{x}$, and $\mathbf{U}_t \in \mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ is an “upper triangular” block matrix with zero entries on its first d rows. We suppose that the coefficients satisfy the following assumption ($\mathbf{A}^{\text{linear}}$):

- The diffusion coefficient $A_t := \Sigma_t \Sigma_t^*$, $t \geq 0$, is uniformly elliptic and bounded, i.e. $\exists \Lambda \geq 1, \forall \xi \in \mathbb{R}^d, \Lambda^{-1} |\xi|^2 \leq \langle A_t \xi, \xi \rangle \leq \Lambda |\xi|^2$.
- For each $i \in \llbracket 1, n-1 \rrbracket$, there exists a closed convex subset $\mathcal{E}_i \subset GL_d(\mathbb{R})$ s.t. for all $t \geq 0$, the matrix α_t^i belongs to \mathcal{E}_i .

Denoting by $(\mathbf{R}(s, t))_{0 \leq s, t}$ the resolvent associated to $(\mathbf{L}_t)_{t \geq 0}$, i.e. $\partial_t \mathbf{R}(t, s) = \mathbf{L}_t \mathbf{R}(t, s)$, $\mathbf{R}(s, s) = \mathbf{I}_{nd}$, we have for $0 \leq s < t$, $\mathbf{x} \in \mathbb{R}^{nd}$, $\mathbf{G}_t^{s, \mathbf{x}} = \mathbf{R}(t, s)\mathbf{x} + \int_s^t \mathbf{R}(t, u) \mathbf{B} \Sigma_u dW_u$ and $\text{Cov}[\mathbf{G}_t^{s, \mathbf{x}}] = \int_s^t \mathbf{R}(t, u) \mathbf{B} A_u \mathbf{B}^* \mathbf{R}(t, u)^* du := \mathbf{K}(s, t)$.

From Propositions 3.1 and 3.4 in [3], the family $(\mathbf{K}(s, t))_{t \in (s, T]}$ of covariance matrices associated to the Gaussian process $(\mathbf{G}_t^{s, \mathbf{x}})_{t \in (s, T]}$ satisfies, under $(\mathbf{A}^{\text{linear}})$, a “good scaling property” in the following sense:

Definition 2.1 (Good scaling property). *Fix $T > 0$. We say that a family $(\mathbf{K}(s, t))_{t \in [s, T]}$, $s \in [0, T)$ of $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ matrices satisfies a good scaling property with constant $C \geq 1$ (see also Definition 3.2 and Proposition 3.4 of [3]) if for all $(t, s) \in (\mathbb{R}^+)^2$, $0 < t - s \leq T$, $\forall \mathbf{y} \in \mathbb{R}^{nd}$, $C^{-1}(t - s)^{-1} |\mathbb{T}_{t-s} \mathbf{y}|^2 \leq \langle \mathbf{K}(s, t) \mathbf{y}, \mathbf{y} \rangle \leq C(t - s)^{-1} |\mathbb{T}_{t-s} \mathbf{y}|^2$.*

Precisely the family $(\mathbf{K}(s, t))_{t \in (s, T]}$ satisfies under $(\mathbf{A}^{\text{linear}})$ a good scaling property with constant $C := C(T, (\mathbf{A}^{\text{linear}}))$.

Remark 2.1. *We point out that it is precisely the second assumption of $(\mathbf{A}^{\text{linear}})$ concerning the existence of convex subsets $(\mathcal{E}_i)_{i \in \llbracket 1, n-1 \rrbracket}$ of $GL_d(\mathbb{R})$ that guarantees the good scaling property (see Propositions 3.1 and 3.4 in [3] for details).*

The density at time $t > s$ in $\mathbf{y} \in \mathbb{R}^{nd}$ of $\mathbf{G}_t^{s, \mathbf{x}}$ writes

$$q(s, t, \mathbf{x}, \mathbf{y}) = \frac{1}{(2\pi)^{nd/2} \det(\mathbf{K}_{s,t})^{1/2}} \exp\left(-\frac{1}{2} \langle \mathbf{K}(s, t)^{-1} (\mathbf{R}(t, s)\mathbf{x} - \mathbf{y}), \mathbf{R}(t, s)\mathbf{x} - \mathbf{y} \rangle\right). \quad (2.2)$$

Since under $(\mathbf{A}^{\text{linear}})$, $(\mathbf{K}(s, t))_{t \in (s, T]}$ satisfies a good scaling property in the sense of Definition 2.1, we then derive from (2.2):

Proposition 2.1. *Under $(\mathbf{A}^{\text{linear}})$, for all $T > 0$ there exists a constant $C_{2.1} := C_{2.1}(T, (\mathbf{A}^{\text{linear}})) \geq 1$ s.t. for all $0 < t - s \leq T$:*

$$\begin{aligned} C_{2.1}^{-1}(t - s)^{-n^2 d/2} \exp(-C_{2.1}(t - s) |\mathbb{T}_{t-s}^{-1} (\mathbf{R}(t, s)\mathbf{x} - \mathbf{y})|^2) &\leq q(s, t, \mathbf{x}, \mathbf{y}) \\ &\leq C_{2.1}(t - s)^{-n^2 d/2} \exp(-C_{2.1}^{-1}(t - s) |\mathbb{T}_{t-s}^{-1} (\mathbf{R}(t, s)\mathbf{x} - \mathbf{y})|^2). \end{aligned} \quad (2.3)$$

This means that the off-diagonal bound of Gaussian processes with dynamics (2.1) and fulfilling $(\mathbf{A}^{\text{linear}})$ is homogeneous to the square of the difference between the final point \mathbf{y} and $\mathbf{R}(t, s)\mathbf{x}$ (which corresponds to the transport of the initial condition by the deterministic system deriving from (2.1), that is $\dot{\boldsymbol{\theta}}_t = \mathbf{L}_t \boldsymbol{\theta}_t$) rescaled by the intrinsic time-scale of each component. We here recall that the component $i \in \llbracket 1, d \rrbracket$ has characteristic time scale $t^{(2i-1)/2}$.

2.2 Linearization of the initial dynamics and associated estimates

The crucial feature of the parametrix method described in the introduction was to choose a “good” process to approximate the density of the diffusion. In the uniformly elliptic case, with bounded coefficients, one could take, as a first approximation, the Gaussian process with coefficients frozen in space at the fixed final spatial point where we wanted to estimate the density. The choice is natural since it makes the kernel H (defined in (1.2)) “compatible” with the bounds of the frozen density. It is precisely the off diagonal term in $\exp(-c|x - y|^2/(t - s))$ that allows to equilibrate the singularity in $|x - y|^n/(t - s)$ coming from the second order spatial derivatives. In their work, [2], Bass and Perkins exactly exploited the specific behavior of the singular kernel H which

has an integrable singularity in time at 0 (see their Proposition 2.3), to derive uniqueness of the martingale problem in the non-degenerate time-homogeneous framework. This approach provides a natural link between parametrix expansions and the study of martingale problems for uniformly Hölder continuous coefficients.

Parametrix expansions, to derive density estimates on systems of the form (1.5), have been discussed in [3]. We thus have in the current degenerate framework of assumption **(A- η)** some natural candidate defined below. The key idea is to consider a “degenerate” Gaussian process whose density anyhow has a specific “off-diagonal” behavior similar to the one exhibited in equation (2.3) and to choose the freezing process in order that the singularity deriving from H is still compatible with the “off-diagonal” bound in the sense that it will be sufficient to remove the time-singularity. We follow the same line of reasoning in our current framework.

For fixed parameters $T > 0, \mathbf{y} \in \mathbb{R}^{nd}$, introduce the linear equation:

$$d\tilde{\mathbf{X}}_t^{T,\mathbf{y}} = [\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y})) + D\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))(\tilde{\mathbf{X}}_t^{T,\mathbf{y}} - \boldsymbol{\theta}_{t,T}(\mathbf{y}))] dt + B\sigma(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))dW_t, \quad 0 \leq t \leq T, \quad (2.4)$$

where $(\boldsymbol{\theta}_{t,T}(\mathbf{y}))_{t \geq 0}$ solves the ODE $[d/dt]\boldsymbol{\theta}_{t,T}(\mathbf{y}) = \mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))$, $t \geq 0$, with the boundary condition $\boldsymbol{\theta}_{T,T}(\mathbf{y}) = \mathbf{y}$ and $\forall (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$,

$$D\mathbf{F}(t, \mathbf{x}) := \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ D_{\mathbf{x}_1} \mathbf{F}_2(t, \mathbf{x}) & 0 & \cdots & \cdots & 0 \\ 0 & D_{\mathbf{x}_2} \mathbf{F}_3(t, \mathbf{x}) & 0 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & D_{\mathbf{x}_{n-1}} \mathbf{F}_n(t, \mathbf{x}) & 0 \end{pmatrix}$$

is the subdiagonal of the Jacobian matrix $D_{\mathbf{x}}\mathbf{F}$. Write $\tilde{p}^{T,\mathbf{y}}(t, T, \mathbf{x}, \cdot)$ for the density of $\tilde{\mathbf{X}}_T^{T,\mathbf{y}}$ starting from \mathbf{x} at time t .

The deterministic ODE associated with $\tilde{\mathbf{X}}^{T,\mathbf{y}}$ has the form

$$\frac{d}{dt} \tilde{\boldsymbol{\phi}}_t = \mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y})) + D\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))[\tilde{\boldsymbol{\phi}}_t - \boldsymbol{\theta}_{t,T}(\mathbf{y})], \quad t \geq 0. \quad (2.5)$$

We denote by $(\tilde{\boldsymbol{\theta}}_{t,s}^{T,\mathbf{y}})_{s,t \geq 0}$ the associated flow, i.e. $\tilde{\boldsymbol{\theta}}_{t,s}^{T,\mathbf{y}}(\mathbf{x})$ is the value of $\tilde{\boldsymbol{\phi}}_t$ when $\tilde{\boldsymbol{\phi}}_s = \mathbf{x}$. It is affine:

$$\tilde{\boldsymbol{\theta}}_{t,s}^{T,\mathbf{y}}(\mathbf{x}) = \tilde{\mathbf{R}}^{T,\mathbf{y}}(t, s)\mathbf{x} + \int_s^t \tilde{\mathbf{R}}^{T,\mathbf{y}}(t, u)(\mathbf{F}(u, \boldsymbol{\theta}_{u,T}(\mathbf{y})) - D\mathbf{F}(u, \boldsymbol{\theta}_{u,T}(\mathbf{y}))\boldsymbol{\theta}_{u,T}(\mathbf{y})) du. \quad (2.6)$$

Above, $(\tilde{\mathbf{R}}^{T,\mathbf{y}}(t, s))_{s,t \geq 0}$ stands for the resolvent associated with the matrices $(D\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y})))_{t \geq 0}$. We now claim

Lemma 2.1. *Let $T_0 > 0$ be fixed. There exists a constant $C_{2.1} \geq 1$, depending on **(A)** and T_0 such that, for any $t \in [0, T)$, $T \leq T_0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{nd}$,*

$$C_{2.1}^{-1} |\mathbb{T}_{T-t}^{-1}[\mathbf{x} - \boldsymbol{\theta}_{t,T}(\mathbf{y})]| \leq |\mathbb{T}_{T-t}^{-1}[\tilde{\boldsymbol{\theta}}_{T,t}^{T,\mathbf{y}}(\mathbf{x}) - \mathbf{y}]| \leq C_{2.1} |\mathbb{T}_{T-t}^{-1}[\mathbf{x} - \boldsymbol{\theta}_{t,T}(\mathbf{y})]|.$$

This means that we can compare the rescaled “forward” transport of the initial condition \mathbf{x} from t to T by the linear flow and the rescaled “backward” transport from T to t of the final point \mathbf{y} by the original deterministic differential dynamics. We refer to Lemma 5.3 of [3] for a proof. Furthermore, under $(\mathbf{A}\text{-}\eta)$ we have that $DF(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))$ satisfies $(\mathbf{A}^{\text{linear}})$. We thus derive from Lemma 2.1 and a direct extension of Proposition 2.1 (the mean of $\tilde{\mathbf{X}}_T^{t,\mathbf{y}}$ starting from \mathbf{x} at time t being $\tilde{\boldsymbol{\theta}}_{T,t}^{t,\mathbf{y}}(\mathbf{x})$) the following result.

Lemma 2.2. *Let $T_0 > 0$ be fixed. There exists a constant $C_{2,2} > 0$, depending on $(\mathbf{A}\text{-}\eta)$ and T_0 such that, for all $0 \leq t < T \leq T_0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{nd}$,*

$$\tilde{p}^{t,\mathbf{y}}(t, T, \mathbf{x}, \mathbf{y}) \leq C_{2,2} g_{C_{2,2}, T-t}(\mathbf{x} - \boldsymbol{\theta}_{t,T}(\mathbf{y})),$$

where for all $a > 0, t > 0$, $g_{a,t}(\mathbf{y}) = t^{-n\frac{d}{2}} \exp(-a^{-1}t|\mathbb{T}_t^{-1}\mathbf{y}|^2)$, $a, t > 0, \mathbf{y} \in \mathbb{R}^{nd}$.

For all $0 \leq s < t \leq T$, $\mathbf{z}, \mathbf{y} \in \mathbb{R}^{nd}$, define now the kernel H as:

$$\begin{aligned} H(s, t, \mathbf{z}, \mathbf{y}) &= \{ \langle \mathbf{F}(s, \mathbf{z}), \mathbf{D}_z \tilde{p}^{t,\mathbf{y}}(s, t, \mathbf{z}, \mathbf{y}) \rangle + \frac{1}{2} \text{Tr}(a(s, \mathbf{z}) D_{z_1}^2 \tilde{p}^{t,\mathbf{y}}(s, t, \mathbf{z}, \mathbf{y})) \} - \\ &\quad \{ \langle \mathbf{F}(s, \boldsymbol{\theta}_{s,t}(\mathbf{y})) + DF(s, \boldsymbol{\theta}_{s,t}(\mathbf{y}))(\mathbf{z} - \boldsymbol{\theta}_{s,t}(\mathbf{y})), \mathbf{D}_z \tilde{p}^{t,\mathbf{y}}(s, t, \mathbf{z}, \mathbf{y}) \rangle \\ &\quad + \frac{1}{2} \text{Tr}(a(s, \boldsymbol{\theta}_{s,t}(\mathbf{y})) D_{z_1}^2 \tilde{p}^{t,\mathbf{y}}(s, t, \mathbf{z}, \mathbf{y})) \} \\ &= (L_{s,\mathbf{z}} - \tilde{L}_{s,\mathbf{z}}^{t,\mathbf{y}}) \tilde{p}^{t,\mathbf{y}}(s, t, \mathbf{z}, \mathbf{y}), \end{aligned} \quad (2.7)$$

where L is the generator of the initial diffusion (1.5) defined in (1.7), $\tilde{L}_{s,\mathbf{z}}^{t,\mathbf{y}}$ and $\tilde{p}^{t,\mathbf{y}}(s, t, \mathbf{z}, \cdot)$ respectively stand for the generator at time s and the density at time t of $\tilde{\mathbf{X}}^{t,\mathbf{y}}$, $\tilde{\mathbf{X}}_s^{t,\mathbf{y}} = \mathbf{z}$ with coefficients “frozen” w.r.t. t, \mathbf{y} . The lower script in \mathbf{z} is to emphasize that \mathbf{z} is the differentiation parameter in the operator.

We have the following control on the kernel (see Lemma 5.5 of [3] for a proof).

Lemma 2.3. *Let $T_0 > 0$ be fixed. There exists a constant $C_{2,3} > 0$, depending on $(\mathbf{A}\text{-}\eta)$ and T_0 such that, for all $t \in [0, T]$, $T \leq T_0$ and $\mathbf{z}, \mathbf{y} \in \mathbb{R}^{nd}$,*

$$|H(t, T, \mathbf{z}, \mathbf{y})| \leq C_{2,3} (T - t)^{\frac{n}{2}-1} g_{C_{2,3}, T-t}(\mathbf{z} - \boldsymbol{\theta}_{t,T}(\mathbf{y})).$$

We conclude this section with a technical Lemma whose proof is postponed to Appendix A.

Lemma 2.4. *Let h be a $C_0^0([0, T] \times \mathbb{R}^{nd}, \mathbb{R})$ function. Define for all $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$,*

$$\forall \varepsilon > 0, \Xi^\varepsilon(s, \mathbf{x}) := \int_{\mathbb{R}^{nd}} d\mathbf{y} h(s, \mathbf{y}) \tilde{p}^{s+\varepsilon, \mathbf{y}}(s, s + \varepsilon, \mathbf{x}, \mathbf{y}).$$

Then $\Xi^\varepsilon(s, \mathbf{x})$ converges boundedly and pointwise to $h(s, \mathbf{x})$ when $\varepsilon \rightarrow 0$.

3 Proof of Theorem 1.1

Suppose we are given two solutions $\mathbb{P}_1, \mathbb{P}_2$ of the martingale problem associated to $(L_t)_{t \in [s, T]}$ starting in \mathbf{x} at time s . W.l.o.g. we can suppose here that $T \leq 1$. Define for a bounded Borel function $f : [0, T] \times \mathbb{R}^{nd} \rightarrow \mathbb{R}$:

$$S^i f := \mathbb{E}_i \left[\int_s^T dt f(t, \mathbf{X}_t) \right], \quad i \in \llbracket 1, 2 \rrbracket,$$

where $(X_t)_{t \in [s, T]}$ stands here for the canonical process associated to $(\mathbb{P}_i)_{i \in \llbracket 1, 2 \rrbracket}$. Let us specify (as indicated in [2]) that $S^i f$ is only a linear functional and not a function since \mathbb{P}_i does not need to come from a Markov process. Let us now introduce

$$S^\Delta f := S^1 f - S^2 f, \Theta := \sup_{\|f\|_\infty \leq 1} |S^\Delta f|.$$

Clearly $\Theta \leq T - s$.

If $f \in C_0^{1,2}([0, T] \times \mathbb{R}^{nd}, \mathbb{R})$ then by definition of the martingale problem we have:

$$f(s, \mathbf{x}) + \mathbb{E}_i \left[\int_s^T dt (\partial_t + L_t) f(t, \mathbf{X}_t) \right] = 0, \quad i \in \llbracket 1, 2 \rrbracket. \quad (3.1)$$

For a fixed point $\mathbf{y} \in \mathbb{R}^{nd}$ and $\varepsilon \geq 0$, introduce $\forall f \in C_0^{1,2}([0, T] \times \mathbb{R}^{nd}, \mathbb{R})$ the Green function

$$\forall (s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}, \quad G^{\varepsilon, \mathbf{y}} f(s, \mathbf{x}) := \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{z} \tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t, \mathbf{x}, \mathbf{z}) f(t, \mathbf{z}). \quad (3.2)$$

We insist that in the above equation $\tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t, \mathbf{x}, \mathbf{z})$ stands for the density at time t and point \mathbf{z} of the process $\tilde{\mathbf{X}}^{t+\varepsilon, \mathbf{y}}$ defined in (2.4) starting from \mathbf{x} at time s with coefficients depending on the backward transport of the freezing point \mathbf{y} by $(\theta_{u, t+\varepsilon})_{u \in [s, t]}$. In particular, the parameter ε can be equal to 0 in the previous definition.

One easily checks that

$$(\partial_s + \tilde{L}_{s, \mathbf{x}}^{t+\varepsilon, \mathbf{y}}) \tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t, \mathbf{x}, \mathbf{z}) = 0, \quad \forall (s, \mathbf{x}, \mathbf{z}) \in [0, t] \times (\mathbb{R}^{nd})^2, \quad \tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t, \mathbf{x}, \cdot) \xrightarrow[t \downarrow s]{} \delta_{\mathbf{x}}(\cdot). \quad (3.3)$$

Introducing for all $f \in C_0^{1,2}([0, T] \times \mathbb{R}^{nd}, \mathbb{R})$, $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$,

$$M_{s, \mathbf{x}}^{\varepsilon, \mathbf{y}} f(s, \mathbf{x}) := \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{z} \tilde{L}_{s, \mathbf{x}}^{t+\varepsilon, \mathbf{y}} \tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t, \mathbf{x}, \mathbf{z}) f(t, \mathbf{z}),$$

we get from equations (3.2), (3.3)

$$(\partial_s G^{\varepsilon, \mathbf{y}} f + M_{s, \mathbf{x}}^{\varepsilon, \mathbf{y}} f)(s, \mathbf{x}) = -f(s, \mathbf{x}), \quad \forall (s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}. \quad (3.4)$$

Define now for a smooth function $h \in C_0^{1,2}([0, T] \times \mathbb{R}^{nd}, \mathbb{R})$ and for all $(s, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$:

$$\begin{aligned} \Phi^{\varepsilon, \mathbf{y}}(s, \mathbf{x}) &:= \tilde{p}^{s+\varepsilon, \mathbf{y}}(s, s+\varepsilon, \mathbf{x}, \mathbf{y}) h(s, \mathbf{y}), \\ \Psi_\varepsilon(s, \mathbf{x}) &:= \int_{\mathbb{R}^{nd}} d\mathbf{y} G^{\varepsilon, \mathbf{y}}(\Phi^{\varepsilon, \mathbf{y}})(s, \mathbf{x}). \end{aligned}$$

Observe that (3.2) yields:

$$\begin{aligned} \Psi_\varepsilon(s, \mathbf{x}) &:= \int_{\mathbb{R}^{nd}} d\mathbf{y} \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{z} \tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t, \mathbf{x}, \mathbf{z}) \Phi^{\varepsilon, \mathbf{y}}(t, \mathbf{z}) \\ &= \int_{\mathbb{R}^{nd}} d\mathbf{y} \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{z} \tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t, \mathbf{x}, \mathbf{z}) \tilde{p}^{t+\varepsilon, \mathbf{y}}(t, t+\varepsilon, \mathbf{z}, \mathbf{y}) h(t, \mathbf{y}) \\ &= \int_{\mathbb{R}^{nd}} d\mathbf{y} \int_s^T dt \tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t+\varepsilon, \mathbf{x}, \mathbf{y}) h(t, \mathbf{y}), \end{aligned} \quad (3.5)$$

exploiting the semigroup property of the frozen density $\tilde{p}^{t+\varepsilon, \mathbf{y}}$ for the last inequality. Write now for all $(s, \mathbf{x}) \in [0, T) \times \mathbb{R}^{nd}$,

$$\begin{aligned} (\partial_s + L_s)\Psi_\varepsilon(s, \mathbf{x}) &= \int_{\mathbb{R}^{nd}} d\mathbf{y}(\partial_s + L_s)(G^{\varepsilon, \mathbf{y}}\Phi^{\varepsilon, \mathbf{y}})(s, \mathbf{x}) \\ &= \int_{\mathbb{R}^{nd}} d\mathbf{y}(\partial_s G^{\varepsilon, \mathbf{y}}\Phi^{\varepsilon, \mathbf{y}} + M_{s, \mathbf{x}}^{\varepsilon, \mathbf{y}}\Phi^{\varepsilon, \mathbf{y}})(s, \mathbf{x}) \\ &\quad + \int_{\mathbb{R}^{nd}} d\mathbf{y}(L_s G^{\varepsilon, \mathbf{y}}\Phi^{\varepsilon, \mathbf{y}} - M_{s, \mathbf{x}}^{\varepsilon, \mathbf{y}}\Phi^{\varepsilon, \mathbf{y}})(s, \mathbf{x}) \\ &= - \int_{\mathbb{R}^{nd}} d\mathbf{y}\Phi^{\varepsilon, \mathbf{y}}(s, \mathbf{x}) + \int_{\mathbb{R}^{nd}} d\mathbf{y}(L_s G^{\varepsilon, \mathbf{y}}\Phi^{\varepsilon, \mathbf{y}} - M_{s, \mathbf{x}}^{\varepsilon, \mathbf{y}}\Phi^{\varepsilon, \mathbf{y}})(s, \mathbf{x}) := I_1^\varepsilon + I_2^\varepsilon, \end{aligned}$$

exploiting (3.4) for the last but one identity. Now Lemma 2.4 gives $I_1^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} -h(s, \mathbf{x})$. On the other hand using the notations of Section 2.2, we derive from (3.5) that the term I_2^ε writes:

$$\begin{aligned} I_2^\varepsilon &= \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y}(L_s - \tilde{L}_{s, \mathbf{x}}^{t+\varepsilon, \mathbf{y}})\tilde{p}^{t+\varepsilon, \mathbf{y}}(s, t + \varepsilon, \mathbf{x}, \mathbf{y})h(t, \mathbf{y}) \\ &= \int_s^T dt \int_{\mathbb{R}^{nd}} d\mathbf{y}H(s, t + \varepsilon, \mathbf{x}, \mathbf{y})h(t, \mathbf{y}). \end{aligned}$$

Thus, from Lemma 2.3,

$$\begin{aligned} |I_2^\varepsilon| &\leq C_{2,3}\|h\|_\infty \int_s^T dt(t + \varepsilon - s)^{-1+\eta/2} \int_{\mathbb{R}^{nd}} d\mathbf{y}g_{C_{2,3}, t+\varepsilon-s}(\mathbf{x} - \boldsymbol{\theta}_{s, t+\varepsilon}(\mathbf{y})) \\ &\leq \|h\|_\infty C_{3,6} \int_s^T dt(t + \varepsilon - s)^{-1+\eta/2} \int_{\mathbb{R}^{nd}} d\mathbf{y}g_{C_{3,6}, t+\varepsilon-s}(\boldsymbol{\theta}_{t+\varepsilon, s}(\mathbf{x}) - \mathbf{y}) \\ &\leq C_{3,6}\{(T-s) \vee \varepsilon\}^{\eta/2}\|h\|_\infty, \end{aligned} \tag{3.6}$$

using the bi-Lipschitz property of the flow for the last but one inequality and up to a modification of $C_{3,6}$ in the last one. Anyhow, the constant $C_{3,6}$ only depends on known parameters in $(\mathbf{A}-\eta)$. Thus for T and ε sufficiently small we have from (3.6)

$$|I_2^\varepsilon| \leq \frac{1}{2}\|h\|_\infty. \tag{3.7}$$

Now, equation (3.1) and the above definition of S^Δ yield:

$$S^\Delta((\partial_s + L_s)\Psi_\varepsilon) = 0 \Rightarrow |S^\Delta I_1^\varepsilon| = |S^\Delta I_2^\varepsilon|.$$

From the bounded convergence part of Lemma 2.4 and (3.7), we have:

$$|S^\Delta h| = \lim_{\varepsilon \rightarrow 0} |S^\Delta I_1^\varepsilon| = \lim_{\varepsilon \rightarrow 0} |S^\Delta I_2^\varepsilon| \leq \Theta \limsup_{\varepsilon \rightarrow 0} |I_2^\varepsilon| \leq \Theta \frac{\|h\|_\infty}{2}.$$

By a monotone class argument, the previous inequality remains valid for bounded measurable functions h compactly supported in $[0, T) \times \mathbb{R}^{nd}$. Taking the supremum over $\|h\|_\infty \leq 1$, we obtain $\Theta \leq \frac{1}{2}\Theta$ which gives $\Theta = 0$ since $\Theta < +\infty$. Hence, $\mathbb{E}_1 \left[\int_s^T dth(t, \mathbf{X}_t) \right] = \mathbb{E}_2 \left[\int_s^T dth(t, \mathbf{X}_t) \right]$ which proves the result on the interval $[0, T]$. Regular conditional probabilities then allow to extend the result on \mathbb{R}^+ , see Chapter 6.2 of [17] for details.

A Proof of Lemma 2.4

Let us denote by $\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,y}$ the covariance matrix associated to equation (2.4) for the process $\tilde{\mathbf{X}}^{s+\varepsilon,y}$ (starting from \mathbf{x} at s) and by $\mathbf{K}_{s,s+\varepsilon}^{s,x}$ the covariance matrix associated to the linear diffusion with dynamics:

$$d\tilde{\mathbf{X}}_t = D\mathbf{F}(t, \boldsymbol{\theta}_{t,s}(\mathbf{x}))\tilde{\mathbf{X}}_t dt + B\sigma(t, \boldsymbol{\theta}_{t,s}(\mathbf{x}))dW_t, t \geq s, \quad (\text{A.1})$$

that is $\mathbf{K}_{s,s+\varepsilon}^{s,x} = \int_s^{s+\varepsilon} du \mathbf{R}^{s,x}(s+\varepsilon, u) B a^*(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) B^* \mathbf{R}^{s,x}(s+\varepsilon, u)^* du$ where $\mathbf{R}^{s,x}$ stands for the resolvent associated to the linear part of (A.1).

Under (A- η), the matrices $\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,y}$, $\mathbf{K}_{s,s+\varepsilon}^{s,x}$ admit a good scaling property in the sense of the previous Definition 2.1, i.e.

$$\begin{aligned} \exists C := C((\mathbf{A}-\eta)) \geq 1, \forall \xi \in \mathbb{R}^{nd}, \quad C^{-1} \varepsilon^{-1} |\mathbb{T}_\varepsilon \xi|^2 \leq \langle \tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,y} \xi, \xi \rangle \leq C \varepsilon^{-1} |\mathbb{T}_\varepsilon \xi|^2, \\ C^{-1} \varepsilon^{-1} |\mathbb{T}_\varepsilon \xi|^2 \leq \langle \mathbf{K}_{s,s+\varepsilon}^{s,x} \xi, \xi \rangle \leq C \varepsilon^{-1} |\mathbb{T}_\varepsilon \xi|^2. \end{aligned} \quad (\text{A.2})$$

We introduce the following decomposition:

$$\begin{aligned} \Xi^\varepsilon(s, \mathbf{x}) &:= \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(2\pi)^{nd/2}} h(s, \mathbf{y}) \exp\left(-\frac{1}{2} \langle (\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,y})^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y}), \tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y} \rangle\right) \\ &\times \left\{ \frac{1}{\det(\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,y})^{1/2}} - \frac{1}{\det(\mathbf{K}_{s,s+\varepsilon}^{s,x})^{1/2}} \right\} \\ &+ \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(2\pi)^{nd/2}} \frac{h(s, \mathbf{y})}{\det(\mathbf{K}_{s,s+\varepsilon}^{s,x})^{1/2}} \left[\exp\left(-\frac{1}{2} \langle (\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,y})^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y}), \tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y} \rangle\right) - \right. \\ &\left. \exp\left(-\frac{1}{2} \langle (\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1} (\boldsymbol{\theta}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y}), \boldsymbol{\theta}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y} \rangle\right) \right] \\ &+ \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(2\pi)^{nd/2}} \frac{h(s, \mathbf{y})}{\det(\mathbf{K}_{s,s+\varepsilon}^{s,x})^{1/2}} \left[\exp\left(-\frac{1}{2} \langle (\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1} (\boldsymbol{\theta}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y}), \boldsymbol{\theta}_{s+\varepsilon,s}^{s+\varepsilon,y}(\mathbf{x}) - \mathbf{y} \rangle\right) - \right. \\ &\left. \exp\left(-\frac{1}{2} \langle (\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1} (\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}), \boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y} \rangle\right) \right] \\ &+ \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(2\pi)^{nd/2}} \frac{h(s, \mathbf{y})}{\det(\mathbf{K}_{s,s+\varepsilon}^{s,x})^{1/2}} \exp\left(-\frac{1}{2} \langle (\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1} (\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}), \boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y} \rangle\right) \\ &:= \sum_{i=1}^4 \Xi_i^\varepsilon(s, \mathbf{x}). \end{aligned} \quad (\text{A.3})$$

Let us now control the $(\Xi_i^\varepsilon(s, \mathbf{x}))_{i \in \llbracket 1,4 \rrbracket}$. Set $\tilde{\mathbf{y}} := (\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1/2} (\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})$ where $(\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1/2}$ denotes the upper triangular matrix obtained through the Cholesky factorization, i.e. $(\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1} = ((\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1/2})^* (\mathbf{K}_{s,s+\varepsilon}^{s,x})^{-1/2}$. We get from the bounded convergence theorem:

$$\Xi_4^\varepsilon(s, \mathbf{x}) = \int_{\mathbb{R}^{nd}} \frac{d\tilde{\mathbf{y}}}{(2\pi)^{nd/2}} h(s, -(\mathbf{K}_{s,s+\varepsilon}^{s,x})^{1/2} \tilde{\mathbf{y}} + \boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x})) \exp\left(-\frac{|\tilde{\mathbf{y}}|^2}{2}\right) \xrightarrow{\varepsilon \rightarrow 0} h(s, \mathbf{x}). \quad (\text{A.4})$$

For $\Xi_3^\varepsilon(s, \mathbf{x})$, we first observe, from Lemma 2.1, the good scaling property (A.2) and the bi-Lipschitz

property of the flow θ , that there exists $C_1 := C_1((\mathbf{A}\cdot\eta)) \geq 1$ s.t.

$$\begin{aligned} C_1^{-1}\varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})| &\leq |(\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1/2}(\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})| \leq C_1\varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|, \\ C_1^{-1}\varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})| &\leq |(\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1/2}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})| \leq C_1\varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|, \\ C_1^{-1}\varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})| &\leq \varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})| \leq C_1\varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|. \end{aligned} \quad (\text{A.5})$$

Write now,

$$|\Xi_3^\varepsilon(s, \mathbf{x})| \leq \|h\|_\infty \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(2\pi)^{nd/2} \det(\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{1/2}} \int_0^1 d\delta |(\varphi_{s,\mathbf{x},\mathbf{y}}^\varepsilon)'(\delta)|, \quad \forall \delta \in [0, 1], \quad (\text{A.6})$$

$$\begin{aligned} \varphi_{s,\mathbf{x},\mathbf{y}}^\varepsilon(\delta) &= \exp\left(-\frac{1}{2} \left\{ \langle (\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1}(\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle + \right. \right. \\ &\delta \left. \left[\langle (\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}), \theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y} \rangle - \langle (\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1}(\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right] \right\} \Big), \end{aligned}$$

$$\begin{aligned} |(\varphi_{s,\mathbf{x},\mathbf{y}}^\varepsilon)'(\delta)| &\leq \left| (\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1/2} \left\{ (\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}) + (\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}) \right\} \right| \\ &\times \left| (\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1/2} \left\{ (\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}) - (\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}) \right\} \right| \varphi_{s,\mathbf{x},\mathbf{y}}^\varepsilon(\delta), \end{aligned}$$

using the Cauchy-Schwarz inequality for the last assertion. Equations (A.5) now yield that there exists $C_2 := C_2((\mathbf{A}\cdot\eta)) \geq 1$ s.t.:

$$\begin{aligned} |(\varphi_{s,\mathbf{x},\mathbf{y}}^\varepsilon)'(\delta)| &\leq C_2\varepsilon^{1/2}|\mathbb{T}_\varepsilon^{-1}(\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \theta_{s+\varepsilon,s}(\mathbf{x}))| \exp(-C_2^{-1}\varepsilon|\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|^2) \\ &:= C_2|D_\varepsilon| \exp(-C_2^{-1}\varepsilon|\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|^2). \end{aligned} \quad (\text{A.7})$$

Let us now recall the differential dynamics of $\theta_{s+\varepsilon,s}(\mathbf{x})$, $\tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x})$, that is:

$$\begin{aligned} \theta_{s+\varepsilon,s}(\mathbf{x}) &= \mathbf{x} + \int_s^{s+\varepsilon} du \mathbf{F}(u, \theta_{u,s}(\mathbf{x})), \\ \tilde{\theta}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) &= \mathbf{x} + \int_s^{s+\varepsilon} du \{ \mathbf{F}(u, \theta_{u,s+\varepsilon}(\mathbf{y})) + D\mathbf{F}(u, \theta_{u,s+\varepsilon}(\mathbf{y}))(\tilde{\theta}_{u,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \theta_{u,s+\varepsilon}(\mathbf{y})) \}. \end{aligned}$$

Let us now set for all $(u, \mathbf{z}) \in [s, s+\varepsilon] \times \mathbb{R}^{nd}$, $\mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \mathbf{z}) := (\mathbf{F}_1(u, \theta_{u,s+\varepsilon}(\mathbf{y})), \mathbf{F}_2(u, \mathbf{z}_1, (\theta_{u,s+\varepsilon}(\mathbf{y}))^{2,n}), \mathbf{F}_3(u, \mathbf{z}_2, (\theta_{u,s+\varepsilon}(\mathbf{y}))^{3,n}), \dots, \mathbf{F}_n(u, \mathbf{z}_{n-1}, (\theta_{u,s+\varepsilon}(\mathbf{y}))_n))^*$. Observe in particular that with the previous definition $\mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \theta_{u,s+\varepsilon}(\mathbf{y})) = \mathbf{F}(u, \theta_{u,s+\varepsilon}(\mathbf{y}))$.

We get:

$$\begin{aligned}
D_\varepsilon &:= \varepsilon^{1/2} \mathbb{T}_\varepsilon^{-1} \left\{ \boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) \right\} = \\
&\varepsilon^{1/2} \mathbb{T}_\varepsilon^{-1} \left\{ \int_s^{s+\varepsilon} du \left[\left(\mathbf{F}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) - \mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) \right) \right. \right. \\
&\quad \left. \left. + \left(D\mathbf{F}(u, \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})) (\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \tilde{\boldsymbol{\theta}}_{u,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x})) \right) \right] \right. \\
&\quad \left. + \left(\int_0^1 d\delta \left(D\mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})) + \delta(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})) \right) \right. \right. \\
&\quad \left. \left. - D\mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})) \right) (\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})) \right) \left. \right\} \\
&:= D_\varepsilon^1 + D_\varepsilon^2 + D_\varepsilon^3, \tag{A.8}
\end{aligned}$$

where for $(u, \mathbf{z}) \in [s, s + \varepsilon] \times \mathbb{R}^{nd}$, $D\mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \mathbf{z})$ is the $(nd) \times (nd)$ matrix with only non zero $d \times d$ matrix entries $(D\mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \mathbf{z}))_{j,j-1} := D_{\mathbf{x}_{j-1}} \mathbf{F}_j(u, \mathbf{z}_{j-1}, \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})^{j,n})$, $j \in \llbracket 2, n \rrbracket$, so that in particular $D\mathbf{F}^{s+\varepsilon,\mathbf{y}}(u, \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})) = D\mathbf{F}(u, \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y}))$.

The structure of the ‘‘partial gradient’’ $D\mathbf{F}^{s+\varepsilon,\mathbf{y}}$ associated to the η -Holder continuity of the mapping $\mathbf{x}_{j-1} \in \mathbb{R}^d \mapsto D_{\mathbf{x}_{j-1}} \mathbf{F}_j(\mathbf{x}_{j-1}, \mathbf{x}^{j,n})$, $\forall \mathbf{x}^{j,n} \in \mathbb{R}^{(n-j+1)d}$ yield that there exists $C_3 := C_3(\mathbf{A}-\eta)$ s.t. for all $j \in \llbracket 2, d \rrbracket$:

$$\begin{aligned}
|(D_\varepsilon^3)_j| &\leq C_3 \varepsilon^{1/2-j} \int_s^{s+\varepsilon} du |\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})|_{j-1}^{1+\eta} \\
&\leq C_3 \varepsilon^{-1} \int_s^{s+\varepsilon} du \left(\sum_{k=2}^n \varepsilon^{1/2-(k-1)} |\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})|_{k-1} \right)^{1+\eta} \varepsilon^{((j-1)-1/2)\eta} \\
&\leq C_3 \varepsilon^{-1+\eta(j-3/2)} \int_s^{s+\varepsilon} du (\varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1}(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y}))|)^{1+\eta} \\
&\leq C_3 \varepsilon^{-1+\eta(j-3/2)} \int_s^{s+\varepsilon} du (\varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1}(\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|)^{1+\eta} \\
&\leq C_3 \varepsilon^{\eta(j-3/2)} (\varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1}(\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|)^{1+\eta}, \tag{A.9}
\end{aligned}$$

up to a modification of C_3 and using the bi-Lipschitz property of the flow $\boldsymbol{\theta}$ for the last but one inequality (see the end of the proof of Proposition 5.1 in [3] for details).

On the other hand, the term D_ε^1 can be seen as a remainder w.r.t. the characteristic time scales. Precisely, there exists $C_4 := C_4(\mathbf{A}-\eta)$ s.t. for all $j \in \llbracket 1, n \rrbracket$:

$$\begin{aligned}
|(D_\varepsilon^1)_j| &\leq C_4 \varepsilon^{1/2-j} \int_s^{s+\varepsilon} du \sum_{k=j}^n |(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y}))_k| \\
&\leq C_4 \int_s^{s+\varepsilon} du \varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1}(\boldsymbol{\theta}_{u,s}(\mathbf{x}) - \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y}))| \\
&\leq C_4 \varepsilon (\varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1}(\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|) \tag{A.10}
\end{aligned}$$

using once again the bi-Lipschitz property of the flow θ for the last inequality. Recall now that D_2^ε is the linear part of equation (A.8), i.e. it can be rewritten

$$\begin{aligned} D_2^\varepsilon &= \int_s^{s+\varepsilon} du \left\{ \varepsilon^{1/2} \mathbb{T}_\varepsilon^{-1} D\mathbf{F}(u, \theta_{u,s+\varepsilon}(\mathbf{y})) (u-s)^{-1/2} \mathbb{T}_{u-s} \right\} \\ &\quad \times \left((u-s)^{1/2} \mathbb{T}_{u-s}^{-1} (\theta_{u,s}(\mathbf{x}) - \tilde{\theta}_{u,s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x})) \right) \\ &:= \int_s^{s+\varepsilon} du \alpha_\varepsilon^y(u, s) \left((u-s)^{1/2} \mathbb{T}_{u-s}^{-1} (\theta_{u,s}(\mathbf{x}) - \tilde{\theta}_{u,s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x})) \right), \end{aligned}$$

where there exists a constant $\tilde{C} := \tilde{C}(\mathbf{A}-\eta)$ independent of ε s.t. $\int_s^{s+\varepsilon} du |\alpha_\varepsilon^y(u, s)| \leq \tilde{C}$. From (A.10), (A.9), (A.8) and Gronwall's Lemma we derive

$$\exists C_5 := C_5(\mathbf{A} - \eta), |D_\varepsilon| \leq C_5 \varepsilon^{\eta/2} ((\varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon, s}(\mathbf{x}) - \mathbf{y})|)^{1+\eta} + 1).$$

Plugging this estimate into (A.7), we then get from (A.6), using as well the good scaling property (A.2), that there exists $C_6 := C_6(\mathbf{A}-\eta)$,

$$\begin{aligned} |\Xi_3^\varepsilon(s, \mathbf{x})| &\leq C_6 \varepsilon^{\eta/2} \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{\varepsilon^{n^2 d/2}} ((\varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon, s}(\mathbf{x}) - \mathbf{y})|)^{1+\eta} + 1) \\ &\quad \times \exp(-C_2^{-1} \varepsilon |\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon, s}(\mathbf{x}) - \mathbf{y})|^2) \leq C_6 \varepsilon^{\eta/2}, \end{aligned} \quad (\text{A.11})$$

up to a modification of C_6 in the last inequality.

Let us consider now $\Xi_2^\varepsilon(s, \mathbf{x})$. Write:

$$|\Xi_2^\varepsilon(s, \mathbf{x})| \leq C_7 \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{(2\pi)^{nd/2}} \frac{h(s, \mathbf{y})}{\det(\mathbf{K}_{s, s+\varepsilon}^{s, \mathbf{x}})^{1/2}} \int_0^1 d\delta |(\psi_{s, \mathbf{x}, \mathbf{y}}^\varepsilon)'(\delta)|, \quad C_7 := C_7(\mathbf{A} - \eta), \quad (\text{A.12})$$

$$\begin{aligned} \forall \delta \in [0, 1], \psi_{s, \mathbf{x}, \mathbf{y}}^\varepsilon(\delta) &= \exp \left(-\frac{1}{2} \left\{ \langle (\mathbf{K}_{s, s+\varepsilon}^{s, \mathbf{x}})^{-1} (\tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right. \right. \\ &\quad + \delta \left[\langle (\tilde{\mathbf{K}}_{s, s+\varepsilon}^{s+\varepsilon, \mathbf{y}})^{-1} (\tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right. \\ &\quad \left. \left. - \langle (\mathbf{K}_{s, s+\varepsilon}^{s, \mathbf{x}})^{-1} (\tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right] \right\} \Bigg), \\ |(\psi_{s, \mathbf{x}, \mathbf{y}}^\varepsilon)'(\delta)| &\leq \left| \langle (\tilde{\mathbf{K}}_{s, s+\varepsilon}^{s+\varepsilon, \mathbf{y}})^{-1} - (\mathbf{K}_{s, s+\varepsilon}^{s, \mathbf{x}})^{-1} \rangle (\tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right| \\ &\quad \times \psi_{s, \mathbf{x}, \mathbf{y}}^\varepsilon(\delta). \end{aligned}$$

Equations (A.5) (that according to (A.2) hold for $\tilde{\mathbf{K}}_{s+\varepsilon, s}^{s, \mathbf{y}}$ as well) yield:

$$\begin{aligned} |(\psi_{s, \mathbf{x}, \mathbf{y}}^\varepsilon)'(\delta)| &\leq C \left| \langle ((\tilde{\mathbf{K}}_{s, s+\varepsilon}^{s+\varepsilon, \mathbf{y}})^{-1} - (\mathbf{K}_{s, s+\varepsilon}^{s, \mathbf{x}})^{-1}) (\tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\theta}_{s+\varepsilon, s}^{s+\varepsilon, \mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right| \\ &\quad \times \exp(-C \varepsilon |\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon, s}(\mathbf{x}) - \mathbf{y})|^2) \\ &:= C |Q_\varepsilon| \exp(-C \varepsilon |\mathbb{T}_\varepsilon^{-1}(\theta_{s+\varepsilon, s}(\mathbf{x}) - \mathbf{y})|^2), \end{aligned} \quad (\text{A.13})$$

for $C := C((\mathbf{A}-\eta))$. From the scaling Lemma 3.6 in [3], we can write:

$$\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,\mathbf{y}} = \varepsilon^{-1} \mathbb{T}_\varepsilon \widehat{\mathbf{K}}_1^{s,s+\varepsilon,s+\varepsilon,\mathbf{y}} \mathbb{T}_\varepsilon, \quad \mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}} = \varepsilon^{-1} \mathbb{T}_\varepsilon \widehat{\mathbf{K}}_1^{s,s+\varepsilon,s,\mathbf{x}} \mathbb{T}_\varepsilon,$$

where $\widehat{\mathbf{K}}_1^{s,s+\varepsilon,s+\varepsilon,\mathbf{y}}, \widehat{\mathbf{K}}_1^{s,s+\varepsilon,s,\mathbf{x}}$ are uniformly elliptic and bounded matrices of $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$. Now, the covariance matrices explicitly write

$$\begin{aligned} \tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,\mathbf{y}} &= \int_s^{s+\varepsilon} du \tilde{\mathbf{R}}^{s+\varepsilon,\mathbf{y}}(s+\varepsilon, u) Ba(u, \boldsymbol{\theta}_{u,s+\varepsilon}(\mathbf{y})) B^* \tilde{\mathbf{R}}^{s+\varepsilon,\mathbf{y}}(s+\varepsilon, u)^*, \\ \mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}} &= \int_s^{s+\varepsilon} du \mathbf{R}^{s,\mathbf{x}}(s+\varepsilon, u) Ba(u, \boldsymbol{\theta}_{u,s}(\mathbf{x})) B^* \mathbf{R}^{s,\mathbf{x}}(s+\varepsilon, u)^*, \end{aligned}$$

where $\tilde{\mathbf{R}}^{s+\varepsilon,\mathbf{y}}, \mathbf{R}^{s,\mathbf{x}}$ respectively denote the resolvents associated to the linear parts of equations (2.4) and (A.1). Thus, our standing smoothness assumptions in (\mathbf{A}) (i.e. $a, (\nabla_{i-1} \mathbf{F}_i)_{i \in \llbracket 2, n \rrbracket}$ are supposed to be uniformly η -Hölder continuous) and the bi-Lipschitz property of the flow give:

$$\begin{aligned} & \left| \langle (\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,\mathbf{y}} - \mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}}) (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right| \\ &= \left| \langle (\widehat{\mathbf{K}}_1^{s,s+\varepsilon,s+\varepsilon,\mathbf{y}} - \widehat{\mathbf{K}}_1^{s,s+\varepsilon,s,\mathbf{x}}) (\varepsilon^{-1/2} \mathbb{T}_\varepsilon^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})), \varepsilon^{-1/2} \mathbb{T}_\varepsilon^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}) \rangle \right| \\ &\leq C |\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}|^\eta \varepsilon^{-1} |\mathbb{T}_\varepsilon^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2. \end{aligned}$$

Because of the non degeneracy of a , the inverse matrices $(\widehat{\mathbf{K}}_1^{s,s+\varepsilon,s+\varepsilon,\mathbf{y}})^{-1}, (\widehat{\mathbf{K}}_1^{s,s+\varepsilon,s,\mathbf{x}})^{-1}$ have the same Hölder regularity. Indeed, up to a change of coordinates one can assume that one of the two matrices is diagonal at the point considered and that the other has dominant diagonal if $|\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}|$ is small enough (depending on the ellipticity bound and the dimension). This reduces to the scalar case. Hence,

$$\begin{aligned} |Q_\varepsilon| &:= \left| \langle ((\tilde{\mathbf{K}}_{s,s+\varepsilon}^{s+\varepsilon,\mathbf{y}})^{-1} - (\mathbf{K}_{s,s+\varepsilon}^{s,\mathbf{x}})^{-1}) (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}), \tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y} \rangle \right| \\ &= \left| \langle ((\widehat{\mathbf{K}}_1^{s,s+\varepsilon,s+\varepsilon,\mathbf{y}})^{-1} - (\widehat{\mathbf{K}}_1^{s,s+\varepsilon,s,\mathbf{x}})^{-1}) (\varepsilon^{1/2} \mathbb{T}_\varepsilon^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})), \varepsilon^{1/2} \mathbb{T}_\varepsilon^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y}) \rangle \right| \\ &\leq C |\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}|^\eta \varepsilon |\mathbb{T}_\varepsilon^{-1} (\tilde{\boldsymbol{\theta}}_{s+\varepsilon,s}^{s+\varepsilon,\mathbf{y}}(\mathbf{x}) - \mathbf{y})|^2 \leq C |\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y}|^\eta \varepsilon |\mathbb{T}_\varepsilon^{-1} (\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|^2, \end{aligned}$$

for $C := C((\mathbf{A}-\eta))$ using Lemma 2.1 and the bi-Lipschitz property of the flow $\boldsymbol{\theta}$ for the last inequality. From equations (A.13), (A.12) and (A.2), we eventually get:

$$\begin{aligned} |\Xi_2^\varepsilon(s, \mathbf{x})| &\leq C \varepsilon^{\eta/2} \int_{\mathbb{R}^{nd}} \frac{d\mathbf{y}}{\varepsilon^{n^2 d/2}} (\varepsilon^{1/2} |\mathbb{T}_\varepsilon^{-1} (\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|)^\eta \exp(-C \varepsilon |\mathbb{T}_\varepsilon^{-1} (\boldsymbol{\theta}_{s+\varepsilon,s}(\mathbf{x}) - \mathbf{y})|^2) \\ &\leq C \varepsilon^{\eta/2}, \end{aligned} \tag{A.14}$$

for $C := C((\mathbf{A}-\eta))$. Arguments similar to those employed for $\Xi_2^\varepsilon(s, \mathbf{x})$ can be used to prove $\Xi_1^\varepsilon(s, \mathbf{x}) \xrightarrow{\varepsilon \rightarrow 0} 0$. The proof then follows from (A.4), (A.11), (A.14) recalling the original decomposition (A.3).

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