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CORRECTION TO "THE MEASURABILITY OF HITTING TIMES"

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Abstract

We correct an error in [1].

There is an error in the proof of Theorem 2.1, which was pointed out to me by K. Szczypkowski. On page 101, line 18, although $A_n \setminus L_n \subset \cup_{i=1}^n (A_i \setminus K_i)$, the assertion concerning the projections is not necessarily true.
The following should replace the proof of Theorem 2.1, from line 6 of page 101 through the line that is 7 lines before the end of page 101.

A The correction

A set A is a $\mathbb{P}$-null set if $\mathbb{P}^*(A) = 0$. The following lemma is well known; see, e.g., [2, p. 94].

Lemma A.1. *(a) If $A \subset \Omega$, there exists $C \in \mathscr{F}$ such that $A \subset C$ and $\mathbb{P}^*(A) = \mathbb{P}(C)$.*
(b) Suppose $A_n \uparrow A$. Then $\mathbb{P}^*(A) = \lim_{n\to\infty} \mathbb{P}^*(A_n)$.

Proof. (a) By the definition of $\mathbb{P}^*(A)$, for each n there exists $C_n \in \mathscr{F}$ such that $A \subset C_n$ and $\mathbb{P}(C_n) \le \mathbb{P}^*(A) + (1/n)$. Setting $C = \cap_n C_n$, we have $A \subset C$, $C \in \mathscr{F}$, and $\mathbb{P}(C) \le \mathbb{P}(C_n) \le \mathbb{P}^*(A) + (1/n)$ for each n, hence $\mathbb{P}(C) \le \mathbb{P}^*(A)$.
(b) Choose $C_n \in \mathscr{F}$ with $A_n \subset C_n$ and $\mathbb{P}^*(A_n) = \mathbb{P}(C_n)$. Let $D_n = \cap_{k\ge n} C_k$ and $D = \cup_n D_n$. We see that $D_n \uparrow D$, $D \in \mathscr{F}$, and $A \subset D$. Then

$$\mathbb{P}^*(A) \ge \sup_n \mathbb{P}^*(A_n) = \sup_n \mathbb{P}(C_n) \ge \sup_n \mathbb{P}(D_n) = \mathbb{P}(D) \ge \mathbb{P}^*(A). \qquad \square$$

Let $\mathscr{T}_t = [0,t] \times \Omega$. Given a compact Hausdorff space X, let $\rho^X : X \times \mathscr{T}_t \to \mathscr{T}_t$ be defined by $\rho^X(x,(s,\omega)) = (s,\omega)$. Let

$$\mathscr{L}_0(X) = \{A \times B : A \subset X, A \text{ compact}, B \in \mathscr{K}(t)\},$$

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$\mathscr{L}_1(X)$ the class of finite unions of sets in $\mathscr{L}_0(X)$, and $\mathscr{L}(X)$ the class of intersections of countable decreasing sequences in $\mathscr{L}_1(X)$. Let $\mathscr{L}_\sigma(X)$ be the class of unions of countable increasing sequences of sets in $\mathscr{L}(X)$ and $\mathscr{L}_{\sigma\delta}(X)$ the class of intersections of countable decreasing sequences of sets in $\mathscr{L}_\sigma(X)$.

Lemma A.2. *If $A \in \mathscr{B}[0,t] \times \mathscr{F}_t$, there exists a compact Hausdorff space X and $B \in \mathscr{L}_{\sigma\delta}(X)$ such that $A = \rho^X(B)$.*

Proof. If $A \in \mathscr{K}(t)$, we take $X = [0,1]$, the unit interval with the usual topology and $B = X \times A$. Thus the collection $\mathscr{M}$ of subsets of $\mathscr{B}[0,t] \times \mathscr{F}_t$ for which the lemma is satisfied contains $\mathscr{K}(t)$. We will show that $\mathscr{M}$ is a monotone class.

Suppose $A_n \in \mathscr{M}$ with $A_n \downarrow A$. There exist compact Hausdorff spaces X_n and sets $B_n \in \mathscr{L}_{\sigma\delta}(X_n)$ such that $A_n = \rho^{X_n}(B_n)$. Let $X = \prod_{n=1}^\infty X_n$ be furnished with the product topology. Let $\tau_n : X \times \mathscr{T}_t \to X_n \times \mathscr{T}_t$ be defined by $\tau_n(x,(s,\omega)) = (x_n,(s,\omega))$ if $x = (x_1, x_2, \ldots)$. Let $C_n = \tau_n^{-1}(B_n)$ and let $C = \cap_n C_n$. It is easy to check that $\mathscr{L}(X)$ is closed under the operations of finite unions and intersections, from which it follows that $C \in \mathscr{L}_{\sigma\delta}(X)$. If $(s,\omega) \in A$, then for each n there exists $x_n \in X_n$ such that $(x_n,(s,\omega)) \in B_n$. Note that $((x_1, x_2, \ldots),(s,\omega)) \in C$ and therefore $(s,\omega) \in \rho^X(C)$. It is straightforward that $\rho^X(C) \subset A$, and we conclude $A \in \mathscr{M}$.

Now suppose $A_n \in \mathscr{M}$ with $A_n \uparrow A$. Let X_n and B_n be as before. Let $X' = \cup_{n=1}^\infty X_n \times \{n\}$ with the topology generated by $\{G \times \{n\} : G \text{ open in } X_n\}$. Let X be the one point compactification of X'. We can write $B_n = \cap_m B_{nm}$ with $B_{nm} \in \mathscr{L}_\sigma(X_n)$. Let

$$C_{nm} = \{((x,n),(s,\omega)) \in X \times \mathscr{T}_t : x \in X_n, (x,(s,\omega)) \in B_{nm}\},$$

$C_n = \cap_m C_{nm}$, and $C = \cup_n C_n$. Then $C_{nm} \in \mathscr{L}_\sigma(X)$ and so $C_n \in \mathscr{L}_{\sigma\delta}(X)$.

If $((x,p),(s,\omega)) \in \cap_m \cup_n C_{nm}$, then for each m there exists n_m such that $((x,p),(s,\omega)) \in C_{n_m m}$. This is only possible if $n_m = p$ for each m. Thus $((x,p),(s,\omega)) \in \cap_m C_{pm} = C_p \subset C$. The other inclusion is easier and we thus obtain $C = \cap_m \cup_n C_{nm}$, which implies $C \in \mathscr{L}_{\sigma\delta}(X)$. We check that $A = \rho^X(C)$ along the same lines, and therefore $A \in \mathscr{M}$.

If $\mathscr{I}^0(t)$ is the collection of sets of the form $[a,b) \times C$, where $a < b \le t$ and $C \in \mathscr{F}_t$, and $\mathscr{I}(t)$ is the collection of finite unions of sets in $\mathscr{I}^0(t)$, then $\mathscr{I}(t)$ is an algebra of sets. We note that $\mathscr{I}(t)$ generates the σ-field $\mathscr{B}[0,t] \times \mathscr{F}_t$. A set in $\mathscr{I}^0(t)$ of the form $[a,b) \times C$ is the union of sets in $\mathscr{K}^0(t)$ of the form $[a, b-(1/m)] \times C$, and it follows that every set in $\mathscr{I}(t)$ is the increasing union of sets in $\mathscr{K}(t)$. Since $\mathscr{M}$ is a monotone class containing $\mathscr{K}(t)$, then $\mathscr{M}$ contains $\mathscr{I}(t)$. By the monotone class theorem, $\mathscr{M} = \mathscr{B}[0,t] \times \mathscr{F}_t$. □

The works of Suslin and Lusin present a different approach to the idea of representing Borel sets as projections; see, e.g., [4, p. 88] or [3, p. 284].

Lemma A.3. *If $A \in \mathscr{B}[0,t] \times \mathscr{F}_t$, then A is t-approximable.*

Proof. We first prove that if $H \in \mathscr{L}(X)$, then $\rho^X(H) \in \mathscr{K}_\delta$. If $H \in \mathscr{L}_1(X)$, this is clear. Suppose that $H_n \downarrow H$ with each $H_n \in \mathscr{L}_1(X)$. If $(s,\omega) \in \cap_n \rho^X(H_n)$, there exist $x_n \in X$ such that $(x_n,(s,\omega)) \in H_n$. Then there exists a subsequence such that $x_{n_k} \to x_\infty$ by the compactness of X. Now $(x_{n_k},(s,\omega)) \in H_{n_k} \subset H_m$ for n_k larger than m. For fixed ω, $\{(x,s) : (x,(s,\omega)) \in H_m\}$ is compact, so $(x_\infty,(s,\omega)) \in H_m$ for all m. This implies $(x_\infty,(s,\omega)) \in H$. The other inclusion is easier and therefore $\cap_n \rho^X(H_n) = \rho^X(H)$. Since $\rho^X(H_n) \in \mathscr{K}_\delta(t)$, then $\rho^X(H) \in \mathscr{K}_\delta(t)$. We also observe that for fixed ω, $\{(x,s) : (x,(s,\omega)) \in H\}$ is compact.

Now suppose $A \in \mathscr{B}[0,t] \times \mathscr{F}_t$. Then by Lemma A.2 there exists a compact Hausdorff space X and $B \in \mathscr{L}_{\sigma\delta}(X)$ such that $A = \rho^X(B)$. We can write $B = \cap_n B_n$ and $B_n = \cup_m B_{nm}$ with $B_n \downarrow B$, $B_{nm} \uparrow B_n$, and $B_{nm} \in \mathscr{L}(X)$.
Let $a = \mathbb{P}^*(\pi(A)) = \mathbb{P}^*(\pi \circ \rho^X(B))$ and let $\varepsilon > 0$. By Lemma A.1,

$$\mathbb{P}^*(\pi \circ \rho^X(B \cap B_{1m})) \uparrow \mathbb{P}^*(\pi \circ \rho^X(B \cap B_1)) = \mathbb{P}^*(\pi \circ \rho^X(B)) = a.$$

Take m large enough so that $\mathbb{P}^*(\pi \circ \rho^X(B \cap B_{1m})) > a - \varepsilon$, let $C_1 = B_{1m}$, and $D_1 = B \cap C_1$.
We proceed by induction. Suppose we are given sets $C_1, \dots, C_{n-1}$ and sets $D_1, \dots, D_{n-1}$ with $D_{n-1} = B \cap (\cap_{i=1}^{n-1} C_i)$, $\mathbb{P}^*(\pi \circ \rho^X(D_{n-1})) > a - \varepsilon$, and each $C_i = B_{im_i}$ for some m_i. Since $D_{n-1} \subset B \subset B_n$, by Lemma A.1

$$\mathbb{P}^*(\pi \circ \rho^X(D_{n-1} \cap B_{nm})) \uparrow \mathbb{P}^*(\pi \circ \rho^X(D_{n-1} \cap B_n)) = \mathbb{P}^*(\pi \circ \rho^X(D_{n-1})).$$

We can take m large enough so that $\mathbb{P}^*(\pi \circ \rho^X(D_{n-1} \cap B_{nm})) > a - \varepsilon$, let $C_n = B_{nm}$, and $D_n = D_{n-1} \cap C_n$.
If we let $G_n = C_1 \cap \cdots \cap C_n$ and $G = \cap_n G_n = \cap_n C_n$, then each G_n is in $\mathscr{L}(X)$, hence $G \in \mathscr{L}(X)$. Since $C_n \subset B_n$, then $G \subset \cap_n B_n = B$. Each $G_n \in \mathscr{L}(X)$ and so by the first paragraph of this proof, for each fixed ω and n, $\{(x,s) : (x,(s,\omega)) \in G_n\}$ is compact. Hence by a proof very similar to that of Lemma 2.2, $\pi \circ \rho^X(G_n) \downarrow \pi \circ \rho^X(G)$. Using the first paragraph of this proof and Lemma 2.2, we see that

$$\mathbb{P}(\pi \circ \rho^X(G)) = \lim \mathbb{P}(\pi \circ \rho^X(G_n)) \geq \lim \mathbb{P}^*(\pi \circ \rho^X(D_n)) \geq a - \varepsilon.$$

Using the first paragraph of this proof once again, we see that A is t-approximable. □

Proof of Theorem 2.1. Let E be a progressively measurable set and let $A = E \cap ([0,t] \times \Omega)$. By Lemma A.3, A is t-approximable. By Proposition 2.3, $(D_E \leq t) = \pi(A) \in \mathscr{F}_t$. Because t was arbitrary, we conclude D_E is a stopping time. □

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References

[1] R.F. Bass. The measurability of hitting times. *Electron. Comm. Prob.* **15** (2010) 99–105. MR2606507 , http://www.math.washington.edu/~ejpecp/ECP/viewarticle.php?id=2181&layout=abstract. .

[2] K. Bichteler. *Integration: a Functional Approach.* Birkhäuser, Basel, 1998. MR1635816

[3] D.L. Cohn. *Measure Theory.* Birkhäuser, Boston, 1980. MR0578344

[4] I. Fonseca and G. Leoni. *Modern Methods in the Calculus of Variations: L^p Spaces.* Springer, New York, 2007. MR2341508