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A REFLECTION TYPE PROBLEM FOR THE STOCHASTIC 2-D NAVIER-STOKES EQUATIONS WITH PERIODIC CONDITIONS

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Abstract
We prove the existence of a solution for the Kolmogorov equation associated with a reflection problem for 2-D stochastic Navier-Stokes equations with periodic spatial conditions and the corresponding stream flow in a closed ball of a Sobolev space of the torus $\mathbb{T}^2$.

1 Introduction

We consider here the 2-D stochastic Navier-Stokes equation for an incompressible non-viscous fluid

$$\left\{\begin{array}{l} dX - \nu\Delta X\,dt + (X\cdot\nabla)X\,dt = \nabla p\,dt + dW_t \\ \nabla\cdot X = 0 \end{array}\right. \tag{1}$$

This equation is considered on a 2-D torus, that we identify with the square $\mathbb{T}^2 = [0,2\pi]\times[0,2\pi]$ and with periodic boundary conditions.
Here ν is the viscosity of the fluid, X is the velocity field, p is the pressure and W is a cylindrical Wiener process.

If we denote by $\phi : \mathbb{T}^2 \to \mathbb{R}$ the corresponding stream function, that is

$$X = \nabla^{\perp}\phi, \quad -\Delta\phi = \operatorname{curl} X, \quad \phi(\xi+2\pi) \equiv \phi(\xi) \tag{2}$$

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where $\nabla^\perp = (-D_2, D_1)$, curl $X = D_2X_1 - D_1X_2$, $\quad X = (X_1, X_2)$ we may rewrite (1) in terms of the stream function ϕ (see [1], [2])

$$d(\nabla^\perp \phi) - \nu\Delta\nabla^\perp\phi\, dt + (\nabla^\perp\phi \cdot \nabla)\nabla^\perp\phi\, dt = \nabla p\, dt + dW_t \tag{3}$$

and formulate for (1) the corresponding reflection problem on the set

$$K = \{\phi \in H^{1-\alpha}(\mathbb{T};\mathbb{R}^2) : \|\phi\|_{1-\alpha} \le \ell\} \tag{4}$$

where $H^{1-\alpha}$ is the Sobolev space of order $1-\alpha$ with $\alpha > \frac{3}{2}$, with respect to the natural Gibbs measure μ given by enstrophy (see Section 2 below.)
More precisely, we shall prove that the Kolmogorov equation associated with (1), (2) and (4) has at least one solution $\varphi : \mathbb{T}^2 \to \mathbb{R}$. In terms of coordinates $u_j = \frac{1}{2\pi}\int_{\mathbb{T}^2} e^{\mathrm{i}j\cdot\xi}\phi(\xi)\,d\xi$ this equation has the form

$$\begin{cases} \lambda\varphi - L\varphi = f & \text{in } \mathring{K} \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial K. \end{cases} \tag{5}$$

where L is the Kolmogorov operator

$$L\varphi(u) = \sum_{k\in\mathbb{Z}^2}\left[\frac{1}{2k^2}D_k^2\varphi(u) - \nu k^2 u_k D_k\varphi(u) - B_k(u)D_k\varphi(u)\right], \tag{6}$$

defined on a space $\mathscr{F}C_b^2$ of cylindrical smooth functions. (The function B_k is defined in (10).)
The main result of this work, Theorem 1 below, amounts to saying that the Neumann problem (5) has at least one weak solution φ, but the uniqueness of this solution remains open. It should be said that the uniqueness is still an open problem in the case $K = H^{1-\alpha}$ and it is equivalent in the later case with the unique extension of operator L from $\mathscr{F}C_b^2$ to an m-dissipative operator in $L^2(\mu)$ see [3]. We mention, however, that L is essentially m-dissipative in $L^1(\mu)$ when the viscosity ν is sufficiently large (Stannat [11]). It should mention also that in this way the study of stochastic process $X = X_t$ reduces to a linear infinite dimensional equation in the space $H^{1-\alpha}$ associated to the operator L.

There is a large number of works devoted to infinite dimensional stochastic reflection problems but most of them are, except a few notable works, concerned with Wiener processes W with finite covariance. So the existence theory for (13) is still open.
Here following the way developped in [5], [6], we will treat instead of (1) its associated Kolmogorov equation which as noted in Introduction will lead to an infinite dimensional Neumann problem on the convex K. (The Kolmogorov equation [6] in the special case $K = H^{1-\alpha}$ was previously studied by Flandoli and Gozzi [9].)

Previous results on infinite dimensional reflection problems, starting from [10] are essentially concerned with reversible systems. We believe that the present paper is the first attempt to study non symmetric infinite dimensional Kolmogorov operators with Neumann boundary conditions.

2 The functional setting

Consider the Sobolev space of order $p \in \mathbb{R}$ defined by

$$H^p = \left\{ y(\xi) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}_+^2} u_j e^{\mathrm{i}j\cdot\xi} : \sum_{j\in\mathbb{Z}_+^2} j^{2p}|u_j|^2 < +\infty \right\}$$

where $j=(j_1,j_2)$ and $\mathbb{Z}^2_+=\{j\in\mathbb{Z}^2:\ j_1>0 \text{ or } j_1=0, j_2>0\}$. We set also $\mathbb{Z}^2_0=\mathbb{Z}^2\setminus\{(0,0)\}$, $j^2=j_1^2+j_2^2$ and set $u=\{u_j\}_{j\in\mathbb{Z}^2_0}$, $u_j=\bar u_{-j}$ for $j\in\mathbb{Z}^2_0\setminus\mathbb{Z}^2_+$. The space H^p is a complex Hilbert space with the scalar product

$$\langle y_1,y_2\rangle_p=\sum_{j\in\mathbb{Z}^2_+} j^{2p}(y_1)_j(\bar y_2)_j,\quad y_j=\frac{1}{2\pi}\int_{\mathbb{T}^2} y(\xi)e^{\mathrm{i}j\cdot\xi}\,d\xi.$$

Consider the Gibbs measure $\mu=\mu_\nu$ given by the enstrophy, that is

$$d\mu(u)=\prod_{j\in\mathbb{Z}^2_+} d\mu^i(u_j), d\mu^j(z)=\frac{\nu\, j^4}{2\pi}\exp\Big(-\tfrac12\nu j^4|z|^2\Big)\,dx\,dy, z=x+\mathrm{i}y.$$

We recall (see [1], [3]) that for $\alpha>0$ we have

$$\int_H |u|^2_{1-\alpha}\,d\mu(u)<\infty,$$

and so the probability measure μ is supported by H^p, $p<1$. For each $q\ge 1$ we denote the space $L^q(\Lambda,\mu)$ by $L^q(\mu)$.
We denote by $H^{1,2}(H^\delta,\mu)$ the completion of the space $\mathscr{F}C^2_b$ in the norm

$$\|\varphi\|^2_\delta=\sum_{j\in\mathbb{Z}^2_0}|j|^{2\delta}\int_H|D_j\varphi|^2\,d\mu+\int_H|\varphi|^2\,d\mu.$$

Given a closed convex subset $K\subset H^\delta$ with smooth boundary we denote by $H^{1,2}_\delta(K,\mu)$ the space $\{\varphi\big|_K:\varphi\in H^{1,2}(H^\delta,\mu)\}$ with the norm

$$\|\varphi\|^2_{H^{1,2}(K,\mu)}=\sum_{j\in\mathbb{Z}^2_0}|j|^{2\delta}\int_K|D_j\varphi|^2\,d\mu+\int_K|\varphi|^2\,d\mu.$$

There is a standard way (see [1], [2]) to reduce equation (1) to a differential equation in $H^{1-\alpha}$ we briefly present below. Namely applying the curl operator into (3) we get for $\psi=\operatorname{curl}X$ the equation

$$d\psi-\nu\Delta\psi\,dt+\operatorname{curl}\,[(\nabla^\perp\phi\cdot\nabla)\nabla^\perp\phi]\,dt=d\operatorname{curl}W_t.$$

Now, we expand ϕ in Fourier series

$$\phi(t,\xi)=\frac{1}{2\pi}\sum_{j\in\mathbb{Z}^2_0}u_j(t)e^{\mathrm{i}j\cdot\xi} \tag{7}$$

and take W to be the cylindrical Wiener process

$$W_t=\frac{1}{2\pi}\sum_{j\in\mathbb{Z}^2_0}|j|^{-1}\nabla^\perp(e^{\mathrm{i}j\cdot\xi})W_j(t) \tag{8}$$

where $\{W_j\}_{j\in\mathbb{Z}^2_0}$ are independent Brownian motions in a probability space $\{\Omega,\mathscr{F},\mathbb{P},\{\mathscr{F}_t\}_{t\ge0}\}$. We note that

$$\operatorname{curl}W_t=-\frac{1}{2\pi}\sum_{j\in\mathbb{Z}^2_0}|j|\,e^{\mathrm{i}j\cdot\xi}W_j(t)$$

By (7) we have

$$\psi(t,\xi) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}_0^2} j^2 u_j(t) e^{\mathrm{i} j\cdot\xi}, \quad \Delta\psi(t,\xi) = -\frac{1}{2\pi}\sum_{j\in\mathbb{Z}_0^2} (j^2)^2 u_j(t) e^{\mathrm{i} j\cdot\xi}$$

and (see [2])

$$\text{curl}\,[(\nabla^\perp\phi\cdot\nabla)\nabla^\perp\phi] = \sum_{j\in\mathbb{Z}_0^2} j^2 B_j(u).$$

Then (1) reduces to

$$du_j(t) + \nu j^2 u_j(t)\,dt - B_j(u(t))\,dt = |j|^{-1} dW_j(t). \tag{9}$$

Here we have used the notation

$$B_j(u) = \sum_{\substack{h\neq 0\\ h\neq j}} \alpha_{h,j}\, u_j\, u_{j-h}, \quad \alpha_{h,j} = \frac{1}{2\pi}\left[j^{-2}(j\cdot h^\perp)(j\cdot h) - \frac{1}{2}h^\perp\cdot j\right], \tag{10}$$

and $h^\perp = (-h_2, h_1)$, $h = (h_1, h_2)$. Since the function ϕ is real valued one must have $u_k = \bar{u}_{-k}$ and this implies $\bar{B}_k = B_{-k}$ for all k.
It turns out that if $p < -1$ then the vector field $B = \{B_j\}_{j\in\mathbb{Z}_0^2}$ is L^q-integrable in the norm $|\cdot|_p$ with respect to the Gibbs measure μ for all $q \geq 1$.
One also has (see [7])

$$\sum_{j\in\mathbb{Z}_0^2} j^{2p}\left(\int |B_j(u)|^{2q}\,d\mu\right)^{\frac{1}{2}} < \infty. \tag{11}$$

Moreover, the measure μ is infinitesimally invariant for B (see [1], [7].)
Equation (9) can be written in $H^{1-\alpha}$ as

$$du + \nu QAu\,dt - Bu\,dt = dW_t \tag{12}$$

where

$$Au = \{k^{-(1+\alpha)}u_k\}_{k\in\mathbb{Z}_0^2}, \quad W_t = \{|j|^{-1}W_j(t)\}_{j\in\mathbb{Z}_0^2}, \quad Qv = \{k^{3+\alpha}v_k\}_{k\in\mathbb{Z}_0^2}.$$

We recall (see [1]) that A is a Hilbert-Schmidt operator on H^2 and $|Au|_2 = |u|_{1-\alpha}$.
Now, we associate with (12) the stochastic variational inequality

$$du + \nu QAu\,dt - B(u)\,dt + R\partial I_K(u)\,dt \ni dW_t \tag{13}$$

where $Rv = \{k^{-2\alpha}v_k\}_{k\in\mathbb{Z}_0^2}$, K is a smooth closed and convex subset of $H = H^{1-\alpha}$ and $\partial I_K : K \to 2^H$ is the normal cone to K. Formally (13) can be written as

$$\begin{cases} du(t) + \nu QAu(t)\,dt - Bu(t)\,dt = dW_t & \text{in } \{t \mid u(t)\in \mathring{K}\}\\ du(t) + \nu QAu(t)\,dt - Bu(t)\,dt + \lambda(t)Rn_K(u(t)) = dW_t & \text{in } \{t \mid u(t)\in \partial K\}\\ u(t)\in K \quad \forall t\geq 0 & \end{cases}$$

where $\lambda(t) \geq 0$ and $n_K(u)$ is the unit exterior normal to ∂K.

Coming back to equation (1) and taking into account (2) the variational inequality (13) can be rewritten in terms of the velocity field X under the form

$$\begin{cases} dX - \nu\Delta X\,dt + (X\cdot\nabla)X\,dt + N_{\mathscr{K}}(X)\,dt \ni \nabla p\,dt + dW_t \\ \nabla\cdot X = 0, X = 0 \text{ on } \partial\mathscr{O} \end{cases} \tag{14}$$

where $N_{\mathscr{K}}(X)$ is the normal cone to the closed convex set $\mathscr{K}$ of $\{X \in \left(L^2(0,2\pi)\right)^2; \nabla\cdot X = 0, X(0) = X(2\pi)\}$ defined by,

$$\mathscr{K} = \{X:\ \{\langle\phi, e^{-\mathrm{i}j\cdot\xi}\rangle_{L^2(\mathbb{T}^2)}\}_{j\in\mathbb{Z}_0^2} \in K,\ \phi = (-\Delta)^{-1}\mathrm{curl}\,X\}.$$

This is the reflection problem to the boundary of $\mathscr{K}$ on the oblique normal direction $N_{\mathscr{K}}(x)$. In the special case of K given by (4) its meaning is that the stream value ϕ of the fluid is constrained to the set $\|\phi\|_{1-\alpha} \leq \ell$ and when ϕ reaches the boundary ∂K in the dynamic of fluid arises a convective acceleration oriented toward interior of K along an oblique direction. Indeed we have by definition of the normal cone $N_{\mathscr{K}}(X)$,

$$N_{\mathscr{K}}(X) = \left\{\eta \in \left(L^2(0,2\pi)\right)^2; \int_0^{2\pi}\int_0^{2\pi} \eta(\xi)(X(\xi) - Y(\xi))d\xi \geq 0 \quad \forall Y \in \mathscr{K}\right\}$$

Recalling that by (2), (7),

$$X = \frac{\mathrm{i}}{2\pi}\sum_{j\in\mathbb{Z}_0^2} j^{\perp}\, u_j\, e^{\mathrm{i}\,j\cdot\xi}$$

and setting

$$\eta = \frac{\mathrm{i}}{2\pi}\sum_{j\in\mathbb{Z}_0^2} j^{\perp}\,\eta_j\, e^{\mathrm{i}\,j\cdot\xi}, \quad Y = \frac{\mathrm{i}}{2\pi}\sum_{j\in\mathbb{Z}_0^2} j^{\perp}\, v_j\, e^{\mathrm{i}\,j\cdot\xi}$$

where $\{\eta_j\}_j$, $\{v_j\}_j \in H^{1-\alpha}$, we see that

$$N_{\mathscr{K}}(X) = \left\{\eta; \sum_{j\in\mathbb{Z}_0^2} |j|^2\,\eta_j\,(\bar{u}_j - \bar{v}_j) \geq 0, \forall\{v_j\}_j \in K\right\}$$

On the other hand, the normal cone $N_K(u)$ to K in $H^{1-\alpha}$ is given by

$$N_K(u) = \left\{\tilde{\eta} = \{\tilde{\eta}_j\}_j; \sum_{j\in\mathbb{Z}_0^2} j^{2(1-\alpha)}\,\tilde{\eta}_j\,(\bar{u}_j - \bar{v}_j) \geq 0, \forall\tilde{u} = \{u_j\}_j \in K\right\}$$

Hence

$$N_{\mathscr{K}}(X) = \left\{\eta; \langle\eta, e^{\mathrm{i}\,j\cdot\xi}\rangle_{L^2(\mathbb{T}^2)} = \eta_j = j^{-2\alpha}\tilde{\eta}_j; \{\tilde{\eta}_j\}_j \in N_K(u)\right\}$$

and taking into account (13) and definition of $\mathscr{K}$ this yields (14) as claimed.

3 The Kolmogorov equation

Consider the Kolmogorov operator L corresponding to (9) which is defined by (6) on the space $\mathscr{F}C_b^2$ of cylindrical C^2-functions

$$\mathscr{F}C_b^2 = \{\varphi = \varphi(u_{j_1}, u_{j_2}, \ldots, u_{j_n}) : n \geq 1,\ j_1, u_{j_2},\ \ldots, u_{j_n} \in \mathbb{Z}_0^2,\ \varphi \in C_b^2(\mathbb{C}^n)\}.$$

We recall (see e.g., [1], [2], [3]) that the measure μ is invariant for operator L. As noticed earlier the essential m-dissipativity of L in the space $L^2(\mu)$ is still an open problem.
Our aim here is to study the Neumann problem

$$\begin{cases} \lambda\varphi - L\varphi = f & \text{in } \mathring{K} \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial K =: \Sigma \end{cases} \tag{15}$$

considered in some generalized sense to be precised below.

Definition 1. *The function $\varphi : K \to \mathbb{R}$ is said to be weak solution to* (15) *if*

$$\int_K |\varphi|^2\, d\mu < \infty, \qquad \sum_{j\in\mathbb{Z}_0^2} j^{-2} \int_K |D_j\varphi|^2\, d\mu < \infty, \tag{16}$$

and

$$\lambda \int_K \varphi\,\psi\, d\mu + \frac{1}{2}\sum_{j\in\mathbb{Z}_0^2} j^{-2}\int_K D_j\varphi\, D_j\psi\, d\mu - \sum_{j\in\mathbb{Z}_0^2}\int_K B_j(u)\, D_j\psi(u)\,\varphi(u)\, d\mu(u) = \int_K f\,\psi\, d\mu \tag{17}$$

for all real valued $\psi \in \mathscr{F}C_b^2$.

It is readily seen by (11) that (14) makes sense for all $\psi \in \mathscr{F}C_b^2$.
Theorem 1 below is the main result.

Theorem 1. *Assume that $\alpha > \frac{3}{2}$ and*

$$K = \{u \in H^{1-\alpha} : |u|_{1-\alpha} \le \ell\} \tag{18}$$

then for each real valued $f \in L^2(K,\mu)$ problem (5) *has at least one weak solution $\varphi \in H^{1,2}_{-1}(K,\mu)$ and the following estimates hold*

$$\lambda \int_K |\varphi|^2\, d\mu + \frac{1}{2}\sum_{j\in\mathbb{Z}_0^2} j^{-2}\int_K |D_j\varphi|^2\, d\mu \le C\int_K |f|^2\, d\mu \tag{19}$$

$$\int_K |\varphi|^2\, d\mu \le \frac{1}{\lambda^2}\int_K |f|^2\, d\mu. \tag{20}$$

In (17) as well as in (16),(19) by $D_j\varphi$ we mean of course the distributional derivative D_j of function φ which belongs to $L^2(\mu)$.

Remark 1. If φ is a smooth solution to elliptic problem (15) then it is easily seen via integration by parts that φ is also weak solution in the sense of Definition 1.

4 Proof of Theorem 1

To prove Theorem 1 we consider the approximating equation

$$\lambda\varphi_\varepsilon - L\varphi_\varepsilon + \sum_{j\in\mathbb{Z}_0^2} j^{-4}\beta_j^\varepsilon D_j\varphi_\varepsilon = f, \tag{21}$$

where L is given by (6) and

$$\beta^\varepsilon(u) = \frac{1}{\varepsilon}(u - \Pi_K u) = \frac{u}{\varepsilon}\Big(1 - \frac{\ell}{|u|_{1-\alpha}}\Big), \quad u \in H.$$

(Here Π_K is the projection on K.) We introduce also the measure

$$d\mu_\varepsilon(u) = \prod_k e^{-\frac{k^4 d_K^2(u)}{2\varepsilon}} d\mu_k(u)$$

and note that

$$D_j\Big(e^{-\frac{j^4 d_K^2(u)}{2\varepsilon}}\Big) = -j^4\beta_j^\varepsilon(u)\, e^{-\frac{j^4 d_K^2(u)}{\varepsilon}}.$$

It should be mentioned that equation (21) in spite of its apparent simplicity is still unsolvable for all $f \in L^2(\mu)$ and the reason is that as mentioned earlier we dont know whether the operator L is essentially m-dissipative. In order to circumvent this we shall define just a weak solution concept for (21) and prove the existence of such a solution.

Definition 2. *The function $\varphi_\varepsilon : H = H^{1-\alpha} \to \mathbb{R}$ is said to be weak solution to equation* (21) *if the following conditions hold, $\varphi_\varepsilon \in H^{1,2}_{-1}(\mu)$, that is*

$$\int \varphi_\varepsilon^2\, d\mu_\varepsilon < \infty, \quad \sum_{k\in\mathbb{Z}_0^2} k^{-2}\int |D\varphi_\varepsilon|^2\, d\mu_\varepsilon < \infty \tag{22}$$

and

$$\lambda\int \varphi_\varepsilon\,\psi\, d\mu_\varepsilon + \sum_{k\in\mathbb{Z}_0^2} k^{-2}\int_H D_k\varphi_\varepsilon\, D_k\psi\, d\mu_\varepsilon + \\ + \sum_{k\in\mathbb{Z}_0^2}\int B_k(u)\, D_k\psi\,\varphi_\varepsilon\, d\mu_\varepsilon = \int f\,\psi\, d\mu_\varepsilon \tag{23}$$

for all real valued cylindrical functions $\psi \in \mathscr{F}C_b^2$.

We note that Definition 2 is in the spirit of Definition 1 and that if φ_ε is a smooth solution to (21) then we see by (21) via integration by parts that φ_ε satisfies also (23). We note that

$$\sum_{k\in\mathbb{Z}_0^2}\int B_k(u)\, D_k\varphi_\varepsilon\,\psi\, d\mu_\varepsilon = \\ -\sum_{k\in\mathbb{Z}_0^2}\int B_k(u)\, D_k\psi\,\varphi_\varepsilon\, d\mu_\varepsilon - \sum_{k\in\mathbb{Z}_0^2}\int \psi\,\varphi_\varepsilon[D_kB_k(u) + k^4 B_k(u)\,\bar\beta_k^\varepsilon]\, d\mu_\varepsilon = \\ -\sum_{k\in\mathbb{Z}_0^2}\int B_k(u)\,\varphi_\varepsilon\, D_k\psi\, d\mu_\varepsilon \tag{24}$$

because by enstrophy invariance we have (see e.g., [1], [2])

$$\sum_{k\in\mathbb{Z}_0^2} k^4\,\bar{u}_k\,B_k(u)\equiv 0,\quad D_kB_k(u)\equiv 0,\quad \forall k\in\mathbb{Z}_0^2, \tag{25}$$

and

$$\beta_k^\varepsilon(u)=\frac{u_k}{\varepsilon}\Big(1-\frac{\ell}{|u|_{1-\alpha}}\Big),\quad \forall k\in\mathbb{Z}_0^2. \tag{26}$$

Proposition 1. *For each $f\in L^2(\mu)$, $\lambda>0$ equation* (19) *has at least one weak solution φ_ε which satisfies the estimates*

$$\int|\varphi_\varepsilon|^2\,d\mu_\varepsilon\le\frac{1}{\lambda^2}\int|f|^2\,d\mu_\varepsilon,\quad \forall\varepsilon>0, \tag{27}$$

$$\sum_{k\in\mathbb{Z}_0^2}k^{-2}\int|D_k\varphi_\varepsilon|^2\,d\mu_\varepsilon\le C\int|f|^2\,d\mu_\varepsilon,\quad \forall\varepsilon>0. \tag{28}$$

Proof. We shall use the Galerkin scheme for equation (21). Namely, we introduce the finite dimensional approximation B_k^n of B_k (see [1])

$$B_k^n(u)=\sum_{k,j-k\in I_n}\big[\tfrac{1}{j^2}(k^\perp\cdot j)(k\cdot j)-\tfrac12 k^\perp\cdot j)\big]\,u_k\,u_{j-k}$$

and $I_n=\{m\in\mathbb{Z}_0^2:0<|m|\le n\}$.
Then $B^n=\{B_k^n(u)\}_{k\in I_n}$, like B, has the properties (25) and the operator

$$L_n\varphi=\sum_{j\in I_n}\big[\tfrac{1}{2j^2}D_j^2\varphi-\nu\,j^2u_j\,D_j\varphi\big],$$

defined on the space of smooth functions $\varphi=\varphi(u_1,u_2,\dots,u_n)$ has the invariant measure $\mu^n=\prod_{|j|\le n}\mu_j$.
Then we consider the equation

$$\lambda\,\varphi_\varepsilon^n-L_n\varphi_\varepsilon^n+\sum_{k\in I_n}B_k^n\,D_k\varphi_\varepsilon^n+\sum_{k\in I_n}k^{-4}(\beta_k^n)^\varepsilon\,D_k\varphi_\varepsilon^n=f,\ \text{in } H_n \tag{29}$$

where $(\beta_k^n)^\varepsilon=\frac{1}{\varepsilon}\big(1-\frac{\ell}{|u|_{H^n}}\big)\,u_k$ and $H_n=\{u_j:j\in I_n\}$.
By standard existence theory for Kolmogorov equations associated with stochastic differential equations, the equation (29) has a unique solution φ_ε^n which is precisely the function

$$\varphi_\varepsilon^n(u^0)=\mathbb{E}\int_0^\infty e^{-\lambda t}f(X_\varepsilon^n(t,u^0))\,dt,$$

and $X_\varepsilon^n=\{u_j^n:j\in I_n\}$ is the solution to stochastic equation (see [3])

$$\begin{aligned}&du_j^\varepsilon+\nu\,j^2\,u_j^\varepsilon\,dt-B_j^n(u^\varepsilon)\,dt=\tfrac{1}{|j|}\,dW_j,\quad j\in I_n,\\&u_j^\varepsilon(0)=u_j^0,\quad j\in I_n.\end{aligned}$$

We may assume therefore that φ_ε is smooth and so multiplying (29) by φ_ε^n and integrating with respect to the measure

$$\mu_\varepsilon^n=\prod_{k\in I_n}e^{-\frac{k^2d_K^2}{\varepsilon}}\mu_k,$$

we obtain that

$$\lambda \int |\varphi_\varepsilon^n|^2 \, d\mu_\varepsilon + \tfrac{1}{2} \sum_{k \in I_n} k^{-2} \int |D_k \varphi_\varepsilon^n|^2 \, d\mu_\varepsilon +$$

$$\tfrac{1}{2} \sum_{k \in I_n} \int B_k^n(u) D_k |\varphi_\varepsilon^n|^2 \, d\mu_\varepsilon = \int f \varphi_\varepsilon^n \, d\mu_\varepsilon. \quad (30)$$

On the other hand, taking into account that by (25) we have

$$\sum_{k \in I_n} k^4 B_k^n \bar{u}_k \equiv 0, \quad D_k B_k^n \equiv 0, \quad \forall k \in \mathbb{Z}_0^2,$$

and it follows as in (24) that

$$\sum_{k \in I_n} \int B_k^n(u) D_k^n |\varphi_\varepsilon^n|^2 \, d\mu_\varepsilon = 0$$

and so by (30) we have that

$$\lambda \int |\varphi_\varepsilon^n|^2 \, d\mu_\varepsilon + \tfrac{1}{2} \sum_{k \in I_n} k^{-2} \int |D_k \varphi_\varepsilon^n|^2 \, d\mu_\varepsilon =$$

$$\int f \varphi_\varepsilon^n \, d\mu_\varepsilon \le \left(\int |f|^2 \, d\mu_\varepsilon \right)^{\frac{1}{2}} \left(\int |\varphi_\varepsilon^n|^2 \, d\mu_\varepsilon \right)^{\frac{1}{2}}. \quad (31)$$

Hence, on a subsequence, again denoted by $\{n\}$ we have for $n \to \infty$

$$\varphi_\varepsilon^n \to \varphi_\varepsilon \qquad \text{weakly in } L^2(\mu_\varepsilon) \quad (32)$$

$$\{D_k \varphi_\varepsilon^n\} \to \{D_k \varphi_\varepsilon\} \quad \text{weakly in } L^2(\mu_\varepsilon) \quad (33)$$

and letting n tend to infinity into the weak form of (29), that is

$$\lambda \int \varphi_\varepsilon^n \psi \, d\mu_\varepsilon + \tfrac{1}{2} \sum_{k \in I_n} k^{-2} \int D_k \varphi_\varepsilon^n D_k \psi \, d\mu_\varepsilon$$

$$- \sum_{k \in I_n} \int B_k^n(u) D_k \psi \, \varphi_\varepsilon^n \, d\mu_\varepsilon = \int f \psi \, d\mu_\varepsilon \quad (34)$$

and recalling that $\{B_k^n\}$ is strongly convergent to $\{B_k\}$ in $L^2(\mu)$ (see Lemma 1.3.2 in [7]) we infer that φ_ε is solution to (21) as claimed. Estimates (27), (28) follows by (31), (32), (33). This complete the proof of Proposition 1. □

Proof of Theorem 1 (continued). Let φ_ε be a solution to (19). By estimates (27), (28) we have for $\varepsilon \to 0$

$$\varphi_\varepsilon \to \varphi \qquad \text{weakly in } L^2(K, \mu),$$

$$\{D_k \varphi_\varepsilon\} \to \{D_k \varphi\} \qquad \text{weakly in } L^2(K, \mu; H^2).$$

Then, letting ε tend to zero into (23) we see that φ satisfies (17) for all $\psi \in \mathscr{F}C_b^2$. Estimates (19), (20) follow by (27), (28). This completes the proof. □

Remark 2. Letting ε tend to zero into (29) it follows via integration by parts formula by a similar argument as in [5] that $\varphi^n_\varepsilon \to \varphi^n$, $D_j\varphi^n_\varepsilon \to D_j\varphi^n$ in $L^2(H_n,\mu)$ where φ^n is the solution to Neumann boundary value problem

$$\begin{cases} \lambda\varphi^n - \nu\,\Delta\varphi^n + B^n(u_n)\cdot D\varphi^n = f & \text{in } \mathring{K}_n \\ \frac{\partial\varphi^n}{\partial n_{K_n}} = 0 & \text{on } \partial K_n. \end{cases}$$

where $K_n = K \cup H_n$. Moreover, by elliptic regularity, $\varphi^n \in H^2(\mathring{K}_n)$.
On the other hand, it is clear by the above energetic estimates in $H^{1-\alpha}$ that for $n \to \infty$ $\{\varphi^n\}$ is convergent to a weak solution φ to (15). However, this solution is not necessarily that given by approximating process φ_ε.

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