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WHITE AND COLORED GAUSSIAN NOISES AS LIMITS OF SUMS OF RANDOM DILATIONS AND TRANSLATIONS OF A SINGLE FUNCTION

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Abstract

It is shown that a stochastic process obtained by taking random sums of dilations and translations of a given function converges to Gaussian white noise if a dilation parameter grows to infinity and that it converges to Gaussian colored noise if a scaling parameter for the translations grows to infinity. In particular, the question of when one obtains fractional Brownian motion by integrating this colored noise is studied.

1 Introduction

The purpose of this note is to show that if h is a given scalar function and S_n, A_n, B_n, and T_n are real-valued random variables, then under fairly weak conditions the stochastic process

$$DX_{\gamma,t} = \sum_{n\in\mathbb{Z}} \sqrt{\gamma} S_n A_n h\big(\gamma B_n t - (n + T_n)\big), \tag{1}$$

converges to Gaussian white noise (the distribution derivative of Brownian motion) as $\gamma \to \infty$, and the process

$$DY_{\lambda,t} = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{\lambda}} S_n A_n h\left(B_n t - \frac{1}{\lambda}(n + T_n)\right), \tag{2}$$

converges to a certain kind of colored noise with spectral density depending on the function h and the random variables A_n and B_n as $\lambda \to \infty$.

It is not surprising that the kind of time scaling appearing in (1) gives rise to white noise and it is not claimed here that the kind of scaling of the translations used in (2) is the only possible way to obtain colored noise, just that it is one possibility. Thus the motivation for studying (2) is that it is a fairly simple model which may perhaps be of help in understanding the sources of different kinds of colored noise.

The convergence concept considered is that the random variable $\int_{\mathbb{R}} f(t)Z_t \,\mathrm{d}t$ converges in distribution to a Gaussian random variable for all test functions f, (all square integrable functions for the case $Z_t = DX_{\gamma,t}$ and a subspace of $L^2(\mathbb{R})$ in the case $Z_t = DY_{\lambda,t}$). An important assumption is that $\mathbb{E}(S_n) = 0$ and that the random variables S_n are independent of each other and of all other random variables appearing in the sums.

If one takes $h(t) = \mathbf{1}_{[0,1)}(t)$, $A_n = B_n = 1$, $T_n = 0$, and $S_n = \pm 1$ (each with probability $\frac{1}{2}$), then the process $t \mapsto \int_0^t DX_{\gamma,s}\,\mathrm{d}s$ is the linear interpolation of a random walk jumping up or down with step length $\frac{1}{\sqrt{\gamma}}$ in each time interval of length $\frac{1}{\gamma}$ so in this case it is immediately clear that the limit when $\gamma \to \infty$ is Brownian motion. It follows from Theorem 1 below that this is the case under quite general assumptions. The convergence here means that the distributions of all finite linear combinations of samples of the process converge to the corresponding distributions for Brownian motion.

In many cases, however, it has turned out that Brownian motion is not a satisfactory process to use as a model and that one should rather use, e.g., fractional Brownian motion, see [5] and in particular [12] in the case of ethernet traffic. Fractional Brownian motion B_t^H with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with covariance function $\mathbb{E}(B_s^H B_t^H) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$. Another way to express this is that

$$\mathbb{E}\left(\int_{\mathbb{R}} f(t)dB_t^H \int_{\mathbb{R}} g(t)dB_t^H\right) = (2\pi)^{1-2H}\Gamma(2H+1)\sin(\pi H)\int_{\mathbb{R}} |\xi|^{1-2H}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi,$$

where $B_t^H = \int_0^t dB_s^H$ and $\hat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi\xi t} f(t)\,\mathrm{d}t$, see e.g. [8]. In this note it is shown that for suitable (real) functions f and g one has

$$\lim_{\lambda\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} f(t)DY_{\lambda,t}\,\mathrm{d}t \int_{\mathbb{R}} g(t)DY_{\lambda,t}\,\mathrm{d}t\right) = \int_{\mathbb{R}} \mathbb{E}\left(\frac{A_n^2}{|B_n|}\left|\hat{h}\left(\frac{\xi}{B_n}\right)\right|^2\right)\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi.$$

Thus we see that it is possible to obtain fractional Brownian motion as the limit of the process $\int_0^t DY_{\lambda,s}\,\mathrm{d}s$ but the point is that in order to get this process exactly rather special assumptions on the function h and/or on the distribution of (A_n, B_n) are needed whereas the assumptions that suffice for the convergence of $\int_0^t DX_{\gamma,s}\,\mathrm{d}s$ to Brownian motion as $\gamma \to \infty$ are much more general. But this note does not study the question to what extent fractional Brownian motion may be a reasonable approximation to the limit of $\int_0^t DY_{\lambda,s},\mathrm{d}s$.

Fractional Brownian motion can be approximated in many ways, see e.g. [1], [2], [3], [6], [9], [10], and [11], depending partly on whether one wants to get an efficient simulation tool or whether one wants, as in this note, to see to what extent it is a consequence of some "universal principles" and therefore expected to appear in many connections. One can also get a fractional Brownian motion with Hurst parameter $H = \frac{1}{4}$ as a limit from one-dimensional nearest-neighbor symmetric simple exclusion processes, see e.g. [7].

2 Statement of results

Theorem 1. *Assume that*

(i) $h \in L^1(\mathbb{R};\mathbb{R})$, $\hat{h}(0) = 1$, *and* $\sup_{\xi\in\mathbb{R}} |\hat{h}(\xi)||\xi|^{\alpha/2} < \infty$ *for some number* $\alpha > 1$.

(ii) *The real valued random variables S_n and the $\mathbb{R}^3$-valued random variables (A_n, B_n, T_n), where $n \in \mathbb{Z}$, are independent and the distributions of S_n and (A_n, B_n, T_n) do not depend on n. In addition $\mathbb{P}(B_n = 0) = 0$, $\mathbb{E}(S_n) = 0$, and $\mathbb{E}(S_n^2) = 1$ for all $n \in \mathbb{Z}$.*

(iii) $\mathbb{E}\left(\frac{A_n^2}{B_n}\right) = 1$, $n \in \mathbb{Z}$.

If $f \in L^2(\mathbb{R};\mathbb{R})$, then $\int_{\mathbb{R}} f(t) DX_{\gamma,t}\,\mathrm{d}t$ converges in distribution to an $N(0, \|f\|^2_{L^2(\mathbb{R})})$-distributed random variable as $\gamma \to \infty$ and if $g \in L^2(\mathbb{R};\mathbb{R})$ as well, then

$$\lim_{\gamma\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} f(t) DX_{\gamma,t}\,\mathrm{d}t \int_{\mathbb{R}} g(t) DX_{\gamma,t}\,\mathrm{d}t\right) = \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi.$$

Note that assumption (i) implies that $h \in L^2(\mathbb{R};\mathbb{R})$.

Theorem 2. *Assume that*

(i) *The function $h : \mathbb{R} \to \mathbb{R}$ is such that $\int_{\mathbb{R}} |h(t)|(1 + |t|^m)^{-1}\,\mathrm{d}t < \infty$ for some number $m \geq 0$ and the (distribution) Fourier transform of h is induced by a measurable function $\hat{h}$ such that $\int_{\mathbb{R}} |\hat{h}(\xi)|(1 + |\xi|^{\hat{m}})^{-1}\,\mathrm{d}\xi < \infty$ for some number $\hat{m} \geq 0$.*

(ii) *The real valued random variables S_n and the $\mathbb{R}^3$-valued random variables (A_n, B_n, T_n), where $n \in \mathbb{Z}$, are independent and the distributions of S_n and (A_n, B_n, T_n) do not depend on n. In addition $\mathbb{P}(B_n = 0) = 0$, $\mathbb{E}(S_n) = 0$, and $\mathbb{E}(S_n^2) = 1$ for all $n \in \mathbb{Z}$.*

(iii) *There is a number $\alpha > 1$ and a finite set $K \subset \mathbb{R}$ such that the function*

$$\xi \mapsto \mathbb{E}\left(\frac{A_n^2}{B_n}\left(1 + \left|\frac{\xi}{B_n}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{B_n}\right)\right|^2\right),$$

is bounded on compact subsets of $\mathbb{R} \setminus K$.

(iv) *f and $g \in L^2(\mathbb{R};\mathbb{R})$ are such that*

$$\int_{\mathbb{R}} \mathbb{E}\left(\frac{A_n^2}{|B_n|}\left(1 + \left|\frac{\xi}{B_n}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{B_n}\right)\right|^2\right)(|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2)\,\mathrm{d}\xi < \infty.$$

Then $\int_{\mathbb{R}} f(t) DY_{\lambda,t}\,\mathrm{d}t$ converges in distribution to an $N(0,\sigma^2)$-distributed random variable as $\lambda \to \infty$ where

$$\sigma^2 = \int_{\mathbb{R}} \mathbb{E}\left(\frac{A_n^2}{|B_n|}\left|\hat{h}\left(\frac{\xi}{B_n}\right)\right|^2\right)|\hat{f}(\xi)|^2\,\mathrm{d}\xi,$$

and

$$\lim_{\lambda\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} f(t) DY_{\lambda,t}\,\mathrm{d}t \int_{\mathbb{R}} g(t) DY_{\lambda,t}\,\mathrm{d}t\right) = \int_{\mathbb{R}} \mathbb{E}\left(\frac{A_n^2}{|B_n|}\left|\hat{h}\left(\frac{\xi}{B_n}\right)\right|^2\right)\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi.$$

(In this case the distribution Fourier transform of h is defined by the requirement that $\int_{\mathbb{R}} \hat{h}(\xi)\varphi(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} h(t)\hat{\varphi}(t)\,\mathrm{d}t$ where $\hat{\varphi}(t) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t\xi}\varphi(\xi)\,\mathrm{d}\xi$.)
If the probability density of the random variable B_n is b with $b(x) > 0$ for all $x > 0$ and $b(x) = 0$ for all $x \leq 0$, and $A_n = B_n^{-H+1/2}/\sqrt{b(B_n)}$, (that is, there is a very specific relation between the parts of the model) then

$$\mathbb{E}\left(\frac{A_n^2}{|B_n|}\left|\hat{h}\left(\frac{\xi}{B_n}\right)\right|^2\right) = |\xi|^{1-2H}\int_0^\infty s^{2H-2}|\hat{h}(s)|^2\,\mathrm{d}s,$$

and provided the integral is finite we see that the limit of $DY_{\lambda,t}$ is the distribution derivative of a multiple of fractional Brownian motion with Hurst parameter H.
If, on the other hand $h(t) = \operatorname{sign}(t)|t|^{H-3/2}$, then

$$\mathbb{E}\left(\frac{A_n^2}{|B_n|}\left|\hat{h}\left(\frac{\xi}{B_n}\right)\right|^2\right) = \mathbb{E}(A_n^2|B_n|^{2H-2})\left(\frac{\sin(\frac{\pi}{4}(1-2H))}{\frac{\pi}{4}(1-2H)}\right)^2\frac{\pi^2\Gamma(\frac{1}{2}+H)^2}{(2\pi)^{2H-1}}|\xi|^{1-2H},$$

and if $\mathbb{E}(A_n^2|B_n|^{2H-2}) < \infty$, then we again get the distribution derivative of a multiple of fractional Brownian motion. But the function h is locally integrable in the case $H \in (\frac{1}{2}, 1)$ only. If $H \in (0, \frac{1}{2}]$ we can, however, interpret the integral $\int_{\mathbb{R}} \frac{1}{\sqrt{\lambda}} S_n A_n h(B_n t - \frac{1}{\lambda}(n+T_n)) f(t)\,\mathrm{d}t$ as a Cauchy principal value, or in other words, consider h as a tempered distribution. Then the conclusion still holds but in these cases hypothesis (iv) may be much more restrictive and exclude indicator functions of intervals.
By considering $h\left(B_n t - \frac{1}{\lambda}(n+T_n)\right)$ as dilation and translation of a tempered distribution one can see that the proof of Theorem 2 extends to a proof of the following claim.

Corollary 3. *The conclusion of Theorem 2 remains true provided hypothesis* (i) *is replaced by the assumption*

(i') *h is a real valued tempered distribution such that the distribution Fourier transform of h is induced by a measurable function $\hat{h}$ such that $\int_{\mathbb{R}} |\hat{h}(\xi)|(1+|\xi|^{\hat{m}})^{-1}\,\mathrm{d}\xi < \infty$ for some number $\hat{m} \geq 0$,*

and integrals involving h are interpreted as the value of a tempered distribution at a test function.

3 Proofs

First we derive some results that are common for the processes $DX_{\gamma,t}$ and $DY_{\lambda,t}$ and throughout we assume that all random variables S_n, A_n, B_n, and T_n are defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that f and $g \in L^2(\mathbb{R})$ in the case of Theorem 1 and that hypothesis (iv) of Theorem 2 holds otherwise. Let

$$U(n,\gamma,\lambda,f,\omega) = \sqrt{\frac{\gamma}{\lambda}}A_n(\omega)\int_{\mathbb{R}} h\left(\gamma B_n(\omega)t - \frac{1}{\lambda}(n+T_n(\omega))\right) f(t)\,\mathrm{d}t.$$

(In many cases below we leave out the argument ω of the random variables.) In the case of Theorem 1 we have $h \in L^2(\mathbb{R})$ and it is immediately clear that this is a well defined random variable. In the other case we first make the additional assumption that f and $g \in \mathscr{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing smooth functions, and then combine the argument below with a limiting procedure, and here hypothesis (iii) of Theorem 2 is used, to show that we indeed get a random variable under hypothesis (iv) of Theorem 2. Note that $\int_{\mathbb{R}} f(t)DX_{\gamma,t}\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} S_n U(n,\gamma,1,f)$ and $\int_{\mathbb{R}} f(t)DY_{\lambda,t}\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} S_n U(n,1,\lambda,f)$ so it follows from hypothesis (ii) that

$$\mathbb{E}\left(\int_{\mathbb{R}} f(t)DX_{\gamma,t}\,\mathrm{d}t \int_{\mathbb{R}} g(t)DX_{\gamma,t}\,\mathrm{d}t\right) = \sum_{n\in\mathbb{Z}} \mathbb{E}\big(U(n,\gamma,1,f)U(n,\gamma,1,g)\big), \tag{3}$$

and

$$\mathbb{E}\left(\int_{\mathbb{R}} f(t)DY_{\lambda,t}\,\mathrm{d}t \int_{\mathbb{R}} g(t)DY_{\lambda,t}\,\mathrm{d}t\right) = \sum_{n\in\mathbb{Z}} \mathbb{E}\big(U(n,1,\lambda,f)U(n,1,\lambda,g)\big). \tag{4}$$

If $\hat{h}$ is the Fourier-transform of h then the Fourier-transform of the function $t \mapsto h(\gamma B_n(\omega)t - \frac{1}{\lambda}(n+T_n)\omega)))$ (or the tempered distribution induced by this function) is the function

$$\xi \mapsto \frac{1}{\gamma|B_n(\omega)|}\mathrm{e}^{-\mathrm{i}2\pi\xi n/\gamma\lambda B_n(\omega)}\mathrm{e}^{-\mathrm{i}2\pi\xi T_n(\omega)/\gamma\lambda B_n(\omega)}\hat{h}\left(\frac{\xi}{\gamma B_n(\omega)}\right)$$

(or the tempered distribution induced by this function) and we deduce by Plancherel's theorem (or the definition of the distribution Fourier transform) that

$$U(n,\gamma,\lambda,f,\omega) = \frac{A_n(\omega)}{\sqrt{\gamma\lambda}|B_n(\omega)|}\int_{\mathbb{R}} \overline{\mathrm{e}^{\mathrm{i}2\pi\xi n/\gamma\lambda B_n(\omega)}\mathrm{e}^{\mathrm{i}2\pi\xi T_n(\omega)/\gamma\lambda B_n(\omega)}\hat{h}\left(\frac{\xi}{\gamma B_n(\omega)}\right)}\hat{f}(\xi)\,\mathrm{d}\xi. \tag{5}$$

In order to get an estimate for $\sum_{n\in\mathbb{Z}} \mathbb{E}(U(n,\gamma,\lambda,f)^2)$ we rewrite $U(n,\gamma,\lambda,f,\omega)$ as

$$\begin{aligned} U(n,\gamma,\lambda,f,\omega) &= \sqrt{\gamma\lambda}A_n(\omega)\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi\xi n}\mathrm{e}^{\mathrm{i}2\pi T_n(\omega)\xi}\overline{\hat{h}(\lambda\xi)}\hat{f}\left(\gamma\lambda B_n(\omega)\xi\right)\,\mathrm{d}\xi \\ &= \sqrt{\gamma\lambda}A_n(\omega)\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi\xi n}\sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi T_n(\omega)(\xi+k)}\overline{\hat{h}\big(\lambda(\xi+k)\big)}\hat{f}\left(\gamma\lambda B_n(\omega)(\xi+k)\right)\mathrm{d}\xi. \end{aligned} \tag{6}$$

Let A, B, and T be random variables on a probability space $(\Omega_*, \mathscr{F}_*, \mathbb{P}_*)$ with the same distribution as A_n, B_n, and T_n. It is crucial for the proof that when taking expectations one can use random variables that do not depend on n. In the argument below one should first take $g = f$ and show that the sum $\sum_{n\in\mathbb{Z}} \mathbb{E}(U(n,\gamma,\lambda,f)^2$ is finite and then go through the argument again, using the fact that one may now invoke Fubini's theorem and other results needed, in the case where f and g may be different as well. By (6) we see that when one calculates expectations of the random variables $U(n,\gamma,\lambda,f)$ one gets Fourier-coefficients of a periodic function and hence it follows from

Parseval's theorem that

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbb{E}(U(n,\gamma,\lambda,f)U(n,\gamma,\lambda,g)) &= \sum_{n\in\mathbb{Z}} \int_{\Omega} U(n,\gamma,\lambda,f,\omega)\overline{U(n,\gamma,\lambda,g,\omega)}\,\mathbb{P}(\mathrm{d}\omega) \\
&= \sum_{n\in\mathbb{Z}} \int_{\Omega_*} \gamma\lambda A(\omega)^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi\xi n} \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi T(\omega)(\xi+j)} \overline{\hat{h}\left(\lambda(\xi+j)\right)} \hat{f}\left(\gamma\lambda B(\omega)(\xi+j)\right)\,\mathrm{d}\xi \\
&\quad \times \overline{\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi\xi n} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi T(\omega)(\xi+k)} \overline{\hat{h}\left(\lambda(\xi+k)\right)} \hat{g}\left(\gamma\lambda B(\omega)(\xi+k)\right)\,\mathrm{d}\xi}\,\mathbb{P}_*(\mathrm{d}\omega) \\
&= \int_{\Omega_*} \sum_{n\in\mathbb{Z}} \Bigg(\gamma\lambda A(\omega)^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi\xi n} \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi T(\omega)(\xi+j)} \overline{\hat{h}\left(\lambda(\xi+j)\right)} \hat{f}\left(\gamma\lambda B(\omega)(\xi+j)\right)\,\mathrm{d}\xi \\
&\quad \times \overline{\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi\xi n} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi T(\omega)(\xi+k)} \overline{\hat{h}\left(\lambda(\xi+k)\right)} \hat{g}\left(\gamma\lambda B(\omega)(\xi+k)\right)\,\mathrm{d}\xi}\Bigg)\,\mathbb{P}_*(\mathrm{d}\omega) \\
&= \int_{\Omega_*} \Bigg(\gamma\lambda A(\omega)^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi T(\omega)(\xi+j)} \overline{\hat{h}\left(\lambda(\xi+j)\right)} \hat{f}\left(\gamma\lambda B(\omega)(\xi+j)\right) \\
&\quad \times \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi T(\omega)(\xi+k)} \hat{h}\left(\lambda(\xi+k)\right) \overline{\hat{g}\left(\gamma\lambda B(\omega)(\xi+k)\right)}\,\mathrm{d}\xi\Bigg)\,\mathbb{P}_*(\mathrm{d}\omega) \\
&= \mathbb{E}\Bigg(\gamma\lambda A^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi T(\xi+j)} \overline{\hat{h}\left(\lambda(\xi+j)\right)} \hat{f}\left(\gamma\lambda B(\xi+j)\right) \\
&\qquad\qquad \times \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi T(\xi+k)} \hat{h}\left(\lambda(\xi+k)\right) \overline{\hat{g}\left(\gamma\lambda B(\xi+k)\right)}\,\mathrm{d}\xi\Bigg). \qquad (7)
\end{aligned}
$$

Let $\hat{\mathbb{Z}}$ be either $\mathbb{Z}$ or $\mathbb{Z}\setminus\{0\}$. By the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{j\in\hat{\mathbb{Z}}} \left|\hat{h}\left(\lambda(\xi+j)\right)\right| \left|\hat{f}\left(\gamma\lambda B(\omega)(\xi+j)\right)\right|\right)^2 \mathrm{d}\xi \\
&\le \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{j\in\mathbb{Z}} \frac{1}{1+|\lambda(\xi+j)|^\alpha} \sum_{j\in\hat{\mathbb{Z}}} \left(1+|\lambda(\xi+j)|^\alpha\right) \left|\hat{h}\left(\lambda(\xi+j)\right)\right|^2 \left|\hat{f}\left(\gamma\lambda B(\omega)(\xi+j)\right)\right|^2 \mathrm{d}\xi. \qquad (8)
\end{aligned}
$$

Let $j_0 = 0$ if $\hat{\mathbb{Z}} = \mathbb{Z}$ and $j_0 = 1$ if $\hat{\mathbb{Z}} = \mathbb{Z}\setminus\{0\}$. Changing variables and adding intervals of integration we get,

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{j\in\hat{\mathbb{Z}}} \left(1+|\lambda(\xi+j)|^{\alpha}\right) \left|\hat{h}\left(\lambda(\xi+j)\right)\right|^2 \left|\hat{f}\left(\gamma\lambda B(\omega)(\xi+j)\right)\right|^2 \mathrm{d}\xi \\ = \int_{|\xi|\geq j_0/2} \left(1+|\lambda\xi|^{\alpha}\right) \left|\hat{h}(\lambda\xi)\right|^2 \left|\hat{f}\left(\gamma\lambda B(\omega)\xi\right)\right|^2 \mathrm{d}\xi \\ = \frac{1}{\gamma\lambda|B(\omega)|} \int_{|\xi|\geq\gamma\lambda|B(\omega)|j_0/2} \left(1+\left|\frac{\xi}{\gamma B(\omega)}\right|^{\alpha}\right) \left|\hat{h}\left(\frac{\xi}{\gamma B(\omega)}\right)\right|^2 \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi. \end{aligned} \tag{9}$$

Thus we conclude from (7), (8), and (9) (with $\hat{\mathbb{Z}}=\mathbb{Z}$) that

$$\sum_{n\in\mathbb{Z}} \mathbb{E}(U(n,\gamma,\lambda,f)^2) \leq c_{\alpha} \int_{\mathbb{R}} \mathbb{E}\left(\frac{A^2}{|B|}\left(1+\left|\frac{\xi}{\gamma B}\right|^{\alpha}\right)\left|\hat{h}\left(\frac{\xi}{\gamma B}\right)\right|^2\right) |\hat{f}(\xi)|^2 \mathrm{d}\xi, \tag{10}$$

(provided $\lambda \geq 1$) where $c_{\alpha} = \sup_{\xi\in\mathbb{R}} \sum_{j\in\mathbb{Z}} \frac{1}{1+|(\xi+j)|^{\alpha}}$.
By (7) we have (with $\hat{\mathbb{Z}} = \mathbb{Z}\setminus\{0\}$)

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \mathbb{E}(U(n,\gamma,\lambda,f)U(n,\gamma,\lambda,g)) \\ &= \mathbb{E}\left(\gamma\lambda A^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi T(\xi+0)} \overline{\hat{h}(\lambda(\xi+0))} \hat{f}\left(\gamma\lambda B(\xi+j)\right) \mathrm{e}^{-\mathrm{i}2\pi T(\xi+0)} \hat{h}(\lambda(\xi+k)) \overline{\hat{g}\left(\gamma\lambda B(\xi+0)\right)} \mathrm{d}\xi\right) \\ &\quad + \mathbb{E}\left(\gamma\lambda A^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi T(\xi+0)} \overline{\hat{h}(\lambda(\xi+0))} \hat{f}\left(\gamma\lambda B(\xi+0)\right) \right. \\ &\quad\quad \left. \times \sum_{k\in\hat{\mathbb{Z}}} \mathrm{e}^{-\mathrm{i}2\pi T(\xi+k)} \hat{h}(\lambda(\xi+k)) \overline{\hat{g}\left(\gamma\lambda B(\xi+k)\right)} \mathrm{d}\xi\right) \\ &\quad + \mathbb{E}\left(\gamma\lambda A^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{j\in\hat{\mathbb{Z}}} \mathrm{e}^{\mathrm{i}2\pi T(\xi+j)} \overline{\hat{h}\left(\lambda(\xi+j)\right)} \hat{f}\left(\gamma\lambda B(\xi+j)\right)\right. \\ &\quad\quad \left. \times \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi T(\xi+k)} \hat{h}(\lambda(\xi+k)) \overline{\hat{g}\left(\gamma\lambda B(\xi+k)\right)} \mathrm{d}\xi\right). \end{aligned} \tag{11}$$

By the same change of variable as used above we see that

$$\begin{aligned}&\mathbb{E}\left(\gamma\lambda A^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}2\pi T(\xi+0)} \overline{\hat{h}(\lambda(\xi+0))} \hat{f}\left(\gamma\lambda B(\xi+j)\right)\right. \\ &\quad \left. \times \mathrm{e}^{-\mathrm{i}2\pi T(\xi+0)} \hat{h}(\lambda(\xi+k)) \overline{\hat{g}\left(\gamma\lambda B(\xi+0)\right)} \mathrm{d}\xi\right) \\ &\quad\quad = \mathbb{E}\left(\frac{A^2}{|B|} \int_{-\frac{\gamma\lambda|B|}{2}}^{\frac{\gamma\lambda|B|}{2}} \left|\hat{h}\left(\frac{\xi}{\gamma B}\right)\right|^2 \hat{f}(\xi)\overline{\hat{g}(\xi)} \mathrm{d}\xi\right)\end{aligned}$$

and for the remaining terms in (7) we use the Cauchy-Schwarz inequality together with (8) and (9) in order to conclude that

$$\begin{aligned}
&\left|\sum_{n\in\mathbb{Z}}\mathbb{E}(U(n,\gamma,1,f)U(n,\gamma,1,g))-\mathbb{E}\left(\frac{A^2}{|B|}\int_{-\frac{\gamma\lambda|B|}{2}}^{\frac{\gamma\lambda|B|}{2}}\left|\hat{h}\left(\frac{\xi}{\gamma B}\right)\right|^2\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi\right)\right|\\
&\qquad\le c_\alpha\left(\mathbb{E}\left(\frac{A^2}{|B|}\int_{|\xi|\ge\frac{\gamma\lambda|B|}{2}}\left(1+\left|\frac{\xi}{\gamma B}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{\gamma B}\right)\right|^2\left|\hat{f}(\xi)\right|^2\mathrm{d}\xi\right)\right)^{\frac{1}{2}}\\
&\qquad\quad\times\left(\mathbb{E}\left(\frac{A^2}{|B|}\int_{\mathbb{R}}\left(1+\left|\frac{\xi}{\gamma B}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{\gamma B}\right)\right|^2\left|\hat{g}(\xi)\right|^2\mathrm{d}\xi\right)\right)^{\frac{1}{2}}\\
&\qquad\quad+c_\alpha\left(\mathbb{E}\left(\frac{A^2}{|B|}\int_{|\xi|\ge\frac{\gamma\lambda|B|}{2}}\left(1+\left|\frac{\xi}{\gamma B}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{\gamma B}\right)\right|^2\left|\hat{g}(\xi)\right|^2\mathrm{d}\xi\right)\right)^{\frac{1}{2}}\\
&\qquad\qquad\times\left(\mathbb{E}\left(\frac{A^2}{|B|}\int_{\mathbb{R}}\left(1+\left|\frac{\xi}{\gamma B}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{\gamma B}\right)\right|^2\left|\hat{f}(\xi)\right|^2\mathrm{d}\xi\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

It follows from the dominated convergence theorem and the assumptions (for the cases $\gamma\to\infty$ and $\lambda\to\infty$, respectively) that the right-hand side of this inequality tends to 0 when $\gamma\to\infty$ or $\lambda\to\infty$. Thus we conclude under the assumptions of Theorem 1 that

$$\lim_{\gamma\to\infty}\sum_{n\in\mathbb{Z}}\mathbb{E}(U(n,\gamma,1,f)U(n,\gamma,1,g))=\int_{\mathbb{R}}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi, \tag{12}$$

and under the assumptions of Theorem 2 that

$$\lim_{\lambda\to\infty}\sum_{n\in\mathbb{Z}}\mathbb{E}(U(n,1,\lambda,f)U(n,1,\lambda,g))=\int_{\mathbb{R}}\mathbb{E}\left(\frac{A^2}{|B|}\left|\hat{h}\left(\frac{\xi}{B}\right)\right|^2\right)f(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi, \tag{13}$$

By the Cauchy-Schwarz inequality and (5) we have in the case of Theorem 1 when $\lambda=1$,

$$\begin{aligned}
&\left|U(n,\gamma,1,f,\omega)\right|\\
&\quad\le\frac{|A_n(\omega)|}{\sqrt{\gamma}|B_n(\omega)|}\left(\int_{|\xi|\le\sqrt{\gamma}}\left|\hat{h}\left(\frac{\xi}{\gamma B_n(\omega)}\right)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{|\xi|\le\sqrt{\gamma}}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&\quad+\frac{|A_n(\omega)|}{\sqrt{\gamma}|B_n(\omega)|}\left(\int_{|\xi|>\sqrt{\gamma}}\left|\hat{h}\left(\frac{\xi}{\gamma B_n(\omega)}\right)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{|\xi|>\sqrt{\gamma}}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&\qquad\le\frac{|A_n(\omega)|}{\sqrt{|B_n(\omega)|}}\left(\int_{|\xi|\le\frac{1}{\sqrt{\gamma}B_n(\omega)}}|\hat{h}(\xi)|^2\right)^{\frac{1}{2}}\|\hat{f}\|_{L^2(\mathbb{R})}\\
&\qquad\qquad+\frac{|A_n(\omega)|}{\sqrt{|B_n(\omega)|}}\|\hat{h}\|_{L^2(\mathbb{R})}\left(\int_{|\xi|>\sqrt{\gamma}}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}.
\end{aligned}$$

It follows from this inequality that if $\delta > 0$, $|B_n(\omega)| \geq \delta$, and $|A_n(\omega)| \leq \frac{1}{\delta}$ then for each $\epsilon > 0$ there is a number $\gamma_{\delta,\epsilon}$ so that if $\gamma > \gamma_{\delta,\epsilon}$ then $|U(n,\gamma,1,f,\omega)| < \epsilon$. Thus it follows from (7) and (10) when we replace A_n by $A_n(\mathbf{1}_{\{A>\frac{1}{\delta}\}} + \mathbf{1}_{\{|B|<\delta\}})$ that for each $\delta > 0$ we have

$$\limsup_{\gamma\to\infty} \sum_{n\in\mathbb{Z}} \mathbb{E}(U(n,\gamma,1,f)^2 \mathbf{1}_{|U(n,\gamma,1,f)|>\epsilon}) \leq c_1 \mathbb{E}\left(\frac{A^2}{|B|}\left(\mathbf{1}_{\{A>\frac{1}{\delta}\}} + \mathbf{1}_{\{|B|<\delta\}}\right)^2\right)\int_{\mathbb{R}} |\hat{f}(\xi)|^2\,\mathrm{d}\xi,$$

where $c_1 = c_\alpha \sup_{\xi\in\mathbb{R}}(1+|\xi|^\alpha)|\hat{h}(\xi)|^2$. Since $\mathbb{E}(\frac{A^2}{|B|}) < \infty$ it follows that

$$\lim_{\gamma\to\infty} \sum_{n\in\mathbb{Z}} \mathbb{E}(U(n,\gamma,1,f)^2 \mathbf{1}_{\{|U(n,\gamma,1,f)|>\epsilon\}}) = 0, \tag{14}$$

for each $\epsilon > 0$. This is the well known Lindeberg condition and thus by (12) and the assumption $\mathbb{E}(S_n) = 0$ the limit of $\int_{\mathbb{R}} f(t) DX_{\gamma,t}\,\mathrm{d}t$ is normally distributed with mean 0 and variance $\|f\|^2_{L^2(\mathbb{R})}$, see [4, Thm. 5.12]. The statement about the covariance follows from (3) and (12).
If $\gamma = 1$ and we consider the case of Theorem 2, then we can use the Cauchy-Schwarz inequality and (5) to get

$$\begin{aligned}\left|U(n,1,\lambda,f,\omega)\right| \leq \frac{1}{\sqrt{\lambda}}&\left(\int_{\mathbb{R}}\left(1+\left|\frac{\xi}{B_n(\omega)}\right|^\alpha\right)^{-1}\frac{1}{|B_n(\omega)|}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \\ &\times\left(\int_{\mathbb{R}}\frac{A_n(\omega)^2}{|B_n(\omega)|}\left(1+\left|\frac{\xi}{B_n(\omega)}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{B_n(\omega)}\right)\right|^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}\end{aligned} \tag{15}$$

Let

$$q_n(\omega) = \int_{\mathbb{R}}\frac{A_n(\omega)^2}{|B_n(\omega)|}\left(1+\left|\frac{\xi}{B_n(\omega)}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{B_n(\omega)}\right)\right|^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi,$$

$$q(\omega) = \int_{\mathbb{R}}\frac{A(\omega)^2}{|B(\omega)|}\left(1+\left|\frac{\xi}{B(\omega)}\right|^\alpha\right)\left|\hat{h}\left(\frac{\xi}{B(\omega)}\right)\right|^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi,$$

and we note that q_n and q have the same distributions. From (15) we see (note that the first term on the right hand side is $\int_{\mathbb{R}}(1+|\xi|^\alpha)^{-1}\,\mathrm{d}\xi < \infty$) that for each $\epsilon > 0$ and $m \geq 1$ there is a number $\lambda_{m,\epsilon}$ so that $|U(n,1,\lambda,f,\omega)| \leq \mathbf{1}_{\{q_n(\omega)>m\}}|U(n,1,\lambda,f,\omega)|$ if $|U(n,1,\lambda,f,\omega)| \geq \epsilon$ and $\lambda \geq \lambda_{m,\epsilon}$. Thus we conclude from (10) with the aid of the same argument as above (replace A_n by $A_n\mathbf{1}_{\{q_n>m\}}$) that

$$\limsup_{\lambda\to\infty}\sum_{n\in\mathbb{Z}}\mathbb{E}(U(n,1,\lambda,f)^2\mathbf{1}_{\{|U(n,1,\lambda,f)|>\epsilon\}}) \leq c_\alpha\mathbb{E}(\mathbf{1}_{\{q>m\}}q)$$

and since m was arbitrary we get

$$\lim_{\lambda\to\infty}\sum_{n\in\mathbb{Z}}\mathbb{E}(U(n,1,\lambda,f)^2\mathbf{1}_{\{|U(n,1,\lambda,f)|>\epsilon\}}) = 0. \tag{16}$$

for each $\epsilon > 0$. Since this is again the Lindeberg condition we can deduce from (13) that $\int_{\mathbb{R}} f(t) DY_{\lambda,t}\,\mathrm{d}t$ converges in distribution to a normally distributed random variable with mean 0 and variance $\int_{\mathbb{R}}\mathbb{E}\left(\frac{A^2}{|B|}\left|\hat{h}\left(\frac{\xi}{B}\right)\right|^2\right)|\hat{f}(\xi)|^2\,\mathrm{d}\xi$. The statement about the covariance follows from (4) and (13).

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