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ON THE CONSTRUCTION OF FELLER PROCESSES WITH UNBOUNDED COEFFICIENTS

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Abstract

Construction methods for Feller processes which require bounded coefficients are extended to the case of unbounded coefficients.
Technically speaking, the global condition $\lim_{|\xi|\to 0}\sup_x |q(x,\xi)| = 0$ which is usually posed on the symbol q of the generator of a Feller process can always be replaced by the weaker condition:

$$\lim_{r\to\infty}\sup_{|y|\le r}\sup_{|\xi|\le\frac{1}{r}}|q(y,\xi)| = 0.$$

1 Introduction

Feller semigroups and Feller processes on $\mathbb{R}^d$ are well studied objects, which have many applications (see [5]). But their construction is in general difficult, moreover most of the existing construction methods (see below for details) only allow the construction of Feller processes with bounded coefficients, i.e., they require that the symbol of the generator is bounded uniformly with respect to the space variable.
In contrast, our approach will provide a technique by which any known construction method can be used to construct Feller processes with unbounded coefficients. The main idea is to construct the process with unbounded coefficients as a limit of Feller processes with bounded coefficients. We show that this can be done under general conditions. But first we have a brief look at the known construction methods.
The standard examples of construction methods are:

- Using the Hille-Yosida theorem and Kolmogorov's construction [12, 10].
- Solving the associated evolution equation (Kolmogorov's backwards equation) [4, 3, 14, 15].
- Proving the well-posedness of the martingale problem [2, 10, 23].

- Solving a stochastic differential equation [11, 13, 24].
- Using Dirichlet forms [1, 22].

Based on the martingale problem, there is a Lyapunov function technique (Chapter 5 in [16]) for the construction of processes with unbounded coefficients. This method adapts the underlying function space based on the growth of the coefficients, in contrast we will always consider Feller semigroups on the set of continuous functions vanishing at infinity.
Moreover, also the approach for diffusion processes via SDEs and the approach by Hoh (Chapter 9 in [10]) discuss unbounded coefficients. In the latter also processes with jumps are considered, therein the symbol q of the generator is required to be real valued, and it has to satisfy the condition

$$p(x,\xi) \leq c \frac{1}{\sup_{|\eta|\leq \frac{1}{|x|}} \psi(\eta)} \psi(\xi) \quad \text{for all } |x| \geq 1 \text{ and some constant } c,$$

where ψ is a continuous and negative definite reference symbol ψ.
Our condition is similar to the above, but we also allow complex valued symbols and do not introduce an additional reference function. Furthermore note that the condition in [10] was only applied to a specific construction method.
Besides the construction methods for general Feller processes listed above there is also a special technique for the construction of affine processes (see [8]). Affine processes are Feller processes with linearly growing (and thus unbounded) coefficients. In contrast to this, we allow nonlinearity.
Our condition is strongly related to a condition which ensures that a time homogeneous strong Markov process is conservative, see Theorem 5.5 in [17] and Theorem 2.1 in [25]. But note that in these references the existence of such a process was assumed and no construction methods were given.
Next we will state our main theorem, the required definitions and notations will be explained in Section 2.

Theorem 1.1. *Let $q(.,.) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a function such that*

$$\lim_{r\to\infty} \sup_{|y|\leq r} \sup_{|\xi|\leq \frac{1}{r}} |q(y,\xi)| = 0. \tag{1.1}$$

For each $k \in \mathbb{N}$ let $(X_t^k)_{t\geq 0}$ be a Feller process with semigroup $(T_t^k)_{t\geq 0}$, such that its generator A_k satisfies $C_c^\infty \subset \mathscr{D}(A_k)$ and the symbol $q_k(x,\xi)$ of $A_k\big|_{C_c^\infty}$ satisfies

$$\begin{aligned} |q_k(x,\xi)| &\leq |q(x,\xi)| \quad \text{for all } x,\xi \in \mathbb{R}^d, & (1.2)\\ q_k(x,\xi) &= q(x,\xi) \quad \text{for all } |x| \leq k, \xi \in \mathbb{R}^d. & (1.3) \end{aligned}$$

Then the operator $(-q(x,D), C_c^\infty)$ has an extension which generates a Feller process and the corresponding semigroup is given by

$$T_t u = \lim_{k\to\infty} T_t^k u$$

for $u \in C_\infty$, where the limit is meant in the strong sense, i.e., with respect to $\|.\|_\infty$.

Note that the theorem does not state uniqueness, i.e., $(-q(x,D), C_c^\infty)$ might have further extensions which also generate Feller processes. The uniqueness is equivalent to C_c^∞ being a core for the domain of the generator. Sufficient conditions for this are non trivial, but they can be obtained by various methods (see for example Chapter 5 in [16]).
The theorem will be proved in several steps. The two main steps are:

- The operators A_k converge to $-q(x,D)$ on C_c^∞ with respect to $\|.\|_\infty$. (See Section 3.)
- The semigroups $(T_t^k)_{t\geq 0}$ form a Cauchy sequence on C_∞ with respect to $\|.\|_\infty$. (See Section 4.)

Each step will be proved under conditions on the symbol which appear to be natural for the given setting. In Section 5 it will be shown that these conditions are implied by (1.1) and in fact some are equivalent to (1.1). Based on this the main theorem will be proved. The paper closes with an example in Section 6.

2 Preliminaries

A Feller process is a strong Markov process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ (with $d \in \mathbb{N}$) for which the family of operators $(T_t)_{t\geq 0}$ defined by

$$T_t u(x) := \mathbb{E}_x(u(X_t)) \quad \text{for } u \in C_\infty$$

with

$$C_\infty := \{u : \mathbb{R}^d \to \mathbb{R} \mid u \text{ continuous}, \lim_{|x|\to\infty} u(x) = 0\}$$

is a Feller semigroup, i.e., $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on C_∞ which is positivity preserving. Note that there is a one-to-one correspondence between Feller semigroups and Feller processes.
The generator A of a Feller semigroup is defined by

$$Au = \lim_{t\to 0} \frac{T_t u - u}{t}$$

on the domain

$$\mathscr{D}(A) := \{u \in C_\infty \mid \lim_{t\to 0} \left\| Au - \frac{T_t u - u}{t} \right\|_\infty = 0\},$$

where $\|u\|_\infty := \sup_{x\in\mathbb{R}^d} |u(x)|$. This generator satisfies the positive maximum principle

$$u(x_0) = \sup_x u(x) \geq 0 \ \Rightarrow Au(x_0) \leq 0,$$

since, with $u^+(x) := \max(u(x), 0)$ for all $x \in \mathbb{R}^d$,

$$T_t u(x_0) \leq T_t u^+(x_0) \leq \|T_t u^+\|_\infty \leq \|u^+\|_\infty = u(x_0).$$

Thus, if the arbitrary often differentiable functions with compact support C_c^∞ are a subset of $\mathscr{D}(A)$, the generator has (by [7]), for $u \in C_c^\infty$ a representation as pseudo differential operator

$$Au(x) = -q(x,D)u(x) = -\int_{\mathbb{R}^d} e^{ix\xi} q(x,\xi)\hat{u}(\xi)\, d\xi, \tag{2.1}$$

where $\hat{u}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\xi} u(x)\, dx$ is the Fourier transform of u and the function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, called negative definite symbol, is measurable and locally bounded in (x,ξ). Furthermore q is a continuous and negative definite function (in the sense of Schoenberg) with respect to ξ. That is, $\xi \mapsto q(x,\xi)$ has for each $x \in \mathbb{R}^d$ a Lévy-Khinchine representation

$$q(x,\xi) = k(x) - il(x)\xi + \xi Q(x)\xi + \int_{y\neq 0}\left(1 - e^{iy\xi} + \frac{iy\xi}{1+|y|^2}\right) N(x,dy),$$

where $k(x) \geq 0$, $l(x) \in \mathbb{R}^d$, $Q(x) = (q_{jk}(x))_{j,k=1,\dots,d} \in \mathbb{R}^{d\times d}$ is a positive semidefinite matrix and $N(x,.)$ is a measure which integrates $\frac{|y|^2}{1+|y|^2}$ on $\mathbb{R}^d \backslash \{0\}$. In the following we will use the abbreviation c.n.d.f. for continuous and negative definite functions.
Conversely, one can also define for each negative definite symbol an operator $-q(x,D)$ by (2.1). Moreover, note that by Fourier inversion this operator has for $u \in C_c^\infty$ also the representation

$$\begin{aligned} -q(x,D)u(x) = &- k(x)u(x) + l(x)\nabla u(x) + \sum q_{jk}(x)\partial_j\partial_k u(x) \\ &+ \int_{y\neq 0}\left(u(x+y) - u(x) - \frac{y\nabla u(x)}{1+|y|^2}\right) N(x,dy). \end{aligned} \tag{2.2}$$

Based on this representation we will prove in the next section a condition which ensures that the sequence of generators A_k defined in the main theorem converges to $-q(x,D)$ on C_c^∞ with respect to $\|.\|_\infty$.
The support of a function $u \in C_c^\infty$ will be denoted by supp u, and $B_R(x)$ denotes the open ball around $x \in \mathbb{R}^d$ with radius $R > 0$.

3 The sequence of generators

We start with a general result about the range of an operator with negative definite symbol.

Lemma 3.1. *Let $q(.,.) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a function such that*

$$\xi \mapsto q(x,\xi) \text{ is a c.n.d.f. for all } x \in \mathbb{R}^d. \tag{3.1}$$

Then for each $u \in C_c^\infty$ with $R > 0$ such that supp $u \subset B_R(0)$ *and all $|x| > R$*

$$|q(x,D)u(x)| \leq \|u\|_\infty\, c \sup_{|\xi|\leq \frac{1}{|x|-R}} \operatorname{Re} q(x,\xi), \tag{3.2}$$

where the constant c is independent of q. (In fact c depends only on the dimension d.)

Remark 3.2. Suppose that, in addition to the assumption of the lemma, for all $R > 0$

$$\lim_{|x|\to\infty} \sup_{|\xi|\leq \frac{1}{|x|-R}} \operatorname{Re} q(x,\xi) = 0 \tag{3.3}$$

and

$$x \mapsto q(x,\xi) \text{ is continuous for all } \xi.$$

Then $q(x,D)(C_c^\infty) \subset C_\infty$. The decay is implied by (3.3) and the continuity is a consequence of representation (2.1) and the dominated convergence theorem.

Proof of Lemma 3.1. Let $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a function satisfying (3.1). Then the corresponding operator $-q(x,D)$ can be defined on C_c^∞ by (2.1) and it has also the representation (2.2). Note that the latter simplifies for $x \notin \operatorname{supp} u$ to

$$-q(x,D)u(x) = \int_{y\neq 0} u(x+y)\, N(x,dy).$$

Furthermore note that, as mentioned in Section 4 of [6] and Lemma 6.1 in [19], there exists some exponentially decaying density g on $\mathbb{R}^d$ with

$$\mathbf{1}_{B_R^c(0)}(y) \le 2\frac{\frac{1}{R}|y|^2}{1+\frac{1}{R}|y|^2} = \int_{\mathbb{R}^d} (1-\cos(\eta y/R))g(\eta)\, d\eta$$

for any $R > 0$ and $y \in \mathbb{R}^d$. Thus, by Tonelli's theorem, for any fixed $x \in \mathbb{R}^d$ and $R > 0$

$$\begin{aligned}
N(x, B_R^c(0)) &\le \int_{y\neq 0} 2\frac{\frac{1}{R}|y|^2}{1+\frac{1}{R}|y|^2}\, N(x,dy)\\
&= \int_{y\neq 0}\int_{\mathbb{R}^d} (1-\cos(\eta y/R))g(\eta)\, d\eta\, N(x,dy)\\
&= \int_{\mathbb{R}^d} \left(\operatorname{Re} q(x,\eta/R) - k(x) - \frac{1}{R^2}\eta Q(x)\eta\right) g(\eta)\, d\eta\\
&\le \int_{\mathbb{R}^d} \operatorname{Re} q(x,\eta/R) g(\eta)\, d\eta\\
&\le \int_{\mathbb{R}^d} (1+|\eta|)^2 g(\eta)\, d\eta \sup_{|\xi|\le \frac{1}{R}} \operatorname{Re} q(x,\xi),
\end{aligned}$$

where we used for the last line the fact that also $\eta \mapsto \operatorname{Re} q(x,\eta/R)$ is a c.n.d.f. and any real valued c.n.d.f. ψ satisfies

$$\psi(\eta) \le (1+|\eta|)^2 \sup_{|\xi|\le 1} \psi(\xi)$$

for all η (see for example the proof of Lemma 6.2 in [19]).
Finally let $u \in C_c^\infty$ and $R > 0$ such that $\operatorname{supp} u \subset B_R(0)$. Then, since $B_R(-x) \subset B^c_{|x|-R}(0)$, we get for $x > R$

$$\begin{aligned}
|q(x,D)u(x)| &\le \int_{y\neq 0} |u(x+y)|\, N(x,dy)\\
&\le \|u\|_\infty N(x, B_R(-x))\\
&\le \|u\|_\infty N(x, B^c_{|x|-R}(0))\\
&\le \|u\|_\infty c \sup_{|\xi|\le \frac{1}{|x|-R}} \operatorname{Re} q(x,\xi)
\end{aligned}$$

with $c := \int_{\mathbb{R}^d} (1+|\eta|)^2 g(\eta)\, d\eta$. □

Now we are able to prove a condition for the sequence of generators in the main theorem to converge to $-q(x,D)$ on C_c^∞ with respect to $\|.\|_\infty$.

Proposition 3.3. *Let $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfy for all $R > 0$*

$$\lim_{|x|\to\infty} \sup_{|\xi|\leq \frac{1}{|x|-R}} |q(x,\xi)| = 0, \tag{3.4}$$

and for each $k \in \mathbb{N}$ let there be a function $q_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that

$$\xi \mapsto q_k(x,\xi) \text{ is a c.n.d.f.} \tag{3.5}$$

and (1.2) *and* (1.3) *hold.*
Then for $u \in C_c^\infty$

$$\lim_{k\to\infty} \|q(x,D)u - q_k(x,D)u\|_\infty = 0.$$

Proof. Let q and q_k be given, such that q satisfys (3.4) and q_k satisfy (3.5), (1.2) and (1.3).
Then, by Lemma 3.6.7. in [12] Vol. 1, $\xi \mapsto q(x,\xi)$ is for each $x \in \mathbb{R}^d$ a negative definite function, since it is (by (1.3)) the pointwise limit of negative definite functions. Moreover the limit is locally uniform and $\xi \mapsto q_k(x,\xi)$ is continuous for each k, thus $\xi \mapsto q(x,\xi)$ is a c.n.d.f..
Now fix $u \in C_c^\infty$ and $\varepsilon > 0$. Thus by Lemma 3.1 we have for $|x| > R$, with $R > 0$ such that $\operatorname{supp} u \subset B_R(0)$,

$$|q(x,D)u(x)| \leq \|u\|_\infty \, c \sup_{|\xi|\leq \frac{1}{|x|-R}} |q(x,\xi)|$$

and with (1.2) also

$$|q_k(x,D)u(x)| \leq \|u\|_\infty \, c \sup_{|\xi|\leq \frac{1}{|x|-R}} |q_k(x,\xi)| \leq \|u\|_\infty \, c \sup_{|\xi|\leq \frac{1}{|x|-R}} |q(x,\xi)|$$

with C independent of k. Thus there exists a $\gamma > 0$ such that for all $|x| > \gamma$

$$|q_k(x,D)u(x)| \vee |q(x,D)u(x)| \leq \frac{\varepsilon}{2}.$$

Finally for $k \geq \gamma$ we get

$$\|q(x,D)u - q_k(x,D)u\|_\infty \leq \varepsilon + \sup_{|x|\leq\gamma} \left| \int_{\mathbb{R}^d} e^{ix\xi}(q(x,\xi) - q_k(x,\xi))\hat{u}(\xi)\, d\xi \right| = \varepsilon,$$

where the integral term vanishes by (1.3). □

Next we will look at the corresponding semigroups.

4 The sequence of semigroups

For this section the following result is fundamental.

Theorem 4.1. *Let $(X_t)_{t\geq 0}$ be a Feller process with generator A, $C_c^\infty \subset \mathscr{D}(A)$ and symbol q. Then for any $x \in \mathbb{R}^d$ and $r, t > 0$*

$$\mathbb{P}_x(\tau_{B_r(x)} \leq t) \leq c t \sup_{|y-x|\leq r} \sup_{|\xi|\leq \frac{1}{r}} |q(y,\xi)|,$$

where

$$\tau_{B_r(x)} := \inf\{t > 0 \mid X_t \notin B_r(x)\},$$

and the constant c does not depend on q.

Proof. The statement can be found in [21], Theorem 1.1. But note that we will only apply it to Feller processes with bounded coefficients, and in this case the statement is already a consequence of Lemma 6.1, Corollary 6.2 and Lemma 5.1 in [18] (see also Theorem 5.5 in [17]), for an alternative proof see Proposition 4.3 in [20]. □

Similar to the decay estimate for the generators in Lemma 3.1 we are now able to prove a decay estimate for the semigroups.

Lemma 4.2. *Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(T_t)_{t\geq 0}$, generator A, $C_c^\infty \subset \mathscr{D}(A)$ and symbol q. Then for all $\varepsilon > 0$, $u \in C_\infty$, $R > 0$, such that $|u(y)| \leq \varepsilon$ for all $|y| > R$, and $x \in \mathbb{R}^d$ with $|x| > R$*

$$|T_t u(x)| \leq \|u\|_\infty tc \sup_{|y-x|\leq|x|-R} \sup_{|\xi|\leq\frac{1}{|x|-R}} |q(y,\xi)| + \varepsilon,$$

where the constant c does not depend on q.

Proof. The proof is an adaptation of the proof of Proposition 3.8 in [21]. Let $(X_t)_{t\geq 0}$, $(T_t)_{t\geq 0}$, q, ε, u, R and x be given as in the statement above. Then an application of Theorem 4.1 yields

$$\begin{aligned}
|T_t u(x)| &\leq \int_{\mathbb{R}^d} |u(y)|\, \mathbb{P}_x(X_t \in dy) \\
&\leq \int_{B_R(0)} |u(y)|\, \mathbb{P}_x(X_t \in dy) + \varepsilon \\
&\leq \|u\|_\infty\, \mathbb{P}_x(|X_t| \leq R) + \varepsilon \\
&\leq \|u\|_\infty\, \mathbb{P}_x(|X_t - x| \geq |x| - R) + \varepsilon \\
&\leq \|u\|_\infty\, \mathbb{P}_x(\tau_{B_{|x|-R}(x)} \leq t) + \varepsilon \\
&\leq \|u\|_\infty\, tc \sup_{|y-x|\leq|x|-R} \sup_{|\xi|\leq\frac{1}{|x|-R}} |q(y,\xi)| + \varepsilon.
\end{aligned}$$

□

Now we can prove that the semigroups $(T_t^k)_{t\geq 0}$ form a Cauchy sequence.

Proposition 4.3. *Let $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfy for all $R > 0$*

$$\lim_{|x|\to\infty} \sup_{|y-x|\leq|x|-R} \sup_{|\xi|\leq\frac{1}{|x|-R}} |q(y,\xi)| = 0, \tag{4.1}$$

$$\lim_{r\to\infty} \sup_{|y|\leq r+R} \sup_{|\xi|\leq\frac{1}{r}} |q(y,\xi)| = 0. \tag{4.2}$$

For each k let $(X_t^k)_{t\geq 0}$, $(T_t^k)_{t\geq 0}$, A_k and q_k be as in Theorem 1.1, in particular (1.2) and (1.3) hold. Then for all $t \geq 0$ and $u \in C_\infty$

$$\lim_{n,m\to\infty} \|T_t^n u - T_t^m u\|_\infty = 0.$$

Moreover for any fixed $K > 0$ this convergence is uniformly for all $t \leq K$.

Proof. Let the assumptions from above hold. Fix $u \in C_\infty$, $t \geq 0$ and $\varepsilon > 0$. Then, by Lemma 4.2, there exists a $\gamma > 0$ such that for all $|x| > \gamma$ and uniformly in k

$$|T_t^k u(x)| \leq \frac{\varepsilon}{2},$$

since (4.1) holds and $|q_k(x,\xi)| \le |q(x,\xi)|$ for all $x,\xi \in \mathbb{R}^d$, $k \in \mathbb{N}$ by (1.2).
Now let $n,m \in \mathbb{N}$. Then, without loss of generality, for $n \le m$

$$\begin{aligned}
\|T_t^n u - T_t^m u\|_\infty &\le \sup_{|x|\le\gamma} |\mathbb{E}_x(u(X_t^n)) - \mathbb{E}_x(u(X_t^m))| + \varepsilon \\
&= \sup_{|x|\le\gamma} |\mathbb{E}_x(u(X_t^n)\mathbf{1}_{(\tau_{B_n(x)},\infty)}(t)) - \mathbb{E}_x(u(X_t^m)\mathbf{1}_{(\tau_{B_n(x)},\infty)}(t))| + \varepsilon \\
&\le 2\|u\|_\infty \sup_{|x|\le\gamma} \mathbb{P}_x(\tau_{B_n(x)} \le t) + \varepsilon \\
&\le 2\|u\|_\infty\, tc \sup_{|x|\le\gamma} \sup_{|y-x|\le n} \sup_{|\xi|\le\frac{1}{n}} |q_n(y,\xi)| + \varepsilon \\
&\le 2\|u\|_\infty\, tc \sup_{|x|\le\gamma} \sup_{|y-x|\le n} \sup_{|\xi|\le\frac{1}{n}} |q(y,\xi)| + \varepsilon \\
&\le 2\|u\|_\infty\, tc \sup_{|y|\le n+\gamma} \sup_{|\xi|\le\frac{1}{n}} |q(y,\xi)| + \varepsilon,
\end{aligned}$$

where we used for the second line the fact that the processes have, by (1.3), the same distribution until exiting $B_n(x)$. The same argument also implied the third line, since also the distributions of the exit-times from $B_n(x)$ coincide for both processes. The fourth line was obtained by an application of Theorem 4.1, and the fifth line is a consequence of (1.2).
Now (4.2) and the fact that ε was arbitrary imply the convergence. The uniformity for $t \le K$ is obvious, since we can estimate t by K in the last line. □

Finally this enables us to prove Theorem 1.1.

5 Proof of Theorem 1.1

Let the assumptions of the theorem be satisfied.
First of all, note that any negative definite function ψ satisfies $|\psi(n\xi)| \le n^2|\psi(\xi)|$ for any $n \in \mathbb{N}$ and all ξ (e.g. Lemma 3.6.21 in [12] Vol. 1). This yields

$$\begin{aligned}
\sup_{|y|\le 3r} \sup_{|\xi|\le\frac{1}{3r}} |q(y,\xi)| &\le \sup_{|y|\le 3r} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)| \\
&\le \sup_{|y|\le 3r} \sup_{|\eta|\le\frac{1}{3r}} |q(y,3\eta)| \\
&\le 9 \sup_{|y|\le 3r} \sup_{|\eta|\le\frac{1}{3r}} |q(y,\eta)|.
\end{aligned}$$

Hence

$$\lim_{r\to\infty} \sup_{|y|\le 3r} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)| = 0 \quad \Longleftrightarrow \quad \lim_{r\to\infty} \sup_{|y|\le r} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)| = 0. \tag{5.1}$$

This implies that (1.1)⟺(4.2), since for all $R > 0$

$$\lim_{r\to\infty} \sup_{|y|\le r} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)| \le \lim_{r\to\infty} \sup_{|y|\le r+R} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)| \le \lim_{r\to\infty} \sup_{|y|\le 3r} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)|.$$

Moreover for all $R > 0$

$$\begin{aligned}\lim_{|x|\to\infty} \sup_{|\xi|\le\frac{1}{|x|-R}} |q(x,\xi)| &\le \lim_{|x|\to\infty} \sup_{|y-x|\le|x|-R} \sup_{|\xi|\le\frac{1}{|x|-R}} |q(y,\xi)| \\ &\le \lim_{|x|\to\infty} \sup_{|y|\le 2|x|-R} \sup_{|\xi|\le\frac{1}{|x|-R}} |q(y,\xi)| \\ &= \lim_{r\to\infty} \sup_{|y|\le 2r+R} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)| \\ &\le \lim_{r\to\infty} \sup_{|y|\le 3r} \sup_{|\xi|\le\frac{1}{r}} |q(y,\xi)|,\end{aligned}$$

and therefore (5.1) implies that (1.1)⇒(4.1)⇒(3.4).
Thus by Proposition 3.3

$$\lim_{k\to\infty} \|A_k u - (-q(x,D)u)\|_\infty = 0 \quad \text{for all } u \in C_c^\infty \subset \bigcup_l \bigcap_{n\ge l} \mathscr{D}(A_n)$$

and by Proposition 4.3

$$\lim_{n,m\to\infty} \|T_t^n u - T_t^m u\|_\infty = 0 \quad \text{for all } u \in C_\infty.$$

Now, since C_c^∞ is dense in C_∞, there exists (by Theorem 2 and Remark 2 in [9]) a closed extension of $(-q(x,D), C_c^\infty)$ which generates the semigroup $(T_t)_{t\ge0}$ given by $T_t u = \lim_{k\to\infty} T_t^k u$, where the limit is taken with respect to $\|.\|_\infty$ for all $u \in C_\infty$. The semigroup is defined on C_∞ and for each $t \ge 0$ the operator T_t is a positivity preserving contraction, since each T_t^k is a positivity preserving contraction. Moreover, by Proposition 4.3, the convergence $T_t u = \lim_{k\to\infty} T_t^k u$ is uniform for all $t < 1$. Thus for $u \in C_\infty$, $\varepsilon > 0$ and there exists a k such that

$$\sup_{s\le1} \|T_s u - T_s^k u\|_\infty \le \varepsilon/2.$$

Furthermore, since $(T_t^k)_{t\ge0}$ is a Feller semigroup, there exists an $r < 1$ such that $\|u - T_t^k u\|_\infty \le \varepsilon/2$ for all $t \le r$. Hence for all $t \le r$

$$\|u - T_t u\|_\infty \le \|u - T_t^k u\|_\infty + \sup_{s<1} \|T_s^k u - T_s u\|_\infty \le \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

i.e., $(T_t)_{t\ge0}$ is strongly continuous. Thus $(T_t)_{t\ge0}$ is a Feller semigroup and the theorem is proved.

6 Example

There is a vast amount of possible examples, since basically for any sequence of Feller processes with symbols q_k (with bounded coefficients) we might get, under the assumptions of the main theorem, a Feller processes with unbounded coefficients. Therefore we decided to give only one rather simple example which illustrates the technique. It will also explain, why we speak about *bounded* and *unbounded coefficients*.
Let $\psi : \mathbb{R} \to \mathbb{R}$ be a c.n.d.f. such that $\psi(0) = 0$, $\psi(-\xi) = \psi(\xi)$ and $\psi(\xi)$ increases for $|\xi| \to \infty$. Furthermore, let $\Phi(x) = |x|^\beta \vee 1$ for some $\beta \in (0,1)$. Then condition (1.1) is fulfilled for $q(x,\xi) := \psi(\Phi(x)\xi)$ since

$$\lim_{r\to\infty} \sup_{|y|\le r} \sup_{|\xi|\le\frac{1}{r}} |q(x,\xi)| = \lim_{r\to\infty} \psi(|r|^\beta \frac{1}{r}) = \psi(0) = 0.$$

The symbols $q_k(x,\xi) := \psi((\Phi(x) \wedge k)\xi)$ satisfy (1.2) and (1.3). The corresponding processes are (see [19] Section 3) the solutions to the stochastic differential equations with *bounded coefficients*

$$dX_t^k = (\Phi(X_{t-}^k) \wedge k)\, dZ_t,$$

where $(Z_t)_{t\geq 0}$ is a Lévy process with characteristic exponent ψ (i.e., $(Z_t)_{t\geq 0}$ is a Feller process with symbol $(x,\xi) \mapsto \psi(\xi)$). Moreover these solutions $(X_t^k)_{t\geq 0}$ are Feller processes (see for example Corollary 3.3. in [19]). Hence Theorem 1.1 yields that $-q(x,D)$ generates a Feller process. Note that this process is a solution to the stochastic differential equation with *unbounded coefficient*

$$dX_t = \Phi(X_{t-})\, dZ_t.$$

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