GEOMETRY OF THE RANDOM INTERLACEMENT

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Abstract

We consider the geometry of random interlacements on the d-dimensional lattice. We use ideas
from stochastic dimension theory developed in [BKPS04] to prove the following: Given that two
vertices x, y belong to the interlacement set, it is possible to find a path between x and y contained
in the trace left by at most $\lceil d/2 \rceil$ trajectories from the underlying Poisson point process. Moreover,
this result is sharp in the sense that there are pairs of points in the interlacement set which cannot
be connected by a path using the traces of at most $\lceil d/2 \rceil - 1$ trajectories.

1 Introduction

The model of random interlacements was introduced by Sznitman in [Szn10], on the graph $\mathbb{Z}^d$,
$d \geq 3$. Informally, the random interlacement is the trace left by a Poisson point process on the
space of doubly infinite trajectories modulo time shift on $\mathbb{Z}^d$. The intensity measure of the Poisson
process is given by uv, $u > 0$ and v is a measure on the space of doubly infinite trajectories,
see (7) below. This is a site percolation model that exhibits infinite-range dependence, which for
example presents serious complications when trying to adapt techniques developed for standard
independent site percolation.
In [Szn10], it was proved that the random interlacement on $\mathbb{Z}^d$ is always a connected set. In this
paper we prove a stronger statement (for precise formulation, see Theorem 2.2):

*Given that two vertices $x, y \in \mathbb{Z}^d$ belong to the interlacement set, it is a.s. possible to find a path
between x and y contained in the trace left by at most $\lceil d/2 \rceil$ trajectories from the underlying Poisson*

point process. Moreover, this result is sharp in the sense that a.s. there are pairs of points in the inter-
lacement set which cannot be connected by a path using the traces of at most $\lceil d/2 \rceil - 1$ trajectories.

Our method is based on the concept of stochastic dimension (see Section 2.2 below) introduced by Benjamini, Kesten, Peres and Schramm, [BKPS04]. They studied the geometry of the so called uniform spanning forest, and here we show how their techniques can be adapted to the study of the geometry of the random interlacements.

In Section 2.1 we introduce the model of random interlacements more precisely. In Section 2.2 we give the required background on stochastic dimension and random relations from [BKPS04]. Finally the precise statement and proof of Theorem 2.2 is split in two parts: the lower bound is given in Section 5 and the upper bound in Section 4.

Throughout the paper, c and c' will denote dimension dependant constants, and their values may change from place to place. Dependence of additional parameters will be indicated, for example $c(u)$ will stand for a constant depending on d and u.

During the last stages of this research we have learned that B. Rath and A. Sapozhnikov, see [RS10], have solved this problem independently. Their proof is significantly different from the proof we present in this paper.

2 Preliminaries

In this section we recall the model of random interlacements from [Szn10] and the concept of stochastic dimension from [BKPS04].

2.1 Random interlacements

Let W and W_+ be the spaces of doubly infinite and infinite trajectories in $\mathbb{Z}^d$ that spend only a finite amount of time in finite subsets of $\mathbb{Z}^d$:

$$W = \{\gamma : \mathbb{Z} \to \mathbb{Z}^d; |\gamma(n) - \gamma(n+1)| = 1, \forall n \in \mathbb{Z}; \lim_{n \to \pm\infty} |\gamma(n)| = \infty\},$$

$$W_+ = \{\gamma : \mathbb{N} \to \mathbb{Z}^d; |\gamma(n) - \gamma(n+1)| = 1, \forall n \in \mathbb{Z}; \lim_{n \to \infty} |\gamma(n)| = \infty\}.$$

The canonical coordinates on $\mathscr{W}$ and $\mathscr{W}_+$ will be denoted by X_n, $n \in \mathbb{Z}$ and X_n, $n \in \mathbb{N}$ respectively. Here we use the convention that $\mathbb{N}$ includes 0. We endow W and W_+ with the sigma-algebras $\mathscr{W}$ and $\mathscr{W}_+$, respectively which are generated by the canonical coordinates. For $\gamma \in W$, let range$(\gamma) = \gamma(\mathbb{Z})$. Furthermore, consider the space W^* of trajectories in W modulo time shift:

$$W^* = W / \sim, \text{ where } w \sim w' \iff w(\cdot) = w'(\cdot + k) \text{ for some } k \in \mathbb{Z}.$$

Let π^* be the canonical projection from W to W^*, and let $\mathscr{W}^*$ be the sigma-algebra on W^* given by $\{A \subset W^* : (\pi^*)^{-1}(A) \in \mathscr{W}\}$. Given $K \subset \mathbb{Z}^d$ and $\gamma \in W_+$, let $\tilde{H}_K(\gamma)$ denote the hitting time of K by γ:

$$\tilde{H}_K(\gamma) = \inf\{n \geq 1 : X_n(\gamma) \in K\}. \tag{1}$$

For $x \in \mathbb{Z}^d$, let P_x be the law on $(W_+, \mathscr{W}_+)$ corresponding to simple random walk started at x, and for $K \subset \mathbb{Z}^d$, let P_x^K be the law of simple random walk, conditioned on not hitting K. Define the equilibrium measure of K:

$$e_K(x) = \begin{cases} P_x[\tilde{H}_K = \infty], & x \in K \\ 0, & x \notin K. \end{cases} \tag{2}$$

Define the capacity of a set $K \subset \mathbb{Z}^d$ as

$$\text{cap}(K) = \sum_{x \in \mathbb{Z}^d} e_K(x). \tag{3}$$

The normalized equilibrium measure of K is defined as

$$\tilde{e}_K(\cdot) = e_K(\cdot)/\text{cap}(K). \tag{4}$$

For $x, y \in \mathbb{Z}^d$ we let $|x - y| = \|x - y\|_1$. We will repeatedly make use of the following well-known estimates of hitting-probabilities. For any $x, y \in \mathbb{Z}^d$ with $|x - y| \geq 1$,

$$c|x - y|^{-(d-2)} \leq P_x[\tilde{H}_y < \infty] \leq c'|x - y|^{-(d-2)}, \tag{5}$$

see for example Theorem 4.3.1 in [LL10]. Next we define a Poisson point process on $W^* \times \mathbb{R}_+$. The intensity measure of the Poisson point process is given by the product of a certain measure ν and the Lebesque measure on $\mathbb{R}_+$. The measure ν was constructed by Sznitman in [Szn10], and now we characterize it. For $K \subset \mathbb{Z}^d$, let W_K denote the set of trajectories in W that enter K. Let $W_K^* = \pi^*(W_K)$ be the set of trajectories in W^* that intersect K. Define Q_K to be the finite measure on W_K such that for $A, B \in \mathcal{W}_+$ and $x \in \mathbb{Z}^d$,

$$Q_K[(X_{-n})_{n \geq 0} \in A, X_0 = x, (X_n)_{n \geq 0} \in B] = P_x^K[A]e_K(x)P_x[B]. \tag{6}$$

The measure ν is the unique σ-finite measure such that

$$\mathbb{1}_{W_K^*}\nu = \pi^* \circ Q_K, \ \forall K \subset \mathbb{Z}^d \text{ finite}. \tag{7}$$

The existence and uniqueness of the measure was proved in Theorem 1.1 of [Szn10]. Consider the set of point measures in $W^* \times \mathbb{R}_+$:

$$\Omega = \left\{ \omega = \sum_{i=1}^{\infty} \delta_{(w_i^*, u_i)}; w_i^* \in W^*, u_i \in \mathbb{R}_+, \right.$$
$$\left. \omega(W_K^* \times [0, u]) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d \text{ and } u \in \mathbb{R}_+ \right\}. \tag{8}$$

Also consider the space of point measures on W^*:

$$\tilde{\Omega} = \left\{ \sigma = \sum_{i=1}^{\infty} \delta_{w_i^*}; w_i^* \in W^*, \sigma(W_K^*) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d \right\}. \tag{9}$$

For $u > u' \geq 0$, we define the mapping $\omega_{u',u}$ from Ω into $\tilde{\Omega}$ by

$$\omega_{u',u} = \sum_{i=1}^{\infty} \delta_{w_i^*} \mathbb{1}\{u' \leq u_i \leq u\}, \text{ for } \omega = \sum_{i=1}^{\infty} \delta_{(w_i^*, u_i)} \in \Omega. \tag{10}$$

If $u' = 0$, we write ω_u. Sometimes we will refer trajectories in ω_u, rather than in the support of ω_u. On Ω we let $\mathbb{P}$ be the law of a Poisson point process with intensity measure given by $\nu(dw^*)dx$. Observe that under $\mathbb{P}$, the point process $\omega_{u,u'}$ is a Poisson point process on $\tilde{\Omega}$ with intensity measure $(u - u')\nu(dw^*)$. Given $\sigma \in \tilde{\Omega}$, we define

$$\mathcal{I}(\sigma) = \bigcup_{w^* \in \text{supp}(\sigma)} w^*(\mathbb{Z}). \tag{11}$$

For $0 \leq u' \leq u$, we define

$$\mathscr{I}^{u',u} = \mathscr{I}(\omega_{u',u}), \tag{12}$$

which we call the *random interlacement set* between levels u' and u. In case $u' = 0$, we write $\mathscr{I}^u$. For a point process σ on Ω, we let $\sigma|_A$ stand for σ restricted to $A \subset W^*$.

2.2 Stochastic dimension

In this section, we recall some definitions and results from [BKPS04] and adapt them to our needs. For $x, y \in \mathbb{Z}^d$, let $\langle xy \rangle = 1 + |x - y|$. Suppose $W \subset \mathbb{Z}^d$ is finite and that τ is a tree on W. Let $\langle \tau \rangle = \Pi_{\{x,y\} \in \tau} \langle xy \rangle$. Finally let $\langle W \rangle = \min_\tau \langle \tau \rangle$ where the minimum is taken over all trees on the vertex set W. For example, for n vertices $x_1, ..., x_n$, $\langle x_1...x_n \rangle$ is the minimum of n^{n-2} products with $n - 1$ factors in each.

Definition 2.1. *Let $\mathscr{R}$ be a random subset of $\mathbb{Z}^d \times \mathbb{Z}^d$ with distribution $\mathbf{P}$. We will think of $\mathscr{R}$ as a relation and for $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, we write $x\mathscr{R}y$ if $(x, y) \in \mathscr{R}$. Let $\alpha \in [0, d)$. We say that $\mathscr{R}$ has stochastic dimension $d - \alpha$ if there exists a constant $c = c(\mathscr{R}) < \infty$ such that*

$$c\,\mathbf{P}[x\mathscr{R}y] \geq \langle xy \rangle^{-\alpha}, \tag{13}$$

and

$$\mathbf{P}[x\mathscr{R}y, z\mathscr{R}v] \leq c\langle xy \rangle^{-\alpha}\langle zv \rangle^{-\alpha} + c\langle xyzv \rangle^{-\alpha}, \tag{14}$$

for all $x, y, z, v \in \mathbb{Z}^d$.
If $\mathscr{R}$ has stochastic dimension $d - \alpha$, then we write $dim_S(\mathscr{R}) = d - \alpha$.

Observe that $\inf_{x,y \in \mathbb{Z}^d} \mathbf{P}[x\mathscr{R}y] > 0$ if and only if $dim_S(\mathscr{R}) = d$.

Definition 2.2. *Let $\mathscr{R}$ and $\mathscr{M}$ be any two random relations. We define the composition $\mathscr{R}\mathscr{M}$ to be the set of all $(x, z) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that there exists some $y \in \mathbb{Z}^d$ for which $x\mathscr{R}y$ and $y\mathscr{M}z$ holds. The n-fold composition of a relation $\mathscr{R}$ will be denoted by $\mathscr{R}^{(n)}$.*

Next we restate Theorem 2.4 of [BKPS04], which we will use extensively.

Theorem 2.1. *Let $\mathscr{L}, \mathscr{R} \subset \mathbb{Z}^d$ be two independent random relations. Then*

$$dim_S(\mathscr{L}\mathscr{R}) = \min\left\{dim_S(\mathscr{L}) + dim_S(\mathscr{R}), d\right\},$$

when $dim_S(\mathscr{L})$ and $dim_S(\mathscr{R})$ exist.

For each $x \in \mathbb{Z}^d$, we choose a trajectory $w_x \in W_+$ according to P_x. Also assume that w_x and w_y are independent for $x \neq y$ and that the collection $(w_x)_{x \in \mathbb{Z}}$ is independent of ω.
We will take interest in several particular relations, defined in terms of ω and the collection $(w_x)_{x \in \mathbb{Z}^d}$. For $\omega = \sum_{i=1}^\infty \delta_{(w_i^*, u_i)} \in \Omega$, $t_2 \geq t_1 \geq 0$, and $n \in \mathbb{N}$ let

1.
$$\mathscr{M}_{t_1,t_2} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \exists \gamma \in \text{supp}(\omega_{t_1,t_2}) : x, y \in \gamma(\mathbb{Z})\}. \tag{15}$$

   If $t_1 = 0$, we will write $\mathscr{M}_{t_2}$ as shorthand for $\mathscr{M}_{t_1,t_2}$.

2. $\mathscr{L} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : y \in \text{range}(w_x)\}$

3. $\mathscr{R} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : x \in \text{range}(w_y)\}$

4.

$$\mathscr{C}_{n,u} = \mathscr{L}\left(\prod_{i=2}^{n-1} \mathscr{M}_{u(i-1)/n,ui/n}\right)\mathscr{R}, \, n \geq 3. \tag{16}$$

Theorem 2.2. *For any $d \geq 3$, $u > 0$ and all $x, y \in \mathbb{Z}^d$,*

$$\mathbb{P}\left[x\mathscr{M}_u^{\lceil\frac{d}{2}\rceil}y \,\Big|\, x, y \in \mathscr{I}^u\right] = 1.$$

In addition we have

$$\mathbb{P}\left[\exists x, y \in \mathscr{I}^u, y \notin \{z : x\mathscr{M}_u^{\lceil\frac{d}{2}\rceil-1}z\}\right] = 1.$$

For $d = 3, 4$ the theorem follows easily from the fact that two independent simple random walk trajectories intersect each other almost surely, and we omit the details of these two cases. From now on, we will assume that $d \geq 5$. A key step in the proof of our main theorem, is to show that for every $x, y \in \mathbb{Z}^d$ and $u > 0$, we have $\mathbb{P}[x\mathscr{C}_{\lceil d/2\rceil,u}y] = 1$.

Proposition 2.3. *Under $\mathbb{P}$, for any $0 \leq t_1 < t_2 < \infty$, the relation $\mathscr{M}_{t_1,t_2}$ has stochastic dimension 2.*

Proof. Clearly, it is enough to consider the case $t_1 = 0$ and $t_2 = u \in (0, \infty)$. First recall that the trajectories in $\text{supp}(\omega_u)$ that intersect $x \in \mathbb{Z}^d$ can be sampled in the following way (see for example Theorem 1.1 and Proposition 1.3 of [Szn10]):

1. Sample a Poisson random number N with mean $u\text{cap}(x)$

2. Then sample N independent double sided infinite trajectories, where each such trajectory is given by a simple random walk path started at x, together with a simple random walk path started at x conditioned on never returning to x.

We now establish a lower bound of $\mathbb{P}[x\mathscr{M}_uy]$. Since any trajectory in $\text{supp}(\omega_u)$ intersecting x contains a simple random walk trajectory started at x, we obtain that

$$\mathbb{P}[x\mathscr{M}_uy] \stackrel{(5)}{\geq} c\langle xy\rangle^{-(d-2)}. \tag{17}$$

Thus the condition (13) is established and it remains to establish the more complicated condition (14). For this, fix distinct vertices $x, y, z, v \in \mathbb{Z}^d$ and put $K = \{x, y, z, v\}$. Our next task is to find an upper bound of $\mathbb{P}[x\mathscr{M}_uy, z\mathscr{M}_uv]$. For $\omega_u = \sum_{i\geq 0} \delta_{w_i^*}$, we let $\hat{\omega}_u = \sum_{i\geq 0} \delta_{w_i^*}\mathbb{1}_{\{\text{range}(w_i^*)\supset K\}}$. We now write

$$\mathbb{P}[x\mathscr{M}_uy, z\mathscr{M}_uv] = \mathbb{P}[x\mathscr{M}_uy, z\mathscr{M}_uv, \hat{\omega}_u = 0] + \mathbb{P}[x\mathscr{M}_uy, z\mathscr{M}_uv, \hat{\omega}_u \neq 0], \tag{18}$$

and deal with the two terms on the right hand side of (18) separately. For a point measure $\tilde{\omega} \leq \omega_u$, we write "$x\mathscr{M}_uy$ in $\tilde{\omega}$" if there is a trajectory in $\text{supp}(\tilde{\omega})$ whose range contains both x and y. Observe that if $w^* \in \text{supp}(\omega_u - \hat{\omega}_u)$ and $x, y \in \text{range}(w^*)$, then at least one of z or v does not belong to $\text{range}(w^*)$. Hence, the events $\{x\mathscr{M}_uy$ in $\omega_u - \hat{\omega}_u\}$ and $\{z\mathscr{M}_uv$ in $\omega_u - \hat{\omega}_u\}$ are defined in terms of disjoint sets of trajectories, and thus they are independent under $\mathbb{P}$. We get that

$$\mathbb{P}[x\mathscr{M}_uy, z\mathscr{M}_uv, \hat{\omega}_u = 0] = \mathbb{P}[x\mathscr{M}_uy \text{ in } \omega_u - \hat{\omega}_u, z\mathscr{M}_uv \text{ in } \omega_u - \hat{\omega}_u, \hat{\omega}_u = 0]$$

$$\leq \mathbb{P}[x\mathscr{M}_u y \text{ in } \omega_u - \hat{\omega}_u, z\mathscr{M}_u v \text{ in } \omega_u - \hat{\omega}_u]$$
$$= \mathbb{P}[x\mathscr{M}_u y \text{ in } \omega_u - \hat{\omega}_u]\mathbb{P}[z\mathscr{M}_u v \text{ in } \omega_u - \hat{\omega}_u]$$
$$\leq \mathbb{P}[x\mathscr{M}_u y]\mathbb{P}[z\mathscr{M}_u v]$$
$$\overset{(5)}{\leq} c(u)(\langle x\,y\rangle\langle z\,v\rangle)^{-(d-2)}. \tag{19}$$

where in the second equality we used the independence that was mentioned above. In addition, we have

$$\mathbb{P}[x\mathscr{M}_u y, z\mathscr{M}_u v, \hat{\omega}_u \neq 0] = \mathbb{P}[\hat{\omega}_u \neq 0]. \tag{20}$$

We now find an upper bound on $\mathbb{P}[\hat{\omega}_u \neq 0]$. In view of (19), (20) and (18), in order to establish (14) with $\alpha = d - 2$, it is sufficient to show that

$$\mathbb{P}[\hat{\omega}_u \neq 0] \leq c(u)\langle x\,y\,z\,v\rangle^{-(d-2)}. \tag{21}$$

Recall that the law of a simple random walk started at x conditioned on never returning to x is dominated by the law of a simple random walk started at x (this follows from the fact that the trajectory of a simple random walk after the last time it visits x has the same distribution as a the trajectory of a simple random walk conditioned on not returning to x). Using this and the method described above of sampling the trajectories from ω_u containing x, we obtain that $\mathbb{P}[\hat{\omega}_u \neq 0]$ is bounded from above by the probability that at least one of N independent double sided simple random walks started at x hits each of y, z, v. Here N again is a Poisson random variable with mean $u\text{cap}(x)$. We obtain that

$$\mathbb{P}[\hat{\omega} \neq 0] = 1 - \exp(-u\text{cap}(x)P_x^{\otimes 2}[\{y,z,v\} \subset (X'_n)_{n\geq 0} \cup (X_n)_{n\geq 0}])$$
$$\leq u\text{cap}(x)P_x^{\otimes 2}[\{y,z,v\} \subset (X'_n)_{n\geq 0} \cup (X_n)_{n\geq 0}], \tag{22}$$

where we in the last inequality made use of the inequality $1 - \exp(-x) \leq x$ for $x \geq 0$. Here, $P_x^{\otimes 2}[\{y,z,v\} \subset (X'_n)_{n\geq 0} \cup (X_n)_{n\geq 0}]$ is the probability that a double sided simple random walk starting at x hits each of y, z, v. In order to bound this probability, we first obtain some quite standard hitting estimates. We have

$$P_x[H_y < \infty, H_z < \infty, H_v < \infty] = \sum_{\substack{x_1,x_2,x_3 \in \\ \text{perm}(z,y,v)}} P_x[H_{x_1} < H_{x_2} < H_{x_3} < \infty]$$
$$\leq \sum_{\substack{x_1,x_2,x_3 \in \\ \text{perm}(z,y,v)}} P_x[H_{x_1} < \infty]P_{x_1}[H_{x_2} < \infty]P_{x_2}[H_{x_3} < \infty]$$
$$\overset{(5)}{\leq} c \sum_{\substack{x_1,x_2,x_3 \in \\ \text{perm}(z,y,v)}} \left(\langle x\,x_1\rangle\langle x_1\,x_2\rangle\langle x_2\,x_3\rangle\right)^{-(d-2)}$$
$$\leq c\langle K\rangle^{-(d-2)}, \tag{23}$$

where the sums are over all permutations of z, y, v. Similarly, for any choice of x_1 and x_2 from $\{y,z,v\}$ with $x_1 \neq x_2$,

$$P_x[H_{x_1} < \infty, H_{x_2} < \infty] \leq c((\langle x\,x_1\rangle\langle x_1\,x_2\rangle)^{-(d-2)} + (\langle x\,x_2\rangle\langle x_2\,x_1\rangle)^{-(d-2)}) \tag{24}$$

and

$$P_x[H_{x_1} < \infty] \leq c\langle x\,x_1\rangle^{-(d-2)}. \tag{25}$$

Now set $A = \{\{y, z, v\} \subset (X_n)_{n \geq 0}\}$, $A' = \{\{y, z, v\} \subset (X'_n)_{n \geq 0}\}$,

$$B = \bigcup_{t=y,z,v} \{t \in (X_n)_{n \geq 0}, K \setminus \{t\} \subset (X'_n)_{n \geq 0}\}, \tag{26}$$

and

$$B' = \bigcup_{t=y,z,v} \{t \in (X'_n)_{n \geq 0}, K \setminus \{t\} \subset (X_n)_{n \geq 0}\}. \tag{27}$$

Observe that

$$\{\{y, z, v\} \subset (X'_n)_{n \geq 0} \cup (X_n)_{n \geq 0}\} \subset A \cup A' \cup B \cup B'. \tag{28}$$

We have

$$P_x[A] = P_x[A'] \overset{(23)}{\leq} c \langle K \rangle^{-(d-2)}. \tag{29}$$

Using the independence between $(X_n)_{n \geq 0}$ and $(X'_n)_{n \geq 0}$, it readily follows that

$$P_x^{\otimes 2}[B] = P_x^{\otimes 2}[B'] \overset{(24),\ (25)}{\leq} c \langle K \rangle^{-(d-2)}. \tag{30}$$

From (28), (29) and (30) and a union bound, we obtain

$$P_x^{\otimes 2}[\{y, z, v\} \subset (X'_n)_{n \geq 0} \cup (X_n)_{n \geq 0}] \leq c \langle K \rangle^{-(d-2)}. \tag{31}$$

Combining (22) and (31) gives (21), finishing the proof of the proposition.   □

Now recall the definition of the walks $(w_x)_{x \in \mathbb{Z}^d}$ from Section 2.2.

Lemma 2.4. *The relations $\mathscr{L}$ and $\mathscr{R}$ have stochastic dimension* 2.

Proof. We start with the relation $\mathscr{L}$. For $x, y \in \mathbb{Z}^d$, we have

$$\mathbf{P}[x \mathscr{L} y] = \mathbf{P}[y \in \mathrm{range}(w_x)] = P_x[\tilde{H}_y < \infty] \tag{32}$$

From (32) and (5), we obtain

$$c \langle xy \rangle^{-(d-2)} \leq \mathbf{P}[x \mathscr{L} y] \leq c' \langle xy \rangle^{-(d-2)} \tag{33}$$

In addition, for $x, y, z, w \in \mathbb{Z}^d$ with $x \neq y$, using the independence between the walks w_x and w_y, we get

$$\mathbf{P}[x \mathscr{L} z, y \mathscr{L} w] = \mathbf{P}[x \mathscr{L} z] \mathbf{P}[y \mathscr{L} w] \overset{(33)}{\leq} c \langle xz \rangle^{-(d-2)} \langle yw \rangle^{-(d-2)}. \tag{34}$$

In the case $x = y$, we obtain

$$\mathbf{P}[x \mathscr{L} z, y \mathscr{L} w] \leq c((\langle xz \rangle \langle zw \rangle)^{-(d-2)} + (\langle xw \rangle \langle zw \rangle)^{-(d-2)}). \tag{35}$$

From (33), (35) and (34) we obtain $dim_S(\mathscr{L}) = 2$. The proof of the statement $dim_S(\mathscr{R}) = 2$ is shown by the same arguments.   □

Lemma 2.5. *For any $u > 0$ and $n \geq 3$, $dim_S(\mathscr{C}_{n,u}) = \min(2n, d)$.*

Proof. We have

$$
\begin{aligned}
dim_S(\mathscr{C}_{n,u}) &= dim_S \left(\mathscr{L} \left(\prod_{i=2}^{n-1} \mathscr{M}_{u(i-1)/n,ui/n} \right) \mathscr{R} \right) \\
&= \min \left(dim_S(\mathscr{L}) + \sum_{i=2}^{n-1} dim_S(\mathscr{M}_{u(i-1)/n,ui/n}) + dim_S(\mathscr{R}), d \right) \\
&= \min(2 + 2(n-2) + 2, d) \\
&= \min(2n, d),
\end{aligned} \tag{36}
$$

where we in the second equality used the independence of the relations and Theorem 2.1, and for the third equality used Lemma 2.4 and Proposition 2.3. $\qquad\square$

3 Tail trivialities

The proof of the upper bound of Theorem 2.2, which will be given in Section 4 below, will rely on the use of Corollary 3.4 of [BKPS04]. To be able to use that corollary, we first must discuss tail trivialities of the stochastic relations we work with.

Definition 3.1. *Let $\mathscr{E}$ be a random relation and $v \in \mathbb{Z}^d$. Define the left tail field corresponding to the vertex v to be*

$$
\mathscr{F}_{\mathscr{E}}^L(v) = \bigcap_{K \subset \mathbb{Z}^d \text{ finite}} \sigma\{v\mathscr{E}x : x \notin K\}. \tag{37}
$$

We say that $\mathscr{E}$ is left tail trivial if $\mathscr{F}_{\mathscr{E}}^L(v)$ is trivial for every $v \in \mathbb{Z}^d$.

Definition 3.2. *Let $\mathscr{E}$ be a random relation and $v \in \mathbb{Z}^d$. Define the right tail field corresponding to the vertex v be*

$$
\mathscr{F}_{\mathscr{E}}^R(v) = \bigcap_{K \subset \mathbb{Z}^d \text{ finite}} \sigma\{x\mathscr{E}v : x \notin K\}. \tag{38}
$$

We say that $\mathscr{E}$ is right tail trivial if $\mathscr{F}_{\mathscr{E}}^R(v)$ is trivial for every $v \in \mathbb{Z}^d$.

Definition 3.3. *Let $\mathscr{E}$ be a random relation. Define the remote tail field to be*

$$
\mathscr{F}_{\mathscr{E}}^{Rem} = \bigcap_{K_1, K_2 \subset \mathbb{Z}^d \text{ finite}} \sigma\{x\mathscr{E}y : x \notin K_1, y \notin K_2\}. \tag{39}
$$

We say that $\mathscr{E}$ is remote tail trivial if $\mathscr{F}_{\mathscr{E}}^{Rem}$ is trivial.

3.1 Left and right tail trivialities

Recall the definition of the walks $(w_x)_{x \in \mathbb{Z}^d}$ Section 2.2.

Lemma 3.1. *The relation $\mathscr{L}$ is left tail trivial. The relation $\mathscr{R}$ is right tail trivial.*

Proof. We start with the relation $\mathscr{L}$. For any $x \in \mathbb{Z}^d$, we have

$$
\mathscr{F}_{\mathscr{L}}^L(x) = \bigcap_{R > 1} \sigma \left\{ \text{range}(w_x) \cap B(x, R)^c \right\} \subset \bigcap_{R > 1} \sigma \left\{ (w_x(i))_{i \geq R} \right\}. \tag{40}
$$

Since the σ-algebra on the right hand side of (40) is trivial ([Dur10] Theorem 6.7.5), $\mathscr{F}_{\mathscr{L}}^L(x)$ is trivial for every $x \in \mathbb{Z}^d$. Hence, $\mathscr{L}$ is left tail trivial. Similarly, we obtain that $\mathscr{R}$ is right tail trivial. $\qquad\square$

3.2 Remote tail triviality

We omit the details of the following lemma.

Lemma 3.2. *Fix $\mu_0 \in \mathbb{R}_+$ and $s \in \mathbb{N}$. For $\mu > \mu_0$, let $X \sim Pois(\mu - \mu_0)$ and $Y \sim Pois(\mu)$. Then*

$$\sum_{t=0}^{\infty} |P[X = t - s] - P[Y = t]| \to 0 \text{ as } \mu \to \infty. \tag{41}$$

Definition 3.4. *For a set $K \subset \mathbb{Z}^d$ denote by $\eta_K = \omega(W_K^*) = |\{w \in supp(\omega) : w \cap K \neq \phi\}|$.*

Lemma 3.3. *Let $K \subset \mathbb{Z}^d$ be a finite set. Denote by $B = B(0, \rho)$, the ball of radius ρ around 0. Then for any $s \in \mathbb{N}$,*

$$\sum_{t=0}^{\infty} \left| \mathbb{P}[\eta_B = t | \eta_K = s] - \mathbb{P}[\eta_B = t] \right| \to 0 \text{ as } \rho \to \infty.$$

Proof. Write $\eta_B = (\eta_B - \eta_K) + \eta_K$. Observe that $\eta_B - \eta_K$ and η_K are independent random variables with distributions $Pois(u\mathrm{cap}(B) - u\mathrm{cap}(K))$ and $Pois(u\mathrm{cap}(K))$ respectively. Consequently

$$\mathbb{P}[\eta_B = t | \eta_K = s] = \mathbb{P}[\eta_B - \eta_K = t - s]. \tag{42}$$

Since $u\mathrm{cap}(B) \to \infty$ as $\rho \to \infty$, the lemma follows from (42) and Lemma 3.3, with the choices $\mu_0 = u\mathrm{cap}(K)$, $\mu = u\mathrm{cap}(B)$, $X = \eta_B - \eta_K$ and $Y = \eta_B$.

$\square$

We will need the following lemma, easily deduced from [LL10] Proposition 2.4.2 and Theorem A.4.5. For every $x \in \mathbb{Z}^d$, denote by $\mathrm{par}(x) = \sum_{j=1}^{d} x_j$, and $\mathrm{even}(x) = \delta_{\mathrm{par}(x) \text{ is even}}$.

Lemma 3.4. *Let $k > 0$, $r > 0$, $\epsilon > 0$ and $K = B(0, r) \subset \mathbb{Z}^d$. For every*

$$(x_i, y_i)_{i=1}^{k} \in \partial K \times \partial K$$

we can define $2k$ random walks $(X_n^i)_{i=1}^{k}, (Y_n^i)_{i=1}^{k}$, conditioned on never returning to K, on the same probability space with initial starting points $X_0^i = x_i, Y_0^i = y_i$ for all $1 \leq i \leq k$ such that $((X_n^i)_{n \geq 0}, (Y_n^i)_{n \geq 0})_{i=1}^{k}$ are independent and there is a $n = n(k, \epsilon, K) > 0$ large enough for which

$$P[\forall 1 \leq i \leq k, X_m^i = Y_{m+\mathrm{even}(x_i - y_i)}^i \text{ for all } m \geq n] \geq 1 - \epsilon.$$

Lemma 3.5. *Let $u > 0$. The relation $\mathcal{M}_u$ is remote tail trivial.*

Proof. First we show that it is enough to prove that $\mathcal{F}_{\mathcal{M}_u}^{Rem}$ is independent of $\mathcal{F}_K = \sigma\{x \mathcal{M}_u y : x, y \in K\}$ for every finite $K \subset \mathbb{Z}^d$. So assume this independence. Let $A \in \mathcal{F}_{\mathcal{M}_u}^{R}$ and let K_n be finite sets such that $K_n \subset K_{n+1}$ for every n and $\cup_n K_n = \mathbb{Z}^d$. Let $M_n = \mathbb{P}[A | \mathcal{F}_{K_n}]$. Then M_n is a martingale and $M_n = \mathbb{P}[A]$ a.s, since we assumed independence. From Doob's martingale convergence theorem we get that $M_n \to \mathbb{1}_A$ a.s and thus $\mathbb{P}[A] \in \{0, 1\}$.
Let $K \subset \mathbb{Z}^d$ be finite. Suppose $A \in \mathcal{F}_{\mathcal{M}_u}^{Rem}$, $B \in \mathcal{F}_K$ and that $\mathbb{P}[B] > 0$. According to the above, to obtain the remote tail triviality of $\mathcal{M}_u$ it is sufficient to show that

$$\mathbb{P}[A|B] = \mathbb{P}[A]. \tag{43}$$

Let $0 < r_1 < r_2$ be such that $K \subset B(0, r_1)$. Later, r_1 and r_2 will be chosen to be large. Fix $\epsilon > 0$. Let $N = \eta_{B(0, r_1)}$. Let $C = C(K) > 0$ and $D = D(r_1) > 0$ be so large that

$$\mathbb{P}[\eta_K \geq C] < \epsilon \mathbb{P}[B] \text{ and } \mathbb{P}[N \geq D] < \epsilon \mathbb{P}[B]. \tag{44}$$

Recall that $\omega_u|_{W^*_{B(0,r_1)}} = \sum_{i=1}^N \delta_{\pi^*(w_i)}$ where N is $\mathrm{Pois}(u\mathrm{cap}(B(0, r_1)))$ distributed and conditioned on N, $(w_i(0))_{i=1}^N$ are i.i.d. with distribution $\tilde{e}_{B(0,r_1)}(\cdot)$, $((w_i(k))_{k \geq 0})_{i=1}^N$ are independent simple random walks, and $((w_i(k))_{k \leq 0})_{i=1}^N$ are independent simple random walks conditioned on never returning to $B(0, r_1)$ (see for example Theorem 1.1 and Proposition 1.3 of [Szn10], or (6) in Section 2.1 above). Letting τ_i be the last time $(w_i(k))_{k \geq 0}$ visits $B(0, r_1)$, we have have that $((w_i(k))_{k \geq \tau_i})_{i=1}^N$ are independent simple random walks conditioned on never returning to $B(0, r_1)$. We define the vector

$$\bar{\xi} = \big(w_1(0), \ldots, w_N(0), w_1(\tau_1), \ldots, w_N(\tau_N)\big) \in \partial(B(0, r_1))^{2N}.$$

Let $\kappa_i^+ = \inf\{k > \tau_i : w_i(k) \in \partial B(0, r_2)\}$ and let $\kappa_i^- = \sup\{k < 0 : w_i(k) \in \partial B(0, r_2)\}$. Define the vector

$$\bar{\gamma} = \big(w_1(\kappa_1^+), \ldots, w_N(\kappa_N^+), w_1(\kappa_1^-), \ldots, w_N(\kappa_N^-)\big) \in \partial(B(0, r_2))^{2N}.$$

Observe that since A belongs to $\mathscr{F}_{\mathcal{M}_u}^{Rem}$ and $|\kappa_i^+ - \kappa_i^-| < \infty$ for $i = 1, \ldots, N$ a.s., we get that A is determined by $((w_i(k))_{k \geq \kappa_i^+})_{i=1}^N$ and $((w_i(k))_{k \leq \kappa_i^-})_{i=1}^N$ and $\omega_u|_{W^* \setminus W^*_{B(0,r_1)}}$. On the other hand, B is determined by $((w_i(k))_{0 \leq k \leq \tau_i})_{i=1}^N$ where $0 < \kappa_i^- < \tau_i < \kappa_i^+$ for $i = 1, \ldots, N$. In addition, conditioned on N and $\bar{\gamma}$ we have that $((w_i(k))_{k \geq \kappa_i^+})_{i=1}^N$, $((w_i(k))_{k \leq \kappa_i^-})_{i=1}^N$ and $((w_i(k))_{\kappa_i^- \leq k \leq \kappa_i^+})_{i=1}^N$ are conditionally independent. Therefore, conditioned on N and $\bar{\gamma}$, the events A and B are conditionally independent. It follows that for any $n \in \mathbb{N}$ and any $\bar{x} \in (\partial B(0, r_2))^{2n}$

$$\mathbb{P}[A \cap B | N = n, \bar{\gamma} = \bar{x}] = \mathbb{P}[A | N = n, \bar{\gamma} = \bar{x}] \mathbb{P}[B | N = n, \bar{\gamma} = \bar{x}]. \tag{45}$$

From (45) we easily deduce

$$\mathbb{P}[A | B, N = n, \bar{\gamma} = \bar{x}] = \mathbb{P}[A | N = n, \bar{\gamma} = \bar{x}]. \tag{46}$$

Therefore,

$$|\mathbb{P}[A|B] - \mathbb{P}[A]| = \left| \sum_{n=0}^{\infty} \sum_{\bar{x} \in (\partial B(0,r_2))^{2n}} \mathbb{P}[A|B, N = n, \bar{\gamma} = \bar{x}] \mathbb{P}[N = n, \bar{\gamma} = \bar{x}|B] - \mathbb{P}[A] \right|$$

$$\overset{(46)}{=} \left| \sum_{n=0}^{\infty} \sum_{\bar{x} \in (\partial B(0,r_2))^{2n}} \mathbb{P}[A|N = n, \bar{\gamma} = \bar{x}] \big[\mathbb{P}[N = n, \bar{\gamma} = \bar{x}|B] - \mathbb{P}[N = n, \bar{\gamma} = \bar{x}] \big] \right| \tag{47}$$

$$\leq \sum_{n=0}^{\infty} \sum_{\bar{x} \in (\partial B(0,r_2))^{2n}} \big| \mathbb{P}[N = n, \bar{\gamma} = \bar{x}|B] - \mathbb{P}[N = n, \bar{\gamma} = \bar{x}] \big|.$$

Hence, to obtain (43) it will be enough to show that the double sum appearing in the right hand side of (47) can be made arbitrarily small by choosing r_1 sufficiently large, and then r_2 sufficiently large. This will be done in several steps.

Using Lemma 3.3 we can choose r_1 big enough such that for every $m < C$,

$$\sum_{n=0}^{\infty} |\mathbb{P}[N = n | \eta_K = m] - \mathbb{P}[N = n]| < \epsilon/C. \tag{48}$$

Also observe that since N depends only on $\omega_u|_{W_K^*}$ through η_K, we have

$$\mathbb{P}[N = n | B, \eta_K = m] = \mathbb{P}[N = n | \eta_K = m]. \tag{49}$$

This gives

$$\sum_{n=0}^{\infty} |\mathbb{P}[N = n | B] - \mathbb{P}[N = n]|$$

$$= \sum_{n=0}^{\infty} | \sum_{m=0}^{\infty} (\mathbb{P}[N = n | B, \eta_K = m] - \mathbb{P}[N = n]) \, \mathbb{P}[\eta_K = m | B] |$$

$$\overset{(49)}{\leq} \sum_{n=0}^{\infty} \sum_{m=0}^{C-1} |\mathbb{P}[N = n | \eta_K = m] - \mathbb{P}[N = n]| \, \mathbb{P}[\eta_K = m | B] \tag{50}$$

$$+ \sum_{n=0}^{\infty} \sum_{m=C}^{\infty} |\mathbb{P}[N = n | \eta_K = m] - \mathbb{P}[N = n]| \mathbb{P}[\eta_K = m | B].$$

We now estimate the last two lines of (50) separately. We have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{C-1} |\mathbb{P}[N = n | \eta_K = m] - \mathbb{P}[N = n]| \, \mathbb{P}[\eta_K = m | B] \overset{(48)}{\leq} \epsilon. \tag{51}$$

For the last line of (50), we get

$$\sum_{n=0}^{\infty} \sum_{m=C}^{\infty} |\mathbb{P}[N = n | \eta_K = m] - \mathbb{P}[N = n]| \mathbb{P}[\eta_K = m | B]$$

$$\leq \sum_{n=0}^{\infty} \sum_{m=C}^{\infty} \mathbb{P}[N = n | \eta_K = m] \mathbb{P}[\eta_K = m | B] + \sum_{n=0}^{\infty} \sum_{m=C}^{\infty} \mathbb{P}[N = n] \mathbb{P}[\eta_K = m | B] \tag{52}$$

$$\leq \sum_{n=0}^{\infty} \sum_{m=C}^{\infty} \frac{\mathbb{P}[N = n, \eta_K = m]}{\mathbb{P}[B]} + \mathbb{P}[\eta_K \geq C | B] \leq \frac{\mathbb{P}[\eta_K \geq C]}{\mathbb{P}[B]} + \frac{\mathbb{P}[\eta_K \geq C]}{\mathbb{P}[B]}$$

$$\overset{(44)}{\leq} 2\epsilon.$$

Combining (50), (51) and (52) we obtain that

$$\sum_{n=0}^{\infty} |\mathbb{P}[N = n | B] - \mathbb{P}[N = n]| \leq 3\epsilon. \tag{53}$$

Applying Lemma 3.4 to both backward and forward paths, we can choose $r_2 > r_1$ large enough so that for any $n \leq D$ and for any $\bar{y} \in (\partial B(0, r_1))^{2n}$,

$$\sum_{\bar{x} \in \partial B(0, r_2)^{2n}} |\mathbb{P}[\bar{\gamma} = \bar{x} | B, N = n, \bar{\xi} = \bar{y}] - \mathbb{P}[\bar{\gamma} = \bar{x} | N = n]| =$$

$$\sum_{\bar{x} \in \partial B(0, r_2)^{2n}} |\mathbb{P}[\bar{\gamma} = \bar{x} | N = n, \bar{\xi} = \bar{y}] - \mathbb{P}[\bar{\gamma} = \bar{x} | N = n]| < \epsilon, \tag{54}$$

where the first equality can be shown in a way similar to (49). Thus, for any $n \leq D$,

$$\sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n] \right|$$

$$\leq \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \sum_{\bar{y} \in \partial B(0,r_1)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n, \bar{\xi} = \bar{y}] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n] \right| \mathbb{P}[\bar{\xi} = \bar{y}|B, N = n]$$

$$= \sum_{\bar{y} \in \partial B(0,r_1)^{2n}} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n, \bar{\xi} = \bar{y}] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n] \right| \mathbb{P}[\bar{\xi} = \bar{y}|B, N = n]$$

$$\overset{(54)}{<} \epsilon.$$

(55)

We now have what we need to bound the right hand side of (47):

$$\sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[N = n, \bar{\gamma} = \bar{x}|B] - \mathbb{P}[N = n, \bar{\gamma} = \bar{x}] \right|$$

$$= \sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n]\mathbb{P}[N = n|B] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n]\mathbb{P}[N = n] \right|$$

$$\leq \sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n]\mathbb{P}[N = n|B] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n]\mathbb{P}[N = n|B] \right|$$

$$+ \sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|N = n]\mathbb{P}[N = n|B] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n]\mathbb{P}[N = n] \right|$$

(56)

We now estimate the two last lines in (56) separately. We have

$$\sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n]\mathbb{P}[N = n|B] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n]\mathbb{P}[N = n|B] \right|$$

$$\leq \sum_{n=0}^{D} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n] \right| \mathbb{P}[N = n|B]$$

$$+ \sum_{n=D}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|B, N = n] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n] \right| \mathbb{P}[N = n|B]$$

$$\overset{(55)}{\leq} \epsilon + 2\mathbb{P}[N \geq D|B] \leq \epsilon + 2\frac{\mathbb{P}[N \geq D]}{\mathbb{P}[B]} \overset{(44)}{\leq} 3\epsilon,$$

(57)

where the first inequality follows from the triangle inequality. For the second line of (56) we get

$$\sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[\bar{\gamma} = \bar{x}|N = n]\mathbb{P}[N = n|B] - \mathbb{P}[\bar{\gamma} = \bar{x}|N = n]\mathbb{P}[N = n] \right|$$

$$= \sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \mathbb{P}[\bar{\gamma} = \bar{x}|N = n] \left| \mathbb{P}[N = n|B] - \mathbb{P}[N = n] \right| \overset{(53)}{\leq} 3\epsilon.$$

(58)

From (56), (57) and (58), we obtain

$$\sum_{n=0}^{\infty} \sum_{\bar{x} \in \partial B(0,r_2)^{2n}} \left| \mathbb{P}[N = n, \bar{\gamma} = \bar{x}|B] - \mathbb{P}[N = n, \bar{\gamma} = \bar{x}] \right| \leq 6\epsilon.$$

(59)

Since $\epsilon > 0$ is arbitrary, we deduce that $\mathbb{P}[A|B] = \mathbb{P}[A]$ from (47) and (59). The triviality of the sigma algebra $\mathscr{F}_{\mathcal{M}_u}^{Rem}$ is therefore established.   □

4   Upper bound

In this section, we provide the proof of the upper bound of Theorem 2.2. Throughout this section, fix $n = \lceil d/2 \rceil$ and fix $u \in (0, \infty)$. Recall the definition of the trajectories $(w_x)_{x \in \mathbb{Z}^d}$ from Section 2.2. We have proved in Lemma 2.5 that the random relation $\mathscr{C}_{n,u}$ has stochastic dimension d, and therefore $\inf_{x,y \in \mathbb{Z}^d} \mathbf{P}[x \mathscr{C}_{n,u} y] > 0$. Since $\mathscr{L}$ is left tail trivial, $\mathscr{R}$ is right tail trivial and the relations $\mathcal{M}_{u(i-1)/n, ui/n}$ are remote tail trivial for $i = 1, ..., n$, we obtain from Corollary 3.4 of [BKPS04] that

$$\mathbf{P}[x \mathscr{C}_{n,u} y] = 1 \text{ for every } x, y \in \mathbb{Z}^d. \tag{60}$$

Now fix x and y and let A_1 be the event that $x \in \mathscr{I}^{u/n}$ and A_2 be the event that $y \in \mathscr{I}^{(n-1)u/n,u}$. Put $A = A_1 \cap A_2$. We now use (60) to argue that

$$\mathbb{P}\left[x \mathcal{M}_u^{(n)} y \Big| A\right] = 1. \tag{61}$$

To see this, first observe that A is the event that $\omega_{0,u/n}(W_x^*) \geq 1$ and $\omega_{u(n-1)/n,u}(W_y^*) \geq 1$. Consequently, on A, $\mathscr{I}(\omega_{u/n}|_{W_x^*})$ contains at least one trace of a simple random walk started at x and hence stochastically dominates $\text{range}(w_x)$. In the same way, $\mathscr{I}(\omega_{u(n-1)/n,u/n}|_{W_y^*})$ stochastically dominates $\text{range}(w_y)$. Thus we obtain

$$\mathbb{P}\left[x \mathcal{M}_u^{(n)} y \Big| A\right] \geq \mathbb{P}\left[x \left(\prod_{i=1}^n \mathcal{M}_{u(i-1)/n, ui/n}\right) y \Big| A\right] \geq \mathbf{P}[x \mathscr{C}_{n,u} y] = 1, \tag{62}$$

giving (61). Equation (61) implies that if $x \in \mathscr{I}^{u/n}$ and $y \in \mathscr{I}^{(n-1)u/n,u}$, then x and y are $\mathbb{P}$-a.s. connected in the ranges of at most $\lceil d/2 \rceil$ trajectories from $\text{supp}(\omega_u)$.

Now let $I_1 = [t_1, t_2] \subset [0, u]$ and $I_2 = [t_3, t_4] \subset [0, u]$ be disjoint intervals. Let A_{I_1, I_2} be the event that $x \in \mathscr{I}^{t_1, t_2}$ and $y \in \mathscr{I}^{t_3, t_4}$. The proof of (61) is easily modified to obtain

$$\mathbb{P}\left[x \mathcal{M}_u^{(n)} y \Big| A_{I_1, I_2}\right] = 1. \tag{63}$$

Observe that up to a set of measure 0, we have

$$\{x \in \mathscr{I}^u, y \in \mathscr{I}^u\} = \{x \mathcal{M}_u y\} \cup \left(\bigcup_{I_1, I_2} \{x \in \mathscr{I}^{t_1, t_2}, y \in \mathscr{I}^{t_3, t_4}\}\right), \tag{64}$$

where the union is over all disjoint intervals $I_1 = [t_1, t_2], I_2 = [t_3, t_4] \subset [0, u]$ with rational distinct endpoints. Observe that all the events in the countable union on the right hand side of (64) have positive probability. In addition, due to (63), conditioned on any of them, we have $x \mathcal{M}_u^{(n)} y$ a.s. Therefore, we finally conclude that

$$\mathbb{P}\left[x \mathcal{M}_u^{(n)} y \Big| x, y \in \mathscr{I}^u\right] = 1, \tag{65}$$

finishing the proof of the upper bound of Theorem 2.2.

5 Lower bound

In this section, we provide the proof of the lower bound of Theorem 2.2. More precisely, we show that with probability one, there are vertices x and y contained in $\mathscr{I}^u$ that are not connected by a path using at most $\lceil \frac{d}{2} \rceil - 1$ trajectories from $\text{supp}(\omega_u)$.

We introduce a decomposition of ω_u as follows. Let ω_u^0 be the point measure supported on those $w_i^* \in \text{supp}(\omega_u)$ for which $0 \in w_i^*(\mathbb{Z})$. Then proceed inductively: given $\omega_u^0, ..., \omega_u^{k-1}$, define ω_u^k to be the point measure supported on those $w_i^* \in \text{supp}(\omega_u)$ such that $w_i^* \notin \text{supp}(\sum_{i=0}^{k-1} \omega_u^i)$ and $w_i^*(\mathbb{Z}) \cap \left(\cup_{w_i^* \in \text{supp}(\omega_u^{k-1})} w_i(\mathbb{Z}) \right) \neq \emptyset$.

For $k = 0, 1, ...$, define

$$V_k = \bigcup_{w^* \in \text{supp}(\sum_{i=0}^k \omega_u^i)} w^*(\mathbb{Z}).$$

In addition, let $V_{-1} = \{0\}$ and $V_{-2} = \emptyset$. Observe that with this notation,

$$\omega_u^k = \omega_u|_{(W_{V_{k-1}}^* \setminus W_{V_{k-2}}^*)}, \; k = 0, 1, ...$$

Here $\omega_{u|_A}$ denotes ω_u restricted to the set of trajectories $A \subset W^*$. We also observe that conditioned on $\omega_u^0, ..., \omega_u^{k-1}$, under $\mathbb{P}$,

$$\omega_u^k \text{ is a Poisson point process on } W^* \text{ with intensity measure } u \mathbb{1}_{W_{V_{k-1}}^* \setminus W_{V_{k-2}}^*} \nu(dw^*), \tag{66}$$

see the Appendix for details. We now construct the vector $(\bar{\omega}_u^0, ..., \bar{\omega}_u^k)$ with the same law as the vector $(\omega_u^0, ..., \omega_u^k)$ in the following way. Suppose $\sigma_0, \sigma_1, ...$ are i.i.d. with the same law as ω_u. Let $\bar{\omega}_u^0 = \sigma_0|_{W_{\{0\}}^*}$ and then proceed inductively as follows: Given $\bar{\omega}_u^0, ..., \bar{\omega}_u^k$, define

$$\bar{V}_k = \bigcup_{w^* \in \text{supp}(\sum_{i=0}^k \bar{\omega}_u^i)} w^*(\mathbb{Z}),$$

and let $\bar{V}_{-1} = \{0\}$ and $\bar{V}_{-2} = \emptyset$. Then let

$$\bar{\omega}_u^{k+1} = \sigma_{k+1}|_{(W_{\bar{V}_k}^* \setminus W_{\bar{V}_{k-1}}^*)}.$$

Using (66) one checks that in this procedure, for any $k \geq 0$, the vector $(\bar{\omega}_u^0, ..., \bar{\omega}_u^k)$ has the same law as $(\omega_u^0, ..., \omega_u^k)$.

Let $m = \lceil \frac{d}{2} \rceil - 1$. We now get that

$$\mathbb{P}[0 \mathscr{M}_u^{(m)} x] = \mathbb{P}\left[0 \xleftrightarrow{V_{m-1}} x \right] = \mathbb{P}^{\otimes n}\left[0 \xleftrightarrow{\bar{V}_{m-1}} x \right]. \tag{67}$$

The event $\left\{ 0 \xleftrightarrow{\bar{V}_{m-1}} x \right\}$ is the event that there is some $l \leq m - 1$ and trajectories $\gamma_i \in \bar{\omega}_u^i, i = 0, ..., l$, such that $\gamma_i(\mathbb{Z}) \cap \gamma_{i+1}(\mathbb{Z}) \neq \emptyset$, $0 \in \gamma_0(\mathbb{Z})$ and $x \in \gamma_l(\mathbb{Z})$. Since $\bar{\omega}_u^i \leq \sigma_i$, we obtain

$$\mathbb{P}^{\otimes n}\left[0 \xleftrightarrow{\bar{V}_{m-1}} x \right] \leq \sum_{l=0}^{m-1} \mathbb{P}^{\otimes l}\left[0 \prod_{i=0}^l (\mathscr{M}_u(\sigma_i)) \, x \right], \tag{68}$$

where we use the notation $\mathcal{M}_u(\sigma_i)$ for the random relation defined in the same way as $\mathcal{M}_u$, but using σ_i instead of ω_u. Now use the independence of the σ_i's, Theorem 2.1, and the fact that $\dim_{\mathscr{S}}(\mathcal{M}_u(\sigma_i)) = 2$, to obtain that for any $l \leq m - 1$,

$$\dim_{\mathscr{S}}\left(\prod_{i=0}^{l}(\mathcal{M}_u(\sigma_i))\right) \leq 2m < d. \tag{69}$$

Therefore by (67), (68) and (69),

$$\mathbb{P}[0\mathcal{M}_u^{(m)}x] \to 0 \text{ as } |x| \to \infty. \tag{70}$$

Put $\hat{\omega}_u^m = \sum_{i=0}^{m} \omega_u^i$. Observe that Equation (70) can be restated using the notation of (11) as

$$\mathbb{P}\left[x \in \mathscr{I}(\hat{\omega}_u^{m-1})\right] \to 0 \text{ as } |x| \to \infty. \tag{71}$$

For $n \geq 1$, let $x_n = ne_1$. For $n \geq 1$, we define the events $A_n = \{x_n \in \mathscr{I}^u(\hat{\omega}_u^{m-1})\}$ and $B_n = \{x_n \in \mathscr{I}^u\}$ Using (71) we can find a sequence $(n_k)_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \mathbb{P}[A_{n_k}] < \infty. \tag{72}$$

By the Borel-Cantelli lemma,

$$\mathbb{P}[A_{n_k} \text{ i.o.}] = 0. \tag{73}$$

On the other hand, by ergodicity (see Theorem 2.1 in [Szn10]), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{B_{n_k}} = \mathbb{P}[0 \in \mathscr{I}^u] \text{ a.s.} \tag{74}$$

Since $\mathbb{P}[0 \in \mathscr{I}^u] > 0$, Equation (74) implies that

$$\mathbb{P}[B_{n_k} \text{ i.o.}] = 1. \tag{75}$$

From equations (73) and (75) it readily follows that $\mathbb{P}[\cup_{i\geq 1}(B_i \setminus A_i)] = 1$, which implies

$$\mathbb{P}[\exists y \in \mathscr{I}^u, y \notin \{z : 0\mathcal{M}_u^{\lceil d/2\rceil-1}z\}] = 1. \tag{76}$$

Since the law of $\mathscr{I}^u$ is invariant under the translations of $\mathbb{Z}^d$, we get that for any $x \in \mathbb{Z}^d$,

$$\mathbb{P}[\exists y \in \mathscr{I}^u, y \notin \{z : x\mathcal{M}_u^{\lceil d/2\rceil-1}z\}] = 1. \tag{77}$$

Observe that

$$\{\exists x, y \in \mathscr{I}^u, y \notin \{z : x\mathcal{M}_u^{\lceil d/2\rceil-1}z\}\}$$
$$= \bigcup_{x\in\mathbb{Z}^d} \{\exists y \in \mathscr{I}^u, y \notin \{z : x\mathcal{M}_u^{\lceil d/2\rceil-1}z\}\} \cap \{x \in \mathscr{I}^u\}. \tag{78}$$

Using (77), we see that the probability of the event in the left hand side of (78) equals $\mathbb{P}[\cup_{x\in\mathbb{Z}^d}\{x \in \mathscr{I}^u\}] = 1$. Thus (78) gives

$$\mathbb{P}[\exists x, y \in \mathscr{I}^u, y \notin \{z : x\mathcal{M}_u^{\lceil d/2\rceil-1}z\}] = 1, \tag{79}$$

and the proof the lower bound is complete.

6   Open questions

The following question was asked by Itai Benjamini: Given two points $x, y \in \mathbb{Z}^d$, estimate the probability that x and y are connected by at most $\lceil \frac{d}{2} \rceil$ trajectories intersecting a ball of radius r around the origin.
Answering the first question can help solve the question of how one finds the $\lceil \frac{d}{2} \rceil$ trajectories connecting two points in an efficient manner.

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7   Appendix

Here we show a technical lemma (Lemma 7.2 below) needed in the proof of the lower bound in Section 5. For the proof of Lemma 7.2, we need the following lemma, which is standard and we state without proof.

Lemma 7.1. *Let X be a Poisson point process on W^*, with intensity measure ρ. Let $A \subset W^*$ be chosen at random independently of X. Then, conditioned on A, the point processes $1_A X$ and $1_{A^c} X$ are independent Poisson point processes on W^* with intensity measures $1_A \rho$ and $1_{A^c} \rho$ respectively.*

Write $\omega_u = \sum_{k=0}^\infty \omega_u^k$ where ω_u^k is defined in the end of Section 2.1. Put $V_{-2} = \emptyset$ and $V_{-1} = \{0\}$ and

$$V_k = \mathscr{I}\left(\sum_{i=0}^k \omega_u^i \right), \, k = 0, 1, \ldots \tag{80}$$

Recall that

$$\omega_u^k = \omega_u|_{(W^*_{V_{k-1}} \setminus W^*_{V_{k-2}})}, \, k = 0, 1, \ldots \tag{81}$$

Introduce the point process

$$\tilde{\omega}_u^k = \omega_u|_{W^* \setminus W^*_{V_{k-1}}}, \, k = 0, 1, \ldots \tag{82}$$

For $k \geq 0$, write $\mathbb{P}_k$ for $\mathbb{P}$ conditioned on $\omega_u^0, .., \omega_u^k$.

Lemma 7.2. *Fix $k \geq 0$. Then, conditioned on $\omega_u^0, \ldots, \omega_u^{k-1}$, the point processes ω_u^k and $\tilde{\omega}_u^k$ are independent Poisson point processes on W^*, with intensity measures*

$$u \mathbb{1}_{(W^*_{V_{k-1}} \setminus W^*_{V_{k-2}})} \nu(dw^*) \tag{83}$$

and

$$u \mathbb{1}_{(W^* \setminus W^*_{V_{k-1}})} \nu(dw^*), \tag{84}$$

respectively.

Proof. We will proceed by induction. First consider the case $k = 0$. We have $\omega_u^0 = \omega_u|_{W^*_{\{0\}}}$ and $\tilde{\omega}_u^0 = \omega|_{W^* \setminus W^*_{\{0\}}}$. The sets of trajectories $W^*_{\{0\}}$ and $W^* \setminus W^*_{\{0\}}$ are non-random. Therefore we get that, using for example Proposition 3.6 in [Res08], ω_u^0 and $\tilde{\omega}_u^0$ are Poisson point processes with

intensity measures that agree with (83) and (84) respectively. In addition, the sets of trajectories $W^*_{\{0\}}$ and $W^* \setminus W^*_{\{0\}}$ are disjoint, and therefore ω^0_u and $\tilde{\omega}^0_u$ are independent.

Now fix some $k \geq 0$ and assume that the assertion of the lemma is true for k. Observe that we have

$$\omega^{k+1}_u = \tilde{\omega}^k_u|_{W^*_{\mathscr{I}(\omega^k_u)}} \tag{85}$$

and

$$\tilde{\omega}^{k+1}_u = \tilde{\omega}^k_u|_{W^* \setminus W^*_{\mathscr{I}(\omega^k_u)}}. \tag{86}$$

By the induction assumption, ω^k_u and $\tilde{\omega}^k_u$ are independent Poisson process under $\mathbb{P}_{k-1}$. In particular, under $\mathbb{P}_{k-1}$, $\tilde{\omega}^k_u$ and $W^*_{\mathscr{I}(\omega^k_u)}$ are independent. Therefore, using Lemma 7.1 and (85), we see that if we further condition on ω^k_u, the point process ω^{k+1}_u is a Poisson point process on W^* with intensity measure given by $u \mathbb{1}_{W^*_{\mathscr{I}(\omega^k_u)}} \mathbb{1}_{(W^* \setminus W^*_{V_{k-1}})} \nu(dw^*)$. However,

$$u \mathbb{1}_{W^*_{\mathscr{I}(\omega^k_u)}} \mathbb{1}_{(W^* \setminus W^*_{V_{k-1}})} \nu(dw^*) = u \mathbb{1}_{(W^*_{V_k} \setminus W^*_{V_{k-1}})} \nu(dw^*), \tag{87}$$

and therefore the claim regarding ω^{k+1}_u established. The claim regarding $\tilde{\omega}^{k+1}_u$ follows similarly, by noting that

$$u \mathbb{1}_{W^* \setminus W^*_{\mathscr{I}(\omega^k_u)}} \mathbb{1}_{(W^* \setminus W^*_{V_{k-1}})} \nu(dw^*) = u \mathbb{1}_{(W^* \setminus W^*_{V_k})} \nu(dw^*). \tag{88}$$

Finally, the independence between ω^{k+1}_u and $\tilde{\omega}^{k+1}_u$ under $\mathbb{P}_k$ follows from Lemma 7.1. $\square$

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