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Strong solutions of jump-type stochastic equations*

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Abstract

We establish the existence and uniqueness of strong solutions to some jump-type stochastic equations under non-Lipschitz conditions. The results improve those of Fu and Li [11] and Li and Mytnik [15].

Keywords: Strong solution; jump-type stochastic equation; pathwise uniqueness; non-Lipschitz condition.
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1 Introduction

The problem of existence and uniqueness of solutions to jump-type stochastic equations under non-Lipschitz conditions have been studied by many authors; see, e.g., [1, 2, 3, 10, 11, 13, 15] and the references therein. In particular, some criteria for the existence and pathwise uniqueness of non-negative and general solutions were given in [10, 11, 15]. Stochastic equations have played important roles in the recent progresses in the study of continuous-state branching processes with and without immigration; see, e.g., [5, 6, 7, 14]. The main difficulty of pathwise uniqueness for jump-type stochastic equations usually comes from the compensated Poisson integral term. Let us consider the equation

$$dx(t) = \phi(x(t-))d\tilde{N}(t), \tag{1.1}$$

where $\{\tilde{N}(t) : t \geq 0\}$ is a compensated Poisson process. For each $0 < \alpha < 1$ there is a α-Hölder continuous function ϕ so that the pathwise uniqueness for (1.1) fails. In fact, before the first jump of the Poisson process, the above equation reduces to

$$dx(t) = -\phi(x(t))dt. \tag{1.2}$$

Then to assure the pathwise uniqueness for (1.1) the uniqueness of solution for (1.2) is necessary. If we set $h_\alpha(x) = (1-\alpha)^{-1}x^\alpha 1_{\{x\geq 0\}}$, then both $x_1(t) = 0$ and $x_2(t) = t^{1/(1-\alpha)}$ are solutions of (1.2) with $\phi = -h_\alpha$. From those it is easy to construct two distinct solutions of (1.1). The key of the pathwise uniqueness results in [11, 15] is to consider a non-decreasing kernel for the compensated Poisson integral term in the stochastic equation. The condition was weakened considerably by Fournier [10] for stable driving noises. In fact, as a consequence of Theorem 2.2 in [15], given any $x(0) \in \mathbb{R}$ there is a pathwise unique strong solution to (1.1) with $\phi = h_\alpha$. On the other hand, the monotonicity assumption also excludes some interesting jump-type stochastic equations. Two of them are given below.

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Example 1.1. *Let $z^2\nu(dz)$ be a finite measure on $(0,1]$. Suppose that $\tilde{M}(ds,dz,dr)$ is a compensated Poisson random measure on $(0,\infty)\times(0,1]^2$ with intensity $ds\nu(dz)dr$. Given $0\leq x(0)\leq 1$, we consider the stochastic integral equation*

$$x(t) = x(0) + \int_0^t \int_0^1 \int_0^1 zq(x(s-),r)\tilde{M}(ds,dz,dr), \tag{1.3}$$

where

$$q(x,r) = 1_{\{r\leq 1\wedge x\}} - (1\wedge x)1_{\{x\geq 0\}}.$$

This equation was introduced by Bertoin and Le Gall [4] in their study of generalized Fleming-Viot flows. The existence and uniqueness of a weak solution flow to (1.3) was proved in [4]. The pathwise uniqueness for the equation follows from a result in [7]. The result cannot be derived directly from the those in [11, 15] since $x\mapsto q(x,r)$ is not a non-decreasing function.

Example 1.2. *Let $(1\wedge u^2)\mu(du)$ be a finite measure on $(0,\infty)$. Suppose that $\tilde{N}(ds,du,dr)$ is a compensated Poisson random measure on $(0,\infty)^3$ with intensity $ds\mu(du)dr$. Given $y(0)\geq 0$, we consider the stochastic equation*

$$y(t) = y(0) + \int_0^t \int_0^\infty \int_0^\infty g(y(s-),u,r)\tilde{N}(ds,du,dr), \tag{1.4}$$

where

$$g(x,u,r) = 1_{\{rx\leq 1\}}x(e^{-u}-1).$$

Some generalizations of the above equation were introduced by Döring and Barczy [8] in the study of self-similar Markov processes. From their results it follows that (1.4) has a pathwise unique non-negative strong solution. Since $x\mapsto g(x,u,r)$ is not non-decreasing, one cannot derive the pathwise uniqueness for (1.4) from the results in [11, 15].

In this paper, we give some criteria for the existence and pathwise uniqueness of strong solutions of jump-type stochastic equations. The results improve those in [11, 15] and can be applied to equations like (1.3) and (1.4). In Section 2 we give some basic formulations of the stochastic equations. Two theorems on the pathwise uniqueness of general solutions are presented in Section 3. In Section 4 we prove the existence of weak solutions by a martingale problem approach. The main results on the existence and pathwise uniqueness of general strong solutions are given in Section 5. In Section 6 we give some results on the existence and pathwise uniqueness of non-negative strong solutions. Throughout this paper, we make the conventions

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}$$

for any $b\geq a\geq 0$. Given a function f defined on a subset of $\mathbb{R}$, we write

$$\Delta_z f(x) = f(x+z) - f(x) \quad \text{and} \quad D_z f(x) = \Delta_z f(x) - f'(x)z$$

if the right hand sides are meaningful.

2 Preliminaries

Suppose that $\mu_0(du)$ and $\mu_1(du)$ are σ-finite measures on the complete separable metric spaces U_0 and U_1, respectively. Throughout this paper, we consider a set of parameters (σ, b, g_0, g_1) satisfying the following basic properties:

- $x \mapsto \sigma(x)$ is a continuous function on $\mathbb{R}$;
- $x \mapsto b(x)$ is a continuous function on $\mathbb{R}$ having the decomposition $b = b_1 - b_2$ with b_2 being continuous and non-decreasing;
- $(x, u) \mapsto g_0(x, u)$ and $(x, u) \mapsto g_1(x, u)$ are Borel functions on $\mathbb{R} \times U_0$ and $\mathbb{R} \times U_1$, respectively.

Let $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. Let $\{B(t) : t \geq 0\}$ be a standard $(\mathscr{G}_t)$-Brownian motion and let $\{p_0(t) : t \geq 0\}$ and $\{p_1(t) : t \geq 0\}$ be $(\mathscr{G}_t)$-Poisson point processes on U_0 and U_1 with characteristic measures $\mu_0(du)$ and $\mu_1(du)$, respectively. Suppose that $\{B(t)\}$, $\{p_0(t)\}$ and $\{p_1(t)\}$ are independent of each other. Let $N_0(ds, du)$ and $N_1(ds, du)$ be the Poisson random measures associated with $\{p_0(t)\}$ and $\{p_1(t)\}$, respectively. Let $\tilde{N}_0(ds, du)$ be the compensated measure of $N_0(ds, du)$. By a *solution to* the stochastic equation

$$\begin{aligned} x(t) \;=\; & x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(ds, du) \\ & + \int_0^t b(x(s-))ds + \int_0^t \int_{U_1} g_1(x(s-), u)N_1(ds, du), \end{aligned} \tag{2.1}$$

we mean a càdlàg and $(\mathscr{G}_t)$-adapted real process $\{x(t)\}$ that satisfies the equation almost surely for every $t \geq 0$. Since $x(s-) \neq x(s)$ for at most countably many $s \geq 0$, we can also use $x(s)$ instead of $x(s-)$ for the integrals with respect to $dB(s)$ and ds on the right hand side of (2.1). We say *pathwise uniqueness* holds for (2.1) if for any two solutions $\{x_1(t)\}$ and $\{x_2(t)\}$ of the equation satisfying $x_1(0) = x_2(0)$ we have $x_1(t) = x_2(t)$ almost surely for every $t \geq 0$. Let $(\mathscr{F}_t)_{t \geq 0}$ be the augmented natural filtration generated by $\{B(t)\}$, $\{p_0(t)\}$ and $\{p_1(t)\}$. A solution $\{x(t)\}$ of (2.1) is called a *strong solution* if it is adapted with respect to $(\mathscr{F}_t)$; see [12, p.163] or [16, p.76]. Let $U_2 \subset U_1$ be a set satisfying $\mu_1(U_1 \setminus U_2) < \infty$. We also consider the equation

$$\begin{aligned} x(t) \;=\; & x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(ds, du) \\ & + \int_0^t b(x(s-))ds + \int_0^t \int_{U_2} g_1(x(s-), u)N_1(ds, du). \end{aligned} \tag{2.2}$$

Proposition 2.1. *If (2.2) has a strong solution for every given $x(0)$, so does (2.1). If the pathwise uniqueness holds for (2.2), it also holds for (2.1).*

The above proposition can be proved similarly as Proposition 2.2 in [11]. Then all conditions in the paper only involve U_2 instead of U_1.

3 Pathwise uniqueness

In this section, we prove some results on the pathwise uniqueness for (2.2) under non-Lipschitz conditions. Suppose that (σ, b, g_0, g_1) are given as in the second section. Let us consider the following conditions on the modulus of continuity:

(3.a) for each integer $m \geq 1$ there is a non-decreasing and concave function $z \mapsto r_m(z)$ on $\mathbb{R}_+$ such that $\int_{0+} r_m(z)^{-1}\,dz = \infty$ and

$$|b_1(x) - b_1(y)| + \int_{U_2} |l_1(x, y, u)|\mu_1(du) \leq r_m(|x - y|), \qquad |x|, |y| \leq m,$$

where $l_1(x, y, u) = g_1(x, u) - g_1(y, u)$;

(3.b) the function $x \mapsto x + g_0(x,u)$ is non-decreasing for all $u \in U_0$ and for each integer $m \geq 1$ there is a constant $K_m \geq 0$ such that

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} l_0(x,y,u)^2 \mu_0(du) \leq K_m |x-y|, \qquad |x|,|y| \leq m,$$

where $l_0(x,y,u) = g_0(x,u) - g_0(y,u)$.

Let us define a sequence of functions $\{\phi_k\}$ as follows. For each integer $k \geq 0$ define $a_k = \exp\{-k(k+1)/2\}$. Then $a_k \to 0$ decreasingly as $k \to \infty$ and

$$\int_{a_k}^{a_{k-1}} z^{-1} dz = k, \qquad k \geq 1.$$

Let $x \mapsto \psi_k(x)$ be a non-negative continuous function supported by (a_k, a_{k-1}) so that

$$\int_{a_k}^{a_{k-1}} \psi_k(x) dx = 1 \quad \text{and} \quad \psi_k(x) \leq 2(kx)^{-1} \tag{3.1}$$

for every $a_k < x < a_{k-1}$. For $z \in \mathbb{R}$ let

$$\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x) dx. \tag{3.2}$$

It is easy to see that the sequence $\{\phi_k\}$ has the following properties:

(i) $\phi_k(z) \mapsto |z|$ non-decreasingly as $k \to \infty$;
(ii) $0 \leq \phi_k'(z) \leq 1$ for $z \geq 0$ and $-1 \leq \phi_k'(z) \leq 0$ for $z \leq 0$;
(iii) $0 \leq |z|\phi_k''(z) = |z|\psi_k(|z|) \leq 2k^{-1}$ for $z \in \mathbb{R}$.

By Taylor's expansion, for any $h, \zeta \in \mathbb{R}$ we have

$$D_h \phi_k(\zeta) = h^2 \int_0^1 \psi_k(|\zeta + th|)(1-t) dt \leq \frac{2}{k} h^2 \int_0^1 \frac{(1-t)}{|\zeta + th|} dt. \tag{3.3}$$

Lemma 3.1. *Suppose that $x \mapsto x + g_0(x,u)$ is non-decreasing for $u \in U_0$. Then, for any $x \neq y \in \mathbb{R}$,*

$$D_{l_0(x,y,u)} \phi_k(x-y) \leq \frac{2}{k} \int_0^1 \frac{l_0(x,y,u)^2 (1-t)}{|x-y+tl_0(x,y,u)|} dt \leq \frac{2 l_0(x,y,u)^2}{k|x-y|}. \tag{3.4}$$

Proof. The first inequality follows from (3.3). Since $x \mapsto x + g_0(x,u)$ is non-decreasing, for $x > y \in \mathbb{R}$ we have $x - y + l_0(x,y,u) \geq 0$, and hence $x - y + tl_0(x,y,u) \geq 0$ for $0 \leq t \leq 1$. It is elementary to see

$$\begin{aligned}
&\int_0^1 \frac{l_0(x,y,u)^2 (1-t)}{x-y+tl_0(x,y,u)} dt \\
&\quad = l_0(x,y,u) \int_0^1 \Big[\frac{x-y+l_0(x,y,u)}{x-y+tl_0(x,y,u)} - 1\Big] dt \\
&\quad = \big[x-y+l_0(x,y,u)\big] \log\Big(1 + \frac{l_0(x,y,u)}{x-y}\Big) - l_0(x,y,u) \\
&\quad \leq [x-y+l_0(x,y,u)] \frac{l_0(x,y,u)}{x-y} - l_0(x,y,u) \\
&\quad = \frac{l_0(x,y,u)^2}{x-y}.
\end{aligned}$$

Then the second inequality in (3.4) follows by symmetry. □

Theorem 3.2. *Suppose that conditions (3.a,b) are satisfied. Then the pathwise uniqueness for (2.2) holds.*

Proof. proof By condition (3.b) and Lemma 3.1, for $x \neq y \in \mathbb{R}$ satisfying $|x|, |y| \leq m$ we have

$$\phi_k''(x-y)[\sigma(x)-\sigma(y)]^2 \leq K_m \phi_k''(x-y)|x-y| \leq \frac{2K_m}{k}$$

and

$$\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) \leq \int_{U_0} \frac{2l_0(x,y,u)^2}{k|x-y|}\mu_0(du) \leq \frac{2K_m}{k}.$$

The right-hand sides of both inequalities tend to zero uniformly on $|x|, |y| \leq m$ as $k \to \infty$. Then the pathwise uniqueness for (2.2) follows by a simple modification of Proposition 3.1 in [15]; see also Theorem 3.1 in [11]. □

We next introduce some condition that is particularly useful in applications to stochastic equations driven by Lévy processes. The condition is given as follows:

(3.c) there is a constant $0 \leq c \leq 1$ such that $x \mapsto cx + g_0(x,u)$ is non-decreasing for all $u \in U_0$ and for each integer $m \geq 1$ there are constants $K_m \geq 0$ and $p_m > 0$ such that

$$|\sigma(x)-\sigma(y)|^2 \leq K_m|x-y| \quad \text{and} \quad |l_0(x,y,u)| \leq |x-y|^{p_m} f_m(u)$$

for $|x|, |y| \leq m$, where $l_0(x,y,u) = g_0(x,u) - g_0(y,u)$ and $u \mapsto f_m(u)$ is a strictly positive function on U_0 satisfying

$$\int_{U_0} [f_m(u) \wedge f_m(u)^2]\mu_0(du) < \infty.$$

For each $m \geq 1$ and the function f_m specified in (3.c) we define the constant

$$\alpha_m = \inf\Big\{\beta > 1 : \lim_{x\to 0+} x^{\beta-1}\int_{U_0} f_m(u)1_{\{f_m(u)\geq x\}}\mu_0(du) = 0\Big\}. \tag{3.5}$$

By Lemma 2.1 in [15] we have $1 \leq \alpha_m \leq 2$.

Lemma 3.3. *Suppose that condition (3.c) holds. Then for any $h \geq 0$ and $|x|, |y| \leq m$ we have*

$$\begin{aligned}\int_{U_0} & D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) \\ &\leq \frac{2}{k}|x-y|^{2p_m-1}1_{\{(1-c)|x-y|<a_{k-1}\}}\int_{U_0} f_m(u)^2 1_{\{f_m(u)\leq h\}}\mu_0(du) \\ &\quad + 2|x-y|^{p_m}1_{\{(1-c)|x-y|<a_{k-1}\}}\int_{U_0} f_m(u)1_{\{f_m(u)>h\}}\mu_0(du).\end{aligned}$$

Proof. We first consider $x > y \in \mathbb{R}$. Since $x \mapsto cx + g_0(x,u)$ is non-decreasing, we have $c(x-y) + l_0(x,y,u) \geq 0$, and hence $c(x-y) + tl_0(x,y,u) \geq 0$ for $0 \leq t \leq 1$. It follows that $x - y + tl_0(x,y,u) \geq (1-c)(x-y)$ for $0 \leq t \leq 1$. Then $(1-c)(x-y) \geq a_{k-1}$ implies $x - y + tl_0(x,y,u) \geq a_{k-1}$ for $0 \leq t \leq 1$. In view of the equality in (3.3) we have

$$D_{l_0(x,y,u)}\phi_k(x-y) = 0 \quad \text{if} \quad (1-c)(x-y) \geq a_{k-1}.$$

By the symmetry of ϕ_k it is follows that, for arbitrary $x, y \in \mathbb{R}$,

$$D_{l_0(x,y,u)}\phi_k(x-y) = 0 \quad \text{if} \quad (1-c)|x-y| \geq a_{k-1}. \tag{3.6}$$

Then we can use condition (3.c) to get

$$\begin{aligned} D_{l_0(x,y,u)}\phi_k(x-y) &\leq 2|l_0(x,y,u)|1_{\{(1-c)(x-y)<a_{k-1}\}} \\ &\leq 2|x-y|^{p_m} f_m(u) 1_{\{(1-c)|x-y|<a_{k-1}\}}. \end{aligned}$$

Similarly, by (3.4) we have

$$\begin{aligned} D_{l_0(x,y,u)}\phi_k(x-y) &\leq \frac{2l_0(x,y,u)^2}{k|x-y|} 1_{\{(1-c)|x-y|<a_{k-1}\}} \\ &\leq \frac{2}{k}|x-y|^{2p_m-1} f_m(u)^2 1_{\{(1-c)|x-y|<a_{k-1}\}}. \end{aligned}$$

Those give the desired result. □

Theorem 3.4. *Suppose that conditions (3.a,c) hold with: (i) $c = 1, \alpha_m = 2, p_m = 1/2$; or (ii) $c < 1, \alpha_m < 2, 1 - 1/\alpha_m < p_m \leq 1/2$. Then the pathwise uniqueness holds for (2.2).*

Proof. Let us consider the case (i). By Lemma 3.3, for any $h \geq 1$ and $|x|, |y| \leq m$ we have

$$\begin{aligned} &\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) \\ &\quad \leq \frac{2}{k}\int_{U_0} f_m(u)^2 1_{\{f_m(u)\leq h\}}\mu_0(du) + 2\sqrt{2m}\int_{U_0} f_m(u)1_{\{f_m(u)>h\}}\mu_0(du) \\ &\quad \leq \frac{2h}{k}\int_{U_0} [f_m(u) \wedge f_m(u)^2]\mu_0(du) + 2\sqrt{2m}\int_{U_0} f_m(u)1_{\{f_m(u)>h\}}\mu_0(du). \end{aligned}$$

By letting $k \to \infty$ and $h \to \infty$ one can see

$$\lim_{k\to\infty}\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) = 0.$$

Then the pathwise uniqueness for (2.2) follows by a modification of Proposition 3.1 in [15]; see also Theorem 3.1 in [11]. The case (ii) follows as in the proof of Proposition 3.3 in [15]. □

We remark that our conditions (3.b) and (3.c) improve similar conditions in [11, 15], where it was assumed that $x \mapsto g_0(x,u)$ is non-decreasing for all $u \in U_0$. The following example shows that the global monotonicity of the functions $x \mapsto x + g_0(x,u)$ and $x \mapsto cx + g_0(x,u)$ in conditions (3.b) and (3.c) are necessary to assure the pathwise uniqueness.

Example 3.5. *Let us consider the equation (1.1). Let $0 < \alpha < 1$ be a constant and define the bounded positive α-Hölder continuous function*

$$\phi(x) = (1-\alpha)^{-1}(|x|^\alpha \wedge |x-1|^\alpha)1_{\{0\leq x\leq 1\}}, \qquad x \in \mathbb{R}. \tag{3.7}$$

Clearly, this function is nondecreasing in the interval $(-\infty, 1/2)$ and nonincreasing in the interval $(1/2, \infty)$. Let $y_1(t) = 1$ for $t \geq 0$ and let

$$y_2(t) = \begin{cases} 1 - t^{1/(1-\alpha)} & \text{for } 0 \leq t < 2^{\alpha-1}, \\ (2^\alpha - t)^{1/(1-\alpha)} & \text{for } 2^{\alpha-1} \leq t < 2^\alpha, \\ 0 & \text{for } t \geq 2^\alpha. \end{cases}$$

It is elementary to show that both $\{y_1(t)\}$ and $\{y_2(t)\}$ are solutions of (1.2) satisfying $y_1(0) = y_2(0) = 1$. Based on $\{y_1(t)\}$ and $\{y_2(t)\}$, it is easy to construct infinitely many solutions of (1.2) satisfying $y(0) = 1$. Therefore (1.1) has infinitely many solutions $\{x(t)\}$ satisfying $x(0) = 1$.

4 Weak solutions

In this section, we prove the existence of the weak solution to (2.2) by considering the corresponding martingale problem. Let (σ, b, g_0, g_1) be given as in the second section. Let $C^2(\mathbb{R})$ be the set of twice continuously differentiable functions on $\mathbb{R}$ which together with their derivatives up to the second order are bounded. For $x \in \mathbb{R}$ and $f \in C^2(\mathbb{R})$ we define

$$\begin{aligned} Af(x) &= \frac{1}{2}\sigma(x)^2 f''(x) + \int_{U_0} D_{g_0(x,u)} f(x)\mu_0(du) \\ &\quad + b(x)f'(x) + \int_{U_2} \Delta_{g_1(x,u)} f(x)\mu_1(du). \end{aligned} \tag{4.1}$$

To simplify the statements we introduce the following condition:

(4.a) there is a constant $K \geq 0$ such that

$$\begin{aligned} &|b(x)| + \sigma(x)^2 + \int_{U_0} g_0(x,u)^2 \mu_0(du) \\ &\quad + \int_{U_2} \big[|g_1(x,u)| \vee g_1(x,u)^2\big]\mu_1(du) \leq K, \qquad x \in \mathbb{R}. \end{aligned}$$

Proposition 4.1. *Suppose that condition (4.a) holds. Then a càdlàg process $\{x(t) : t \geq 0\}$ is a weak solution to (2.2) if and only if for every $f \in C^2(\mathbb{R})$,*

$$f(x(t)) - f(x(0)) - \int_0^t Af(x(s))ds, \qquad t \geq 0 \tag{4.2}$$

is a locally bounded martingale.

Proof. Without loss of generality, we assume $x(0) \in \mathbb{R}$ is deterministic. If $\{x(t) : t \geq 0\}$ is a solution to (2.2), by Itô's formula it is easy to see that (4.2) is a locally bounded martingale. Conversely, suppose that (4.2) is a martingale for every $f \in C^2(\mathbb{R}_+)$. By a standard stopping time argument, we have

$$x(t) = x(0) + \int_0^t b(x(s-))ds + \int_0^t ds \int_{U_2} g_1(x(s-),u)\mu_1(du) + M(t)$$

for a square-integrable martingale $\{M(t) : t \geq 0\}$. As in the proof of Proposition 4.2 in [11], we obtain the equation (2.2) on an extension of the probability space by applying martingale representation theorems; see, e.g., [12, p.90 and p.93]. □

Now suppose that conditions (3.a,b) and (4.a) are satisfied. For simplicity, in the sequel we assume the initial value $x(0) \in \mathbb{R}$ is deterministic. Let $\{V_n\}$ be a non-decreasing sequence of Borel subsets of U_0 so that $\cup_{n=1}^\infty V_n = U_0$ and $\mu_0(V_n) < \infty$ for every $n \geq 1$. It is easy to see that

$$x \mapsto \int_{V_n} g_0(x,u)\mu_0(du)$$

is a bounded continuous function on $\mathbb{R}$. For $n \geq 1$ and $x \in \mathbb{R}$ let

$$\chi_n(x) = \begin{cases} n, & \text{if } x > n, \\ x, & \text{if } |x| \leq n, \\ -n, & \text{if } x < -n. \end{cases} \tag{4.3}$$

By the result on continuous-type stochastic equations, there is a weak solution to

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t b(x(s))ds \\ &\quad - \int_0^t ds \int_{V_n} g_0(\chi_n(x(s)), u)\mu_0(du); \end{aligned} \tag{4.4}$$

see, e.g., [12, p.169]. We can rewrite (4.4) into

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t [b_1(x(s)) + \mu_0(V_n)\chi_n(x(s))]ds \\ &\quad - \int_0^t \Big\{ b_2(x(s)) + \int_{V_n} [\chi_n(x(s)) + g_0(\chi_n(x(s)), u)]\mu_0(du) \Big\} ds, \end{aligned} \tag{4.5}$$

where

$$x \mapsto b_2(x) + \int_{V_n} [\chi_n(x) + g_0(\chi_n(x), u)]\mu_0(du)$$

is a bounded continuous non-decreasing function on $\mathbb{R}$. By Theorem 3.2 the pathwise uniqueness holds for (4.5), so it also holds for (4.4). Then there is a pathwise unique strong solution to (4.4). Let $\{W_n\}$ be a non-decreasing sequence of Borel subsets of U_2 so that $\cup_{n=1}^\infty W_n = U_2$ and $\mu_1(W_n) < \infty$ for every $n \geq 1$. Following the proof of Proposition 2.2 in [11] one can see for every integer $n \geq 1$ there is a strong solution to

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t \int_{V_n} g_0(\chi_n(x(s-)), u)\tilde{N}_0(ds, du) \\ &\quad + \int_0^t b(x(s))ds + \int_0^t \int_{W_n} g_1(x(s-), u)N_1(ds, du). \end{aligned} \tag{4.6}$$

By Theorem 3.2 the pathwise uniqueness holds for (4.6), so the equation has a unique strong solution; see, e.g., [16, p.104]. Let us denote the strong solution to (4.6) by $\{x_n(t) : t \geq 0\}$. By Proposition 4.1, for every $f \in C^2(\mathbb{R})$,

$$f(x_n(t)) = f(x_n(0)) + \int_0^t A_n f(x_n(s))ds + \text{mart.}, \tag{4.7}$$

where

$$\begin{aligned} A_n f(x) &= \frac{1}{2}\sigma(x)^2 f''(x) + \int_{V_n} D_{g_0(\chi_n(x),u)} f(x)\mu_0(du) \\ &\quad + b(x)f'(x) + \int_{W_n} \Delta_{g_1(x,u)} f(x)\mu_1(du). \end{aligned}$$

Lemma 4.2. *Suppose that conditions (4.a) and (3.a,b) are satisfied. If $x_n \to x$ as $n \to \infty$, then $A_n f(x_n) \to Af(x)$ as $n \to \infty$.*

Proof. Let $M \geq 0$ be a constant so that $|x|, |x_n| \leq M$ for all $n \geq 1$. Under the conditions, it is easy to see that

$$x \mapsto \int_{V_k^c} g_0(x, u)^2 \mu_0(du) + \int_{W_k^c} |g_1(x, u)|\mu_1(du)$$

is a continuous function for each $k \geq 1$. By Dini's theorem we have, as $k \to \infty$,

$$\varepsilon_k := \sup_{|x| \leq M} \Big[\int_{V_k^c} g_0(x, u)^2 \mu_0(du) + \int_{W_k^c} |g_1(x, u)|\mu_1(du) \Big] \to 0.$$

Let $y_n = \chi_n(x_n)$. For $n \geq k$ we have

$$\begin{aligned}
\Big| \int_{V_n} & D_{g_0(y_n,u)} f(x_n)\mu_0(du) - \int_{U_0} D_{g_0(x,u)} f(x)\mu_0(du)\Big| \\
& \leq \int_{V_k} \Big| D_{g_0(y_n,u)} f(x_n) - D_{g_0(x,u)} f(x)\Big| \mu_0(du) + \|f''\|\varepsilon_k \\
& \leq \int_{V_k} \Big| f(x_n + g_0(y_n,u)) - f(x + g_0(x,u))\Big| \mu_0(du) \\
& \quad + \int_{V_k} |f(x_n) - f(x)|\mu_0(du) + \|f''\|\varepsilon_k \\
& \quad + \int_{V_k} \Big| f'(x_n) g_0(y_n,u) - f'(x) g_0(x,u)\Big| \mu_0(du) \\
& \leq \|f'\| \int_{V_k} \Big| (x_n + g_0(y_n,u)) - (x + g_0(x,u))\Big| \mu_0(du) \\
& \quad + \int_{V_k} |f(x_n) - f(x)|\mu_0(du) + \|f''\|\varepsilon_k \\
& \quad + \|f'\| \int_{V_k} |g_0(y_n,u) - g_0(x,u)|\mu_0(du) \\
& \quad + \int_{V_k} |f'(x_n) - f'(x)||g_0(x,u)|\mu_0(du) \\
& \leq 2\|f'\| \int_{V_k} |g_0(y_n,u) - g_0(x,u)|\mu_0(du) \\
& \quad + \Big[\|f'\||x_n - x| + |f(x_n) - f(x)|\Big]\mu_0(V_k) + \|f''\|\varepsilon_k \\
& \quad + |f'(x_n) - f'(x)|\mu_0(V_k)^{1/2}\Big[\int_{U_0} g_0(x,u)^2\mu_0(du)\Big]^{1/2},
\end{aligned} \tag{4.8}$$

where

$$\int_{V_k} |g_0(y_n,u) - g_0(x,u)|\mu_0(du) \leq \Big[\mu_0(V_k)\int_{U_0} |g_0(y_n,u) - g_0(x,u)|^2\mu_0(du)\Big]^{1/2}. \tag{4.9}$$

By letting $n \to \infty$ and $k \to \infty$ in (4.8) and using condition (3.b) one can see that

$$\lim_{n\to\infty} \int_{V_n} D_{g_0(y_n,u)} f(x_n)\mu_0(du) = \int_{U_0} D_{g_0(x,u)} f(x)\mu_0(du). \tag{4.10}$$

Similarly, for $n \geq k$ we have

$$\begin{aligned}
\Big| \int_{W_n} & \Delta_{g_1(x_n,u)} f(x_n)\mu_0(du) - \int_{U_2} \Delta_{g_1(x,u)} f(x)\mu_1(du)\Big| \\
& \leq \|f'\| \int_{U_2} |g_1(x_n,u) - g_1(x,u)|\mu_1(du) + 2\|f'\|\varepsilon_k \\
& \quad + \Big[\|f'\||x_n - x| + |f(x_n) - f(x)|\Big]\mu_1(W_k).
\end{aligned}$$

Then letting $n \to \infty$ and $k \to \infty$ and using condition (3.a) one sees

$$\lim_{n\to\infty} \int_{W_n} \Delta_{g_1(x_n,u)} f(x_n)\mu_0(du) = \int_{U_2} \Delta_{g_1(x,u)} f(x)\mu_1(du). \tag{4.11}$$

In view of (4.10) and (4.11), it is obvious that $A_n f(x_n) \to Af(x)$ as $n \to \infty$. □

Proposition 4.3. *Suppose that conditions (4.a) and (3.a,b) are satisfied. Then there exists a weak solution to (2.2).*

Proof. Following the proof of Lemma 4.3 in [11] it is easy to show that $\{x_n(t) : t \geq 0\}$ is a tight sequence in the Skorokhod space $D([0,\infty), \mathbb{R})$. Then there is a subsequence $\{x_{n_k}(t) : t \geq 0\}$ that converges to some process $\{x(t) : t \geq 0\}$ in distribution on $D([0,\infty), \mathbb{R})$. By the Skorokhod representation theorem, we may assume those processes are defined on the same probability space and $\{x_{n_k}(t) : t \geq 0\}$ converges to $\{x(t) : t \geq 0\}$ almost surely in $D([0,\infty), \mathbb{R})$. Let $D(x) := \{t > 0 : \mathbf{P}\{x(t-) = x(t)\} = 1\}$. Then the set $[0,\infty) \setminus D(x)$ is at most countable; see, e.g., [9, p.131]. It follows that $\lim_{k\to\infty} x_{n_k}(t) = x(t)$ almost surely for every $t \in D(x)$; see, e.g., [9, p.118]. From (4.7) and Lemma 4.2 it follows that (4.2) is a locally bounded martingale. Then we get the result by Proposition 4.1. □

Proposition 4.4. *Suppose that conditions (4.a) and (3.a,c) hold with: (i)* $c = 1, \alpha_m = 2, p_m = 1/2$*; or (ii)* $c < 1, \alpha_m < 2, 1-1/\alpha_m < p_m \leq 1/2$*. Then there exists a weak solution to (2.2).*

Proof. In condition (3.c), we can obviously assume $f_m \leq f_{m+1}$ for all $m \geq 1$. Let $V_n = \{u \in U_0 : f_n(u) \geq 1/n\}$. Then the conclusion of Lemma 4.2 remains true. The only necessary modification of the proof is that now we consider $n \geq k \geq M$. Then $|x|, |x_n| \leq M$ implies $|x|, |y_n| \leq k$, so we can replace (4.9) by

$$\begin{aligned}\int_{V_k} |g_0(y_n,u) - g_0(x,u)|\mu_0(du) &\leq |y_n - x|^{p_k} \int_{V_k} f_k(u)\mu_0(du) \\ &\leq k|y_n - x|^{p_k} \int_{U_0} [f_k(u) \wedge f_k(u)^2]\mu_0(du).\end{aligned}$$

Then the result follows as in the proof of Proposition 4.3. □

5 Strong solutions

In this section, we prove the existence of the strong solution to (2.1). Let (σ, b, g_0, g_1) be given as in the second section. We assume the following linear growth condition on the coefficients:

(5.a) there is a constant $K \geq 0$ such that

$$\begin{aligned}&\sigma(x)^2 + \int_{U_0} g_0(x,u)^2\mu_0(du) + \int_{U_2} g_1(x,u)^2\mu_1(du) \\ &\quad + b(x)^2 + \left(\int_{U_2} |g_1(x,u)|\mu_1(du)\right)^2 \leq K(1+x^2), \quad x \in \mathbb{R}.\end{aligned}$$

Theorem 5.1. *Suppose that conditions (5.a) and (3.a,b) are satisfied. Then there is a pathwise unique strong solution to (2.1).*

Proof. By Proposition 4.3 for each integer $m \geq 1$ there is a weak solution to

$$\begin{aligned}x(t) &= x(0) + \int_0^t \sigma(\chi_m(x(s)))dB(s) + \int_0^t \int_{U_0} g_0(\chi_m(x(s-)),u)\tilde{N}_0(ds,du) \\ &\quad + \int_0^t b(\chi_m(x(s)))ds + \int_0^t \int_{U_2} \chi_m \circ g_1(\chi_m(x(s-)),u)N_1(ds,du). \qquad (5.1)\end{aligned}$$

The pathwise uniqueness for the equation follows from Theorem 3.2. Then there is a unique strong solution $\{x_m(t) : t \geq 0\}$ to (5.1); see, e.g., [16, p.104]. Let $\tau_m = \inf\{t \geq 0 : |x_m(t)| \geq m\}$. As in the proof of Proposition 3.4 in [15] it is easy to get

$$\mathbf{E}\Big[1 + \sup_{0\leq s\leq t} x_m(s\wedge\tau_m)^2\Big] \leq (1 + 6\mathbf{E}[x(0)^2])\exp\{6K(4+t)t\}.$$

Then $\tau_m \to \infty$ as $m \to \infty$. Following the proof of Proposition 2.2 in [11] one can show there is a pathwise unique strong solution to (2.2). Then the result follows from Proposition 2.1. □

Theorem 5.2. *Let α_m be the number defined in (3.5). Suppose that conditions (5.a) and (3.a,c) hold with: (i) $c = 1, \alpha_m = 2, p_m = 1/2$; or (ii) $c < 1, \alpha_m < 2, 1 - 1/\alpha_m < p_m \leq 1/2$. Then there exists a pathwise unique strong solution to (2.1).*

Proof. Based on Proposition 4.4, this follows similarly as Theorem 5.1. □

6 Non-negative solutions

In this section, we derive some results on non-negative solutions of the stochastic equation (2.1). Let (σ, b, g_0, g_1) be given as in the second section. In addition, we assume:

- $b(x) \geq 0$ and $\sigma(x) = 0$ for $x \leq 0$;
- for every $u \in U_0$ we have $x + g_0(x, u) \geq 0$ if $x > 0$ and $g_0(x, u) = 0$ if $x \leq 0$;
- $x + g_1(x, u) \geq 0$ for $u \in U_1$ and $x \in \mathbb{R}$.

Then, by Proposition 2.1 in [11], any solution of (5.1) is non-negative. By considering non-negative solutions, we can weaken the linear growth condition of the parameters into the following:

(6.a) there is a constant $K \geq 0$ such that

$$b(x) + \int_{U_2} |g_1(x, u)|\mu_1(du) \leq K(1 + x), \qquad x \geq 0;$$

(6.b) there is a non-decreasing function $x \mapsto L(x)$ on $\mathbb{R}_+$ so that

$$\sigma(x)^2 + \int_{U_0} g_0(x, u)^2 \mu_0(du) \leq L(x), \qquad x \geq 0.$$

Theorem 6.1. *Suppose that conditions (6.a) and (3.a,b) are satisfied. Then for any $x(0) \in \mathbb{R}_+$ there is a pathwise unique non-negative strong solution to (2.1).*

Proof. By conditions (6.a) and (3.b) one can show that the parameters of (5.1) satisfy condition (4.a). Then for each integer $m \geq 1$ there is a non-negative weak solution to (5.1) by Proposition 4.3. The pathwise uniqueness for (5.1) holds by Theorem 3.2, so there is a unique non-negative strong solution to (5.1). Then the result follows as in the proof of Proposition 2.2 in [11]. □

Corollary 6.2. (Dawson and Li [7]) *Given $0 \leq x(0) \leq 1$ there is a pathwise unique strong solution $\{x(t) : t \geq 0\}$ to (1.3) such that $0 \leq x(t) \leq 1$ for all $t \geq 0$.*

Proof. Observe that $q(x, r) = 0$ for $x \leq 0$ and $x \geq 1$. For any $0 \leq x, z, r \leq 1$ we have

$$0 \leq x + zq(x, r) = z1_{\{r \leq x\}} + (1 - z)x \leq 1.$$

Then $0 \leq x(0) \leq 1$ implies $0 \leq x(t) \leq 1$ for all $t \geq 0$. The function $x \mapsto x + q(x, r)$ is clearly non-decreasing and for any $0 \leq x, y \leq 1$,

$$\begin{aligned} \int_0^1 \nu(dz) \int_0^1 z^2 |q(x, r) - q(y, r)|^2 dr &= [|x - y| - (x - y)^2] \int_0^1 z^2 \nu(dz) \\ &\leq |x - y| \int_0^1 z^2 \nu(dz). \end{aligned}$$

Then the result follows by Theorem 6.1. □

Corollary 6.3. (Döring and Barczy [8]) *Given* $x(0) \geq 0$ *there is a unique non-negative strong solution to (1.4).*

Proof. It is easy to see that $x \mapsto x + g(x,u,r)$ is a non-decreasing function. For any $x, y \geq 0$ we have

$$\begin{aligned}\int_0^\infty dr \int_0^\infty (g(x,u,r) - g(y,u,r))^2 \mu_0(du) \\ &= \int_0^\infty (1-e^{-u})^2 \mu_0(du)\big[x + y - 2(x^{-1} \wedge y^{-1})xy\big] \\ &= \int_0^\infty (1-e^{-u})^2 \mu_0(du)|x-y|.\end{aligned}$$

By Theorem 6.1 there is a unique non-negative strong solution to the equation. □

Theorem 6.4. *Suppose that conditions (6.a,b) and (3.a,c) hold with: (i)* $c = 1, \alpha_m = 2, p_m = 1/2$*; or (ii)* $c < 1, \alpha_m < 2, 1 - 1/\alpha_m < p_m \leq 1/2$*. Then there exists a pathwise unique non-negative strong solution to (2.1).*

Proof. This follows similarly as Theorem 6.1. Here condition (6.b) is used to guarantee condition (4.a) is satisfied by the parameters of (5.1). □

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