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# Exponential–uniform identities related to records

Alexander Gnedin* Alexander Marynych†

**Abstract**

We consider a rectangular grid induced by the south-west records from the planar Poisson point process in $\mathbb{R}_+^2$. A random symmetry property of the matrix whose entries are the areas of tiles of the grid implies cute multivariate distributional identities for certain rational functions of independent exponential and uniform random variables.

**Keywords:** records; planar Poisson process; distributional identities.
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## 1 Introduction

Let $E_1, E_2, \dots$ and $U_1, U_2, \dots$ be two independent sequences of independent rate-one exponential and $[0,1]$-uniform random variables, respectively. A prototype of the distributional identities appearing in this note is the identity

$$\left(\frac{E_1}{U_1} + \frac{E_2}{U_1 U_2} + \ldots + \frac{E_n}{U_1 \cdots U_n}\right)(1 - U_1 \cdots U_{n+1}) \stackrel{d}{=} \left(E_1 + \frac{E_2}{U_1} + \ldots + \frac{E_{n+1}}{U_1 \cdots U_n}\right)(1 - U_1 \cdots U_n)\,, \tag{1.1}$$

which was used in [4] to explain coincidence of the values in two quite different problems of optimal stopping. Some probabilities related to the simplest instance of (1.1),

$$\frac{E_1}{U_1}(1 - U_1 U_2) \stackrel{d}{=} \left(E_1 + \frac{E_2}{U_1}\right)(1 - U_1), \tag{1.2}$$

had been evaluated in [7].

We will show that (1.1) along with more general identities for matrix functions in the exponential and uniform variables follow from symmetry properties of the set of *records* (also known as Pareto-extremal points [2]) from the planar Poisson process in the positive quadrant. This continues the line of [5], where it was argued that the planar Poisson process is a natural framework for two gems of combinatorial probability: Ignatov's theorem [6] and the Arratia-Barbour-Tavaré lemma on the scale-invariant Poisson processes on $\mathbb{R}_+$ [1].

*Queen Mary University of London, United Kingdom. E-mail: a.gnedin@qmul.ac.uk
†Eindhoven University of Technology, The Netherlands. E-mail: O.Marynych@tue.nl

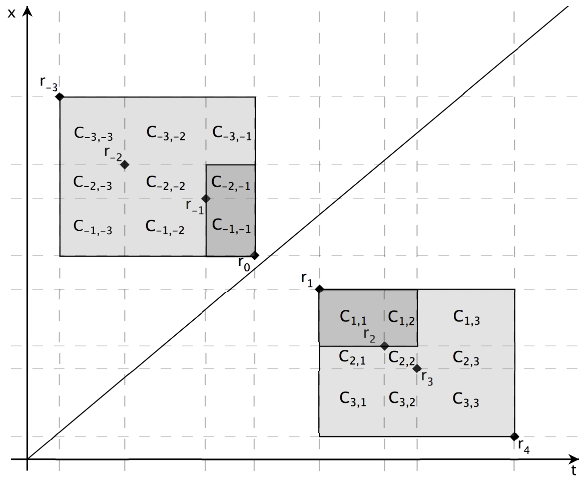


Figure 1: The rectangular tiling and areas with the same distribution.

We shall evaluate the areas of tiles for a rectangular grid induced by the set of records. The identities obtained in this way are genuinely multivariate, albeit they stem from the arrays with identical marginal distributions. In particular, (1.2) appears by a row summation in

$$\begin{pmatrix} (1-U_1)E_1 & (1-U_1)\frac{E_2}{U_1} \\ \\ U_1(1-U_2)E_1 & (1-U_2)E_2 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} (1-U_2)E_2 & (1-U_1)\frac{E_2}{U_1} \\ \\ U_1(1-U_2)E_1 & (1-U_1)E_1 \end{pmatrix}. \tag{1.3}$$

## 2 A random tiling induced by records

We use the self-explaining notations $\nearrow, \searrow, \swarrow, \nwarrow$ for four coordinate-wise partial orders on the positive quadrant. For instance, relations $a \nearrow b$ and $b \swarrow a$ for $a, b \in \mathbb{R}_+^2$ both mean that $b$ is located strictly north-east of $a$.

Let $\mathcal{P}$ be the planar Poisson point process with unit intensity in $\mathbb{R}_+^2$. It will be convenient to understand $\mathcal{P}$ as a random set, rather than counting measure. The event $(t, x) \in \mathcal{P}$ is interpreted as the value $x$ observed at time $t$. Note that with probability one no two atoms of $\mathcal{P}$ lie on the same vertical or horizontal line. An atom $r \in \mathcal{P}$ is said to be a (lower) *record* if there is no earlier observation with a smaller value, that is $a \nearrow r$ for no $a \in \mathcal{P}$. The set of records, denoted henceforth $\mathcal{R}$, is a point process with the intensity function $e^{-tx}$. The collection of records ordered by increase of the time component is a two-sided infinite $\searrow$-chain

$$\ldots \searrow r_{-2} \searrow r_{-1} \searrow r_0 \searrow r_1 \searrow r_2 \searrow \ldots,$$

which we label by nonzero integers in such a way that the records $r_0$ and $r_1$ are separated by the bisectrix $t = x$.

Drawing vertical lines at $t$-locations of records (record times), and drawing horizontal lines at their $x$-locations (record values) divides the positive quadrant into rectangular tiles. Let $C_{ij}$ be the area of the tile whose north-west corner is the intersection point of the horizontal line through $r_i$ and the vertical line through $r_j$ (see Figure 1). In particular, $C_{ii}$ for $i \in \mathbb{Z}$ is the area of a tile spanned on records $r_i, r_{i+1}$.

Given a record at location $(t, x)$, the next record is just the next point of $\mathcal{P}$ south-west of $(t, x)$, hence distributed like $(t+E/x, xU)$, as is easily seen from the independence and homogeneity properties of $\mathcal{P}$. The sequence $r_1, r_2, \dots$ is Markovian with just described transitions, whence

$$(C_{ij}\,;\, i,j = 1,2,\dots) \quad \stackrel{d}{=} \quad \left(U_1 \cdots U_{i-1}(1-U_i)\,\frac{E_j}{U_1 \cdots U_{j-1}}\;;\; i,j = 1,2,\dots\right). \qquad (2.1)$$

Note that the left-hand side of (2.1) is independent of $r_1$. Since the law of $\mathcal{R}$ is not changed by reflection about the bisectrix, we also have

$$(C_{i,j}\,;\, i,j = 1,2,\dots) \stackrel{d}{=} (C_{-j-1,-i-1}\,;\, i,j = 1,2,\dots). \qquad (2.2)$$

As an illustration, the areas of two rectangles (Figure 1) spanned on $r_1, r_4$ and $r_{-3}, r_0$, respectively, have the same distributions.

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ let $M_{k,n} = (C_{k+i-1,k+j-1},\;\; i,j = 1,\dots,n)$ be the $n \times n$ matrix associated with records $r_k, r_{k+1}, \dots, r_{k+n}$. Obviously from the above, $M_{k,n}$ is independent of $r_k$ and satisfies $M_{k,n} \stackrel{d}{=} M_{1,n}$ for $k = 1,2,\dots$. For $k \le 0$, $M_{k,n}$ is not independent of $r_k$, since the transition probability from $r_k = (t,x)$ to $r_{k+1}$ accounts for the condition that $-k$ records must lie south-east of $(t,x)$ above the bisectrix. Also, $M_{k,n} \stackrel{d}{=} M_{1,n}$ fails for $-n < k \le 0$: for instance, $C_{0,0}$ and $C_{1,1}$ have different distributions. Nevertheless, we will show that $M_{k,n} \stackrel{d}{=} M_{1,n}$ holds for $k \le -n$, which by virtue of (2.2) is equivalent to the following random symmetry property of $M_{1,n}$.

Let $M^\star_{1,n}$ be the matrix obtained by reflecting $M_{1,n}$ about the antidiagonal, that is by exchanging each entry $(i,j)$ with entry $(n-j+1, n-i+1)$.

**Proposition 2.1.** *For* $n = 1,2,\dots$

$$M^\star_{1,n} \stackrel{d}{=} M_{1,n}. \qquad (2.3)$$

Identity (1.3) is the $n = 2$ instance of (2.3). Identity (1.1) appears by calculating the sum of all entries of $M_{1,n+1}$ except the entries in the $(n+1)$st row. Further identities can be derived by applying functions, e.g., taking the product of matrix elements in the first row of $M_{1,n}$:

$$\frac{E_1 \cdots E_n (1-U_1)^n}{U_1^{n-1} U_2^{n-2} \cdots U_{n-1}} \stackrel{d}{=} \frac{E_n^n (1-U_1) \cdots (1-U_n)}{U_1 U_2^2 \cdots U_{n-1}^{n-1}}\,.$$

## 3 Records in a finite box

To prove (2.3) we consider records in finite rectangles. Let $A \subset \mathbb{R}^2_+$ be a finite open rectangle with sides parallel to the coordinate axes. Atom $a \in \mathcal{P} \cap A$ will be called $A$-record if no other atom $b \in \mathcal{P} \cap A$ lies south-west of $a$. The set of $A$-records induces a random partition of $A$ in rectangular tiles. Denote by $N = (N_{i,j})$ the random matrix of areas of the tiles. The number of rows (or columns) of $N$ is a random variable that is one plus the number of $A$-records. Let $N^\star = (N^*_{i,j})$ be the array obtained by reflecting $N$ about the antidiagonal, which is defined conditionally on the number of $A$-records.

**Lemma 3.1.** *For every rectangle* $A$ *we have*

(i) $N \stackrel{d}{=} N^\star$,

(ii) *also* $N \stackrel{d}{=} N^\star$ *conditionally given the number of* $A$*-records is* $n$*, for each* $n \ge 0$,

(iii) *the distribution of* $N$ *depends on* $A$ *only through the area of* $A$.

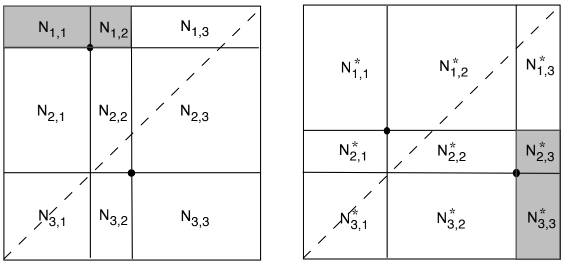


Figure 2: Records in a square

*Proof.* Applying a hyperbolic shift $(t, x) \mapsto (\lambda t, x/\lambda)$ with some $\lambda > 0$, rectangle $A$ can be mapped onto a square. The mapping preserves the area and the coordinate-wise orders, hence preserves the distribution of $N$. For $A$ a square (i) and (ii) follow by symmetry of $A$ about the north-east diagonal (see Figure 2). □

**Lemma 3.2.** *For $n \geq 1$ the following conditional distributions coincide:*

(a) *the distribution of $M_{k,n}$ given the area $v$ of the rectangle spanned on records $r_k$ and $r_{n+k}$, where $k \geq 1$ or $k \leq -n-1$,*

(b) *the distribution of $N$ for a rectangle $A$ of area $v$, given that the number of $A$-records is $n-1$.*

*Proof.* For any fixed rectangle $A$, the set of $A$-records is independent of the Poisson point process outside $A$. On the other hand, given that the north-west and the south-east corners of $A$ are records, $\mathcal{P}$ has no atoms south-east of these corners, hence the set of $A$-records coincides with $\mathcal{R} \cap A$. That is to say, given that two records are located at the corners of $A$, the records inside $A$ are distributed like $A$-records. In the same way, taking, for instance, $k = 1$, we have: given that $r_1$ and $r_{n+1}$ are at the corners of rectangle $A$ (below the line $x = t$), the set $\{r_2, \dots, r_n\}$ has the same distribution as the set of $A$-atoms, conditioned on the event that the number of $A$-records is $n-1$. Now the assertion follows from Lemma 3.1 (iii). □

Proving Proposition 2.1 is now easy. Combining Lemma 3.2 with Lemma 3.1 (ii) we see that the distributional identity (2.3) holds conditionally on the area of the rectangle spanned on $r_1$ and $r_{n+1}$, hence (2.3) also holds unconditionally. Note that the area is equal to the sum of all entries of the $n \times n$ matrix, that is distributed like

$$(1 - U_1 \cdots U_n)\left(E_1 + \frac{E_2}{U_1} + \frac{E_3}{U_1 U_2} + \cdots + \frac{E_n}{U_1 \cdots U_{n-1}}\right).$$

We could not find a proof of (1.1) by computing densities or transforms, or by connecting to other known identities like “beta-gamma algebra” [3]. Spanning a grid on the point process of $k$-corners [5] we were able to show that similar identities hold with uniform distribution replaced by beta$(1, \theta)$.

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