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Equivalence of the Poincaré inequality with a transport-chi-square inequality in dimension one

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Abstract

In this paper, we prove that, in dimension one, the Poincaré inequality is equivalent to a new transport-chi-square inequality linking the square of the quadratic Wasserstein distance with the chi-square pseudo-distance. We also check tensorization of this transport-chi-square inequality.

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For $q \geq 1$, the Wasserstein distance with index q between two probability measures μ and ν on $\mathbb{R}^d$ is denoted by

$$W_q^q(\mu,\nu) = \inf_{\gamma <^{\mu}_{\nu}} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^q d\gamma(x,y) \tag{0.1}$$

where the infimum is taken over all probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ with respective marginals μ and ν. We also introduce the relative entropy and the chi-square pseudo distance

$$H(\nu|\mu) = \begin{cases} \int_{\mathbb{R}^d} \ln\left(\frac{d\nu}{d\mu}(x)\right) d\nu(x) \text{ if } \nu \text{ absolutely continuous w.r.t. } \mu \\ +\infty \text{ otherwise} \end{cases}$$

$$\chi_2^2(\nu|\mu) = \begin{cases} \int_{\mathbb{R}^d} \left(\frac{d\nu}{d\mu}(x) - 1\right)^2 d\mu(x) = \|\frac{d\nu}{d\mu} - 1\|^2_{L^2(\mu)} \text{ if } \nu \text{ absolutely continuous w.r.t. } \mu \\ +\infty \text{ otherwise} \end{cases} .$$

Next, we precise the inequalities that will be discussed in the paper.

Definition 0.1. *The probability measure μ on $\mathbb{R}^d$ is said to satisfy*

the Poincaré inequality $\mathcal{P}(C)$ *with constant C if*

$$\forall \varphi : \mathbb{R}^d \to \mathbb{R}\ C^1 \text{ with a bounded gradient},$$
$$\int_{\mathbb{R}} \varphi^2(x) d\mu(x) - \left(\int_{\mathbb{R}} \varphi(x) d\mu(x)\right)^2 \leq C \int_{\mathbb{R}} |\nabla \varphi(x)|^2 d\mu(x).$$

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the transport-chi-square inequality $\mathcal{T}_\chi(C)$ *with constant* C *if*

$$\forall \nu \text{ probability measure on } \mathbb{R}^d,\ W_2(\mu,\nu) \le \sqrt{C}\chi_2(\nu|\mu).$$

the log-Sobolev inequality $\mathcal{LS}(C)$ *with constant* C *if* $\forall \varphi : \mathbb{R}^d \to \mathbb{R}$ C^2 *compactly supported,*

$$\int_{\mathbb{R}} \varphi^2(x)\ln(\varphi^2(x))d\mu(x) - \int_{\mathbb{R}} \varphi^2(x)d\mu(x)\ln\left(\int_{\mathbb{R}} \varphi^2(x)d\mu(x)\right) \le C\int_{\mathbb{R}} |\nabla\varphi(x)|^2 d\mu(x).$$

the transport-entropy inequality $\mathcal{T}_H(C)$ *with constant* C *if*

$$\forall \nu \text{ probability measure on } \mathbb{R}^d,\ W_2(\mu,\nu) \le \sqrt{C}H(\nu|\mu).$$

According to [9], the log-Sobolev inequality is stronger than the transport-entropy inequality which is itself stronger than the Poincaré inequality and more precisely $\mathcal{LS}(C) \Rightarrow \mathcal{T}_H(C) \Rightarrow \mathcal{P}(C/2)$. The transport-entropy inequality is strictly weaker than the log-Sobolev inequality (see [3, 5] for examples of one-dimensional probability measures μ satisfying the transport-entropy inequality but not the log-Sobolev inequality) and is strictly stronger than the Poincaré inequality (see for example [5] Theorem 1.7).

To obtain some transport inequality equivalent to the Poincaré inequality, one may try to replace either $W_2(\nu,\mu)$ in the left-hand-side by some smaller Wasserstein distance or the relative entropy $H(\nu|\mu)$ in the right-hand-side by some larger pseudo-distance. The first possibility is successfully explored in [2] Corollary 5.1 where the Poincaré inequality is proved to be equivalent to the modified transport-entropy inequality

$$\begin{aligned}&\exists C < +\infty, \forall \nu \text{ probability measure on } \mathbb{R}^d,\\ &\qquad \inf_{\gamma <^{\mu}_{\nu}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(|x-y|^2 \wedge |x-y|\right) d\gamma(x,y) \le CH(\nu|\mu)\end{aligned}$$

with possibly different constants C. The present paper is devoted to the second possibility. More precisely, since the inequality $x\ln(x) \le (x-1)+(x-1)^2$ implies $H(\nu|\mu) \le \chi_2^2(\nu|\mu)$, we consider replacing the transport-entropy inequality $\mathcal{T}_H(C)$ by the weaker transport-chi-square inequality $\mathcal{T}_\chi(C)$. It turns out that, by an easy adaptation of the linearization argument in [9], the transport-chi-square inequality implies the Poincaré inequality. Moreover, in dimension $d=1$, we are able to prove the converse implication so that both inequalities are equivalent. Last, we prove tensorization of the transport-chi-square inequality.

1 Main results

Theorem 1.1. $\forall d \ge 1$, $\mathcal{T}_\chi(C) \Rightarrow \mathcal{P}(C)$. *Moreover, when* $d=1$, $\mathcal{P}(C) \Rightarrow \mathcal{T}_\chi(32C)$ *and the transport-chi-square and Poincaré inequalities are equivalent.*

Before proving Theorem 1.1, we state our second main result dedicated to the tensorization property of the transport-chi-square inequality. Its proof is postponed in Section 4.

Theorem 1.2. *If* μ_1 *and* μ_2 *are probability measures on* $\mathbb{R}^{d_1}$ *and* $\mathbb{R}^{d_2}$ *respectively satisfying* $\mathcal{T}_\chi(C_1)$ *and* $\mathcal{T}_\chi(C_2)$*, then the measure* $\mu_1\otimes\mu_2$ *satisfies* $\mathcal{T}_\chi((C_1+C_2(1+\sqrt{(3d_2+2)d_2}))\wedge(C_2+C_1(1+\sqrt{(3d_1+2)d_1})))$.

Remark 1.3. *According to Proposition 8.4.1 [1], if* μ_1 *and* μ_2 *respectively satisfy* $\mathcal{T}_H(C_1)$ *and* $\mathcal{T}_H(C_2)$*, then* $\mu_1\otimes\mu_2$ *satisfies* $\mathcal{T}_H(C_1\vee C_2)$*. The constant that we obtain in the tensorization of the transport-chi-square inequality is larger than* $C_1 \vee C_2$*.*

The proof of the one-dimensional implication $\mathcal{P}(C) \Rightarrow \mathcal{T}_\chi(32C)$ in Theorem 1.1 relies on the two next propositions, the proof of which are respectively postponed in Sections 2 and 3. When $d = 1$, we denote by $F(x) = \mu((-\infty, x])$ and $G(x) = \nu((-\infty, x])$ the cumulative distribution functions of the probability measures μ and ν. The càg pseudo-inverses of G (resp. F) is defined by $G^{-1} :]0,1[\ni u \mapsto \inf\{x \in \mathbb{R} : G(x) \geq u\}$ (resp. $F^{-1}(u) = \inf\{x \in \mathbb{R} : G(x) \geq u\}$) and satisfies

$$\forall x \in \mathbb{R},\ \forall u \in (0,1),\ x < G^{-1}(u) \Leftrightarrow G(x) < u. \tag{1.1}$$

When μ (resp. ν) admits a density w.r.t. the Lebesgue measure, this density is denoted by f (resp. g). Moreover, the optimal coupling in (0.1) is given by $\gamma = du \circ (F^{-1}, G^{-1})^{-1}$ where du denotes the Lebesgue measure on $(0,1)$ so that

$$W_q^q(\mu,\nu) = \int_0^1 (F^{-1}(u) - G^{-1}(u))^q du$$

(see [10] p107-109). We take advantage of this optimal coupling to work with the cumulative distribution functions and check the following proposition. In higher dimensions, far less is known on the optimal coupling and this is the main reason why we have not been able to check whether the Poincaré inequality implies the transport-chi-square inequality.

Proposition 1.4. *If a probability measure μ on the real line admits a positive probability density f, then, for any probability measure ν on $\mathbb{R}$,*

$$W_2^2(\mu,\nu) \leq 4 \int_{\mathbb{R}} \frac{(F-G)^2}{f}(x)dx. \tag{1.2}$$

Remark 1.5.

- *One deduces that $W_1^2(\mu,\nu) \leq 4\int_{\mathbb{R}} \frac{(F-G)^2}{f}(x)dx$. Notice that since, by (1.1) and Fubini's theorem,*

$$\begin{aligned} W_1(\mu,\nu) &= \int_0^1 \int_{\mathbb{R}} 1_{\{F^{-1}(u) \leq x < G^{-1}(u)\}} + 1_{\{G^{-1}(u) \leq x < F^{-1}(u)\}} dx du \\ &= \int_{\mathbb{R}} \int_0^1 1_{\{G(x) < u \leq F(x)\}} + 1_{\{F(x) < u \leq G(x)\}} du dx = \int_{\mathbb{R}} |F(x) - G(x)| dx, \end{aligned}$$

  the stronger bound

$$W_1^2(\mu,\nu) = \left(\int_{\mathbb{R}} \frac{|F-G|}{\sqrt{f}} \times \sqrt{f}(x)dx\right)^2 \leq \int_{\mathbb{R}} \frac{(F-G)^2}{f}(x)dx$$

  is a consequence of the Cauchy-Schwarz inequality.

- *It is not possible to control $\int_{\mathbb{R}} \frac{(F-G)^2}{f}(x)dx$ in terms of $W_2^2(\mu,\nu)$. Indeed for $f(x) = \frac{1}{2}e^{-|x|}$ and $d\nu(x) = \frac{1}{2}e^{-|x-m|}dx$, one has $W_2^2(\mu,\nu) = m^2$,*

$$G(x) = \frac{e^{x-m}}{2} 1_{\{x \leq m\}} + (1 - \frac{e^{m-x}}{2}) 1_{\{x > m\}},$$

  and for $m > 0$, $$\int_{\mathbb{R}} \frac{(F-G)^2}{f}(x)dx \geq \int_m^{+\infty} \frac{(F-G)^2}{f}(x)dx = \frac{e^{-m}}{2}(e^m - 1)^2.$$

Next, when the probability measure μ on the real line admits a positive probability density satisfying a tail assumption known to be equivalent to the Poincaré inequality (see Theorem 6.2.2 [1]), we are able to control the right-hand-side of (1.2) in terms of $\chi_2^2(\nu|\mu)$.

Proposition 1.6. *Let $f(x)$ be a positive probability density on the real line with cumulative distribution function $F(x)=\int_{-\infty}^x f(y)dy$ and median m such that*

$$b \stackrel{\text{def}}{=} \sup_{x\geq m}\int_x^{+\infty} f(y)dy\int_m^x \frac{dy}{f(y)} \vee \sup_{x\leq m}\int_{-\infty}^x f(y)dy\int_x^m \frac{dy}{f(y)} < +\infty. \tag{1.3}$$

Then for any probability density g on the real line with cumulative distribution function $G(x)=\int_{-\infty}^x g(y)dy$,

$$\int_{\mathbb{R}} \frac{(F-G)^2}{f}(x)dx \leq 4b\int_{\mathbb{R}}\frac{(f-g)^2}{f}(x)dx. \tag{1.4}$$

Remark 1.7.

- *The combination of these two propositions implies that any probability measure μ on the real line admitting a positive density f such that $b<+\infty$ satifies $\mathcal{T}_\chi(16b)$.*
- *Proposition 1.6 is a generalization of the last assertion in Lemma 2.3 [7] where f is restricted to the class of probability densities f_∞ solving $f_\infty(x) = -A(F_\infty(x))$ on the real line with*

  $$A:[0,1]\to\mathbb{R}\ C^1,\ \textit{negative on } (0,1) \textit{ and s.t. } A(0)=A(1)=0,\ A'(0)<0,\ A'(1)>0.$$

  The constant b associated with any such density is finite by the proof of Lemma 2.1 [7]. Moreover, in order to investigate the long-time behaviour of the solution f_t of the Fokker-Planck equation

  $$\partial_t f_t(x) = \partial_{xx}f_t(x) + \partial_x(A'(F_t(x))f_t(x)),\ (t,x)\in[0,+\infty)\times\mathbb{R}$$

  to the density f_∞ such that $\int_{\mathbb{R}} xf_\infty(x)dx = \int_{\mathbb{R}} xf_0(x)dx$, [7] first investigates the exponential convergence to 0 of $\int_{\mathbb{R}}\frac{(F_t-F_\infty)^2}{f_\infty}(x)dx$ (Lemma 2.8) before dealing with that of $\int_{\mathbb{R}}\frac{(f_t-f_\infty)^2}{f_\infty}(x)dx$ (Theorem 2.4).
- *It is not possible to control $\int_{\mathbb{R}}\frac{(f-g)^2}{f}(x)dx$ in terms of $\int_{\mathbb{R}}\frac{(F-G)^2}{f}(x)dx$, even when $b<+\infty$. Indeed let $f(x)=\frac{1}{2}e^{-|x|}$ and*

  $$\textit{for } n\in\mathbb{N},\ g_n(x) = \sum_{k\leq n} f(x)1_{[k-1,k)}(|x|) + \sum_{k\geq n}\frac{e^{-\frac{|x|}{2}}}{2}1_{[x_k,k+1)}(|x|)$$

  where $x_k = k+1-2\ln\left(1+\frac{e-1}{2}e^{-\frac{k+1}{2}}\right)$ belongs to $(k,k+1)$ and is such that $\int_{x_k}^{k+1} e^{-\frac{x}{2}}dx = \int_k^{k+1}e^{-x}dx$. One has, using $\forall y\geq 0,\ \ln(1+y)\geq\frac{y}{1+y}$ by concavity of the logarithm and $1+\frac{e-1}{2}e^{-\frac{k+1}{2}}\leq\sqrt{e}$ for the inequality,

  $$\begin{aligned}\int_{\mathbb{R}}\frac{(f-g_n)^2}{f}(x)dx &= 2\int_n^{+\infty}\frac{g_n^2}{f}(x)dx - e^{-n} = 2\sum_{k\geq n}\ln\left(1+\frac{e-1}{2}e^{-\frac{k+1}{2}}\right) - e^{-n}\\ &\geq \frac{(e-1)}{\sqrt{e}}\sum_{k\geq n}e^{-\frac{k+1}{2}} - e^{-n} = (\sqrt{e}+1)e^{-\frac{n+1}{2}} - e^{-n}.\end{aligned}$$

  On the other hand, since for $k\geq n$ and $x\in[k,k+1]$, $1-\frac{e^{-k}}{2}\leq G_n(x)\leq F(x) = 1-\frac{e^{-x}}{2}$,

  $$\int_{\mathbb{R}}\frac{(F-G_n)^2}{f}(x)dx \leq \sum_{k\geq n}\int_k^{k+1}\frac{(e^{-k}-e^{-x})^2}{e^{-x}}dx = \frac{e^2-2e-1}{e-1}e^{-n}.$$

Proof of Theorem 1.1. The implication $\mathcal{T}_\chi(C) \Rightarrow \mathcal{P}(C)$ is obtained by linearization of the transport-chi-square inequality $\mathcal{T}_\chi(C)$. For $\nu_\varepsilon = (1+\varepsilon\phi)\mu$ with $\phi : \mathbb{R}^d \to \mathbb{R}$ a C^2 function compactly supported and such that $\int_{\mathbb{R}^d} \phi(x)d\mu(x) = 0$, according to [9] p394, there is a finite constant K not depending on ε such that

$$\int_{\mathbb{R}^d} \phi^2(x)d\mu(x) \leq \sqrt{\int_{\mathbb{R}^d} |\nabla\phi(x)|^2 d\mu(x)} \times \frac{W_2(\mu,\nu_\varepsilon)}{\varepsilon} + \frac{KW_2^2(\mu,\nu_\varepsilon)}{\varepsilon}.$$

When $\mathcal{T}_\chi(C)$ holds, then $W_2(\mu,\nu_\varepsilon) \leq \varepsilon\sqrt{C\int_{\mathbb{R}^d}\phi^2(x)d\mu(x)}$ and taking the limit $\varepsilon \to 0$, one deduces that

$$\int_{\mathbb{R}^d} \phi^2(x)d\mu(x) \leq \sqrt{\int_{\mathbb{R}^d} |\nabla\phi(x)|^2 d\mu(x)} \times \sqrt{C\int_{\mathbb{R}^d} \phi^2(x)d\mu(x)}.$$

This implies $\int_{\mathbb{R}^d}\phi^2(x)d\mu(x) \leq C\int_{\mathbb{R}^d}|\nabla\phi(x)|^2d\mu(x)$. Let now $\varphi, \phi_n : \mathbb{R}^d \to \mathbb{R}$ be C^2 functions compactly supported with ϕ_n taking its values in $[0,1]$, equal to 1 on the ball centered at the origin with radius n and $\nabla\phi_n$ bounded by 1. Taking the limit $n \to \infty$ in the inequality written with ϕ replaced by $\varphi_n = \varphi - \phi_n \frac{\int_{\mathbb{R}^d}\varphi(x)d\mu(x)}{\int_{\mathbb{R}^d}\phi_n(x)d\mu(x)}$, one deduces that the Poincaré inequality $\mathcal{P}(C)$ holds for φ. The extension to C^1 functions φ with a bounded gradient is obtained by density.

To prove the converse implication, we now suppose that $d = 1$, μ satisfies the Poincaré inequality $\mathcal{P}(C)$ and that $\chi_2(\nu|\mu) < +\infty$. We set $\mu_n = \rho_n \star \mu$ and $\nu_n = \rho_n \star \nu$ for $n \geq 1$ where

$$\rho_n(x) = \sqrt{\frac{n}{2\pi}} e^{-\frac{nx^2}{2}} \tag{1.5}$$

denotes the density of the centered Gaussian law with variance $1/n$. For φ a C^1 function on $\mathbb{R}$ with a bounded derivative such that $0 = \int_{\mathbb{R}}\varphi(x)d\mu_n(x) = \int_{\mathbb{R}}\rho_n\star\varphi(x)d\mu(x)$, one has

$$\begin{aligned}
\int_{\mathbb{R}} \varphi^2(x)d\mu_n(x) &= \int_{\mathbb{R}} (\rho_n\star\varphi^2)(x) - (\rho_n\star\varphi)^2(x)d\mu(x) + \int_{\mathbb{R}} (\rho_n\star\varphi)^2(x)d\mu(x)\\
&\leq \int_{\mathbb{R}} \frac{1}{n}(\rho_n\star(\varphi')^2)(x)d\mu(x) + C\int_{\mathbb{R}}(\rho_n\star\varphi')^2(x)d\mu(x)\\
&\leq \frac{1+nC}{n}\int_{\mathbb{R}}(\rho_n\star(\varphi')^2)(x)d\mu(x) = \frac{1+nC}{n}\int_{\mathbb{R}}(\varphi')^2(x)d\mu_n(x)
\end{aligned}$$

where we used the Poincaré inequalities for the Gaussian density ρ_n ([1] Théorème 1.5.1 p10) applied to φ and for μ applied to $\rho_n\star\varphi$ for the second inequality then Jensen's inequality. The probability measure μ_n admits a positive density w.r.t. the Lebesgue measure and satisfies $\mathcal{P}(\frac{1+nC}{n})$. According to Théorème 6.2.2 [1], this property is equivalent to the fact that the constant associated with μ_n through (1.3) is $b_n \leq 2\frac{1+nC}{n}$. Combining Propositions 1.4 and 1.6, one deduces that

$$W_2^2(\mu_n,\nu_n) \leq 32\frac{1+nC}{n}\chi_2^2(\nu_n|\mu_n).$$

To conclude, let us check that $W_2^2(\mu,\nu) \leq \liminf_{n\to\infty} W_2^2(\mu_n,\nu_n)$ and that $\chi_2^2(\nu_n|\mu_n) \leq \chi_2^2(\nu|\mu)$. First, the probability measures μ_n with c.d.f. $F_n(x) = \mu_n((-\infty,x])$ (resp ν_n with c.d.f. $G_n(x) = \nu_n((-\infty,x])$) converge weakly to μ (resp. ν) which ensures that du a.e. on $(0,1)$, $(F_n^{-1}(u), G_n^{-1}(u))$ tends to $(F^{-1}(u), G^{-1}(u))$ as $n\to\infty$. With Fatou lemma, one deduces that

$$W_2^2(\mu,\nu) = \int_0^1 (F^{-1}(u) - G^{-1}(u))^2 du$$
$$\le \liminf_{n\to\infty} \int_0^1 (F_n^{-1}(u) - G_n^{-1}(u))^2 du = \liminf_{n\to\infty} W_2^2(\mu_n,\nu_n).$$

On the other hand, by Jensen's inequality,

$$\chi_2^2(\nu_n|\mu_n) = \int_{\mathbb{R}} \left(\frac{\int_{\mathbb{R}} (\frac{d\nu}{d\mu}(y) - 1)\rho_n(x-y)d\mu(y)}{\int_{\mathbb{R}} \rho_n(x-y)d\mu(y)} \right)^2 \int_{\mathbb{R}} \rho_n(x-z)d\mu(z)dx$$
$$\le \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{d\nu}{d\mu}(y) - 1\right)^2 \rho_n(x-y)d\mu(y)dx = \chi_2^2(\nu|\mu).$$

□

Remark 1.8. *Since*

$$W_2^2(\mu_n,\nu_n) \le \inf_{\gamma<^{\mu}_{\nu}} \int_{\mathbb{R}^3} ((x+z) - (y+z))^2 d\gamma(x,y)\rho_n(z)dz = W_2^2(\mu,\nu),$$

one has $\lim_{n\to\infty} W_2(\mu_n,\nu_n) = W_2(\mu,\nu)$.
Moreover, when $\chi_2^2(\nu|\mu) < +\infty$*, then interpreting* μ_n *and (resp* ν_n*) as the distribution at time* $\frac{1}{n}$ *of a Brownian motion initially distributed according to* μ *(resp.* ν*) and using Theorem 1.7 [4], one obtains* $\lim_{n\to\infty} \chi_2^2(\nu_n|\mu_n) = \chi_2^2(\nu|\mu)$.

2 Proof of Proposition 1.4

To prove the proposition, one first needs to express the Wasserstein distance in terms of the cumulative distribution functions F and G instead of their pseudo-inverses.

Lemma 2.1.

$$W_2^2(\mu,\nu) = \int_{\mathbb{R}^2} \left((F(x\wedge y) - G(x\vee y))^+ + (G(x\wedge y) - F(x\vee y))^+ \right) dydx. \tag{2.1}$$

Proof of Lemma 2.1. Using Fubini's theorem and (1.1) for the third equality, one obtains

$$W_2^2(\mu,\nu) = \int_0^1 (G^{-1}(u) - F^{-1}(u))^2 du$$
$$= 2\int_{[0,1]}\int_{\mathbb{R}^2} \left(1_{\{F^{-1}(u)\le x\le y<G^{-1}(u)\}} + 1_{\{G^{-1}(u)\le x\le y<F^{-1}(u)\}}\right) dxdydu$$
$$= 2\int_{\mathbb{R}^2} 1_{\{x\le y\}} \int_0^1 \left(1_{\{G(y)<u\le F(x)\}} + 1_{\{F(y)<u\le G(x)\}}\right) dudydx$$
$$= 2\int_{\mathbb{R}}\int_x^{+\infty} \left((F(x) - G(y))^+ + (G(x) - F(y))^+\right) dydx. \tag{2.2}$$

By symmetry, one deduces that (2.1) holds. □

Proof of Proposition 1.4. One has

$$\int_x^{+\infty} (F(x) - G(y))^+ dy$$
$$= 1_{\{F(x)>G(x)\}} \int_x^{G^{-1}(F(x))} (F(x) - G(y))dy \quad \le (F(x) - G(x))^+ (G^{-1}(F(x)) - x). \tag{2.3}$$

By Fubini's theorem and a similar argument,

$$\begin{aligned}\int_{\mathbb{R}}\int_x^{+\infty}(G(x)-F(y))^+dydx &= \int_{\mathbb{R}}\int_{-\infty}^x (G(y)-F(x))^+dydx\\ &\le \int_{\mathbb{R}}(G(x)-F(x))^+(x-G^{-1}(F(x)))dx\end{aligned}$$

With (2.2) and (2.3), then using Cauchy-Schwarz inequality and the change of variables $u = F(x)$, one deduces that when μ admits a positive density f w.r.t. the Lebesgue measure, then

$$\begin{aligned}W_2^2(\mu,\nu) &\le 2\int_{\mathbb{R}}|G(x)-F(x)||x-G^{-1}(F(x))|dx\\ &\le 2\left(\int_{\mathbb{R}}\frac{(G(x)-F(x))^2}{f(x)}dx\right)^{1/2}\times\left(\int_{\mathbb{R}}(x-G^{-1}(F(x)))^2f(x)dx\right)^{1/2}\\ &= 2\left(\int_{\mathbb{R}}\frac{(G(x)-F(x))^2}{f(x)}dx\right)^{1/2}\times\left(\int_0^1(F^{-1}(u)-G^{-1}(u))^2du\right)^{1/2}.\end{aligned}$$

Recognizing that the second factor in the r.h.s. is equal to $W_2(\mu,\nu)$, one concludes that (1.4) holds as soon as $W_2(\mu,\nu) < +\infty$. To prove (1.4) without assuming finiteness of $W_2(\mu,\nu)$, one defines a sequence $(G_n)_n$ of cumulative distribution functions converging pointwise to G by setting

$$G_n(x) = \begin{cases} F(x)\wedge\frac{1}{n} \text{ if } x < G^{-1}(\frac{1}{n})\\ G(x) \text{ if } x \in [G^{-1}(\frac{1}{n}), G^{-1}(\frac{n-1}{n}))\\ F(x)\vee\frac{n-1}{n} \text{ if } x \ge G^{-1}(\frac{n-1}{n})\end{cases}$$

For $x < G^{-1}(\frac{1}{n})$, $G(x) < \frac{1}{n}$ and

$$|F(x)-G_n(x)| = (F(x)-\frac{1}{n})^+ \le \min(|F(x)-G(x)|,(F(x)-\frac{1}{n+1})^+) \le |F(x)-G_{n+1}(x)|.$$

Similarly, for $x \ge G^{-1}\left(\frac{n-1}{n}\right)$, $G(x) \ge \frac{n-1}{n}$ and

$$|F(x)-G_n(x)| = (\frac{n-1}{n}-F(x))^+ \le \min(|F(x)-G(x)|,(\frac{n}{n+1}-F(x))^+) \le |F(x)-G_{n+1}(x)|.$$

As a consequence, for fixed $x \in \mathbb{R}$, the sequence $(|G_n(x)-F(x)|)_{n\in\mathbb{N}}$ is non-decreasing and goes to $|G(x)-F(x)|$ as $n \to \infty$. By monotone convergence, one deduces that $\lim_{n\to+\infty}\int_{\mathbb{R}}\frac{(G_n-F)^2}{f}(x)dx = \int_{\mathbb{R}}\frac{(G-F)^2}{f}(x)dx$. Moreover,

$$G_n^{-1}(u) = \begin{cases} F^{-1}(u)\wedge G^{-1}(\frac{1}{n}) \text{ if } u \le \frac{1}{n}\\ G^{-1}(u) \text{ if } u \in (\frac{1}{n},\frac{n-1}{n}]\\ F^{-1}(u)\vee G^{-1}(\frac{n-1}{n}) \text{ if } u > \frac{n-1}{n}\end{cases}.$$

As a consequence, denoting by ν_n the probability measure with c.d.f. G_n,

$$W_2^2(\mu,\nu_n) = \int_0^1(F^{-1}(u)-G_n^{-1}(u))^2du < +\infty$$

and $W_2^2(\mu,\nu) \le \liminf_{n\to\infty}W_2^2(\mu,\nu_n)$ by Fatou Lemma. One concludes by taking the limit $n \to +\infty$ in (1.4) written with (ν_n, G_n) replacing (ν, G). □

3 Proof of Proposition 1.6

Let us assume that $b < +\infty$ and $\int_{\mathbb{R}} \frac{(f-g)^2}{f}(x)dx < +\infty$. By integration by parts, for $n \in \mathbb{N}^*$,

$$\int_{-n}^{n} \frac{(F-G)^2}{f}(x)dx = \left[(F-G)^2(x)\int_m^x \frac{dy}{f(y)}\right]_{-n}^{+n} - 2\int_{-n}^{n}(F-G)(f-g)(x)\int_m^x \frac{dy}{f(y)}dx. \tag{3.1}$$

For x larger than the median m of the density f, by definition of b, then by the equality $(F-G)(x) = \int_x^\infty (g-f)(y)dy$ and Cauchy-Schwarz inequality, one has

$$0 \le (F-G)^2(x)\int_m^x \frac{dy}{f(y)} \le b\frac{(F-G)^2(x)}{\int_x^{+\infty} f(y)dy} = b\frac{\left(\int_x^\infty (f-g)(y)dy\right)^2}{\int_x^{+\infty} f(y)dy} \le b\int_x^\infty \frac{(f-g)^2}{f}(y)dy.$$

where the right-hand-side tends to 0 as $x \to +\infty$ by integrability of $\frac{(f-g)^2}{f}$ on the real line. Similarly, $\lim_{x\to-\infty}(F-G)^2(x)\int_x^m \frac{dy}{f(y)} = 0$. Taking the limit $n \to \infty$ in (3.1) and using again the definition of b, one deduces that

$$\int_{\mathbb{R}} \frac{(F-G)^2}{f}(x)dx \le 2b\int_{\mathbb{R}} |(F-G)(f-g)|(x)\left(\frac{1_{\{x\ge m\}}}{\int_x^\infty f(y)dy} + \frac{1_{\{x<m\}}}{\int_{-\infty}^x f(y)dy}\right)dx. \tag{3.2}$$

The product $|(F-G)(f-g)|(x) \times \left(\frac{1_{\{x\ge m\}}}{\int_x^\infty f(y)dy} + \frac{1_{\{x<m\}}}{\int_{-\infty}^x f(y)dy}\right)$ is locally integrable on $\mathbb{R}$ since the first factor is integrable and the second one is locally bounded. Let $a_n < +\infty$ denote the integral of this function on $[-n,n]$.

By Cauchy Schwarz inequality,

$$a_n \le \sqrt{\int_{\mathbb{R}} \frac{(f-g)^2}{f}(x)dx}\left(\int_{-n}^{n} f(F-G)^2(x)\left(\frac{1_{\{x\ge m\}}}{\int_x^\infty f(y)dy} + \frac{1_{\{x<m\}}}{\int_{-\infty}^x f(y)dy}\right)^2 dx\right)^{1/2}. \tag{3.3}$$

Now, setting $\varepsilon_n = \frac{(F-G)^2(n)}{\int_n^\infty f(y)dy} + \frac{(F-G)^2(-n)}{\int_{-\infty}^{-n} f(y)dy}$, we obtain by integration by parts that for $n \ge |m|$,

$$\begin{aligned}
&\int_{-n}^{n} f(F-G)^2(x)\left(\frac{1_{\{x\ge m\}}}{\int_x^\infty f(y)dy} + \frac{1_{\{x<m\}}}{\int_{-\infty}^x f(y)dy}\right)^2 dx \\
&\quad= \left[\frac{(F-G)^2(x)}{\int_x^\infty f(y)dy}\right]_m^n - 2\int_m^n \frac{(F-G)(f-g)(x)}{\int_x^\infty f(y)dy}dx - \left[\frac{(F-G)^2(x)}{\int_{-\infty}^x f(y)dy}\right]_{-n}^m \\
&\qquad\qquad + 2\int_{-n}^m \frac{(F-G)(f-g)(x)}{\int_{-\infty}^x f(y)dy}dx \\
&\quad= -4(F-G)^2(m) + \varepsilon_n - 2\int_{-n}^{n}(F-G)(f-g)(x)\left(\frac{1_{\{x\ge m\}}}{\int_x^\infty f(y)dy} - \frac{1_{\{x<m\}}}{\int_{-\infty}^x f(y)dy}\right)dx \\
&\quad\le 2a_n + \varepsilon_n.
\end{aligned}$$

Plugging this estimation in (3.3), one deduces that

$$\forall n \ge |m|,\ a_n \le 1_{\{a_n>0\}}\left(2 + \frac{\varepsilon_n}{a_n}\right)\int_{\mathbb{R}} \frac{(f-g)^2}{f}(x)dx.$$

Using that, according to the analysis of the boundary terms in the first integration by parts performed in the proof, $\lim_{n\to+\infty}\varepsilon_n = 0$ and that $(a_n)_n$ is non-decreasing, one may take the limit $n \to \infty$ in this inequality to obtain

$$\int_{\mathbb{R}} |(F-G)(f-g)|(x)\left(\frac{1_{\{x\geq m\}}}{\int_x^{\infty} f(y)dy} + \frac{1_{\{x<m\}}}{\int_{-\infty}^{x} f(y)dy}\right)dx \leq 2\int_{\mathbb{R}} \frac{(f-g)^2}{f}(x)dx.$$

One easily concludes with (3.2).

4 Proof of Theorem 1.2

Let ν be a probability measure on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ with respective marginals ν_1 and ν_2 and such that $\chi_2(\nu|\mu_1\otimes\mu_2) < +\infty$, ρ denote the Radon-Nykodym derivative $\frac{d\nu}{d\mu_1\otimes\mu_2}$ and for $x_1\in\mathbb{R}^{d_1}$, $\rho_1(x_1) = \int_{\mathbb{R}^{d_2}} \rho(x_1,x_2)d\mu_2(x_2)$. Notice that

$$\chi_2^2(\nu,\mu_1\otimes\mu_2) = \int_{\mathbb{R}^{d_1+d_2}} (\rho(x_1,x_2)-1)^2\, d\mu_1(x_1)d\mu_2(x_2).$$

According to the tensorization property of transport costs (see for instance Proposition A.1 [6]),

$$W_2^2(\mu_1\otimes\mu_2,\nu) \leq W_2^2(\mu_1,\nu_1) + \int_{\mathbb{R}^{d_1}} 1_{\{\rho_1(x_1)>0\}} W_2^2\left(\mu_2, \frac{\rho(x_1,.)}{\rho_1(x_1)}\mu_2\right) d\nu_1(x_1) \tag{4.1}$$

By $\mathcal{T}_\chi(C_1)$ satisfied by μ_1, the equality $\frac{d\nu_1}{d\mu_1}(x_1) = \rho_1(x_1) = \int_{\mathbb{R}^{d_2}} \rho(x_1,x_2)d\mu_2(x_2)$ and Jensen's inequality, one has

$$W_2^2(\mu_1,\nu_1) \leq C_1\chi_2^2(\nu_1|\mu_1) = C_1\int_{\mathbb{R}^{d_1}} (\rho_1(x_1)-1)^2 d\mu_1(x_1) \leq C_1\chi_2^2(\nu,\mu_1\otimes\mu_2). \tag{4.2}$$

So the first term of the right-hand-side of (4.1) is controled by $\chi_2^2(\nu,\mu_1\otimes\mu_2)$. By the inequality $\mathcal{T}_\chi(C_2)$ satisfied by μ_2, when $\rho_1(x_1)>0$,

$$W_2^2\left(\mu_2, \frac{\rho(x_1,.)}{\rho_1(x_1)}\mu_2\right) \leq C_2\int_{\mathbb{R}^{d_2}} \left(\frac{\rho(x_1,x_2)}{\rho_1(x_1)} - 1\right)^2 d\mu_2(x_2).$$

Unfortunately, there is no hope to control

$$\begin{aligned}\int_{\mathbb{R}^{d_1+d_2}} & 1_{\{\rho_1(x_1)>0\}} \left(\frac{\rho(x_1,x_2)}{\rho_1(x_1)} - 1\right)^2 d\nu_1(x_1)d\mu_2(x_2)\\ &= \int_{\mathbb{R}^{d_1+d_2}} 1_{\{\rho_1(x_1)>0\}} \left(\frac{\rho(x_1,x_2)}{\rho_1(x_1)} - 1\right)^2 \rho_1(x_1)d\mu_1(x_1)d\mu_2(x_2)\end{aligned}$$

in terms of $\chi_2^2(\nu,\mu_1\otimes\mu_2)$ because of the possible very small values of $\rho_1(x_1)$. Therefore it is not enough to plug the latter inequality into the right-hand-side of (4.1) to conclude that $\mu_1\otimes\mu_2$ satisfies a transport-chi-square inequality. So we are only going to use this inequality for $\rho_1(x_1)\geq\frac{1}{\alpha}$ where α is some constant larger than 1 to be optimized at the end of the proof. Using Lemma 4.1 below with $\beta=\alpha$, one obtains

$$\begin{aligned}\int_{\mathbb{R}^{d_1}} & W_2^2\left(\mu_2, \frac{\rho(x_1,.)}{\rho_1(x_1)}\mu_2\right) 1_{\{\rho_1(x_1)\geq\frac{1}{\alpha}\}} d\nu_1(x_1)\\ &\leq \alpha C_2\int_{\mathbb{R}^{d_1+d_2}} (\rho(x_1,x_2)-1)^2 1_{\{\rho_1(x_1)\geq\frac{1}{\alpha}\}} d\mu_1(x_1)d\mu_2(x_2).\end{aligned} \tag{4.3}$$

For small positive values of ρ_1, we use the estimation of $W_2^2\left(\mu_2, \frac{\rho(x_1,.)}{\rho_1(x_1)}\mu_2\right)$ deduced from the optimal coupling for the total variation distance. If $\nu \neq \mu$, let ε denote a Bernoulli random variable with parameter $p = \int_{\mathbb{R}^{d_2}} \left(\frac{\rho(x_1,x_2)}{\rho_1(x_1)} \wedge 1\right) d\mu_2(x_2)$ and (X, Y, Z) denote an independent $\mathbb{R}^{d_2} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_2}$-valued random vector with X, Y and Z respectively distributed according to $\frac{1}{p}\left(\frac{\rho(x_1,x_2)}{\rho_1(x_1)} \wedge 1\right) d\mu_2(x_2)$, $\frac{1}{1-p}\left(1 - \frac{\rho(x_1,x_2)}{\rho_1(x_1)}\right)^+ d\mu_2(x_2)$ and $\frac{1}{1-p}\left(\frac{\rho(x_1,x_2)}{\rho_1(x_1)} - 1\right)^+ d\mu_2(x_2)$. The random variables $\varepsilon X + (1-\varepsilon)Y$ and $\varepsilon X + (1-\varepsilon)Z$ are respectively distributed according to $d\mu_2(x_2)$ and $\frac{\rho(x_1,x_2)}{\rho_1(x_1)} d\mu_2(x_2)$. As a consequence,

$$\begin{aligned}
W_2^2\left(\mu_2, \frac{\rho(x_1,.)}{\rho_1(x_1)}\mu_2\right) &\leq \mathbb{E}\left((1-\varepsilon)^2|Y-Z|^2\right) = (1-p)\mathbb{E}\left(|Y-Z|^2\right) \\
&\leq 2(1-p)\left[\mathbb{E}\left(\left|Y - \int_{\mathbb{R}^{d_2}} y_2 d\mu_2(y_2)\right|^2\right) + \mathbb{E}\left(\left|Z - \int_{\mathbb{R}^{d_2}} y_2 d\mu_2(y_2)\right|^2\right)\right] \\
&\leq 2\int_{\mathbb{R}^{d_2}} \left|x_2 - \int_{\mathbb{R}^{d_2}} y_2 d\mu_2(y_2)\right|^2 \left|\frac{\rho(x_1,x_2)}{\rho_1(x_1)} - 1\right| d\mu_2(x_2).
\end{aligned}$$

One deduces

$$\begin{aligned}
&\int_{\mathbb{R}^{d_1}} 1_{\{0<\rho_1(x_1)<\frac{1}{\alpha}\}} W_2^2\left(\mu_2, \frac{\rho(x_1,.)}{\rho_1(x_1)}\mu_2\right) d\nu_1(x_1) \\
&\quad\leq 2\int_{\mathbb{R}^{d_1+d_2}} \left|x_2 - \int_{\mathbb{R}^{d_2}} y_2 d\mu_2(y_2)\right|^2 |\rho(x_1,x_2) - \rho_1(x_1)|\, 1_{\{\rho_1(x_1)<\frac{1}{\alpha}\}} d\mu_1(x_1) d\mu_2(x_2) \\
&\quad\leq 2\left(\int_{\mathbb{R}^{d_1+d_2}} \left|x_2 - \int_{\mathbb{R}^{d_2}} y_2 d\mu_2(y_2)\right|^4 1_{\{\rho_1(x_1)<\frac{1}{\alpha}\}} d\mu_1(x_1) d\mu_2(x_2)\right)^{1/2} \\
&\qquad\times\left(\int_{\mathbb{R}^{d_1+d_2}} (\rho(x_1,x_2) - \rho_1(x_1))^2 1_{\{\rho_1(x_1)<\frac{1}{\alpha}\}} d\mu_1(x_1) d\mu_2(x_2)\right)^{1/2} \\
&\quad\leq 2C_2\sqrt{(3d_2+2)d_2}\left(\int_{\mathbb{R}^{d_1}} \frac{\alpha^2(\rho_1(x_1)-1)^2}{(\alpha-1)^2} 1_{\{\rho_1(x_1)<\frac{1}{\alpha}\}} d\mu_1(x_1)\right)^{1/2} \\
&\qquad\times\left(\int_{\mathbb{R}^{d_1+d_2}} [(\rho(x_1,x_2)-1)^2 - (\rho_1(x_1)-1)^2] 1_{\{\rho_1(x_1)<\frac{1}{\alpha}\}} d\mu_1(x_1) d\mu_2(x_2)\right)^{1/2} \\
&\quad\leq \frac{C_2\alpha\sqrt{(3d_2+2)d_2}}{\alpha-1}\int_{\mathbb{R}^{d_1+d_2}} (\rho(x_1,x_2)-1)^2 1_{\{\rho_1(x_1)<\frac{1}{\alpha}\}} d\mu_1(x_1) d\mu_2(x_2),
\end{aligned}$$

where we used Cauchy Schwarz inequality for the second inequality, then Lemma 4.2 below and an explicit computation of the third factor for the third inequality and last the inequality $\sqrt{b}\sqrt{a-b} \leq \frac{a}{2}$ for any $a \geq b \geq 0$.

Inserting this estimation together with (4.2) and (4.3) into (4.1), one obtains

$$W_2^2(\mu_1 \otimes \mu_2, \nu) \leq C_1\chi_2^2(\nu_1, \mu_1) + C_2\alpha\left(1 \vee \frac{\sqrt{(3d_2+2)d_2}}{\alpha-1}\right)\chi_2^2(\nu, \mu_1 \otimes \mu_2).$$

For the optimal choice $\alpha = 1 + \sqrt{(3d_2+2)d_2}$, one concludes that the measure $\mu_1 \otimes \mu_2$ satisfies $\mathcal{T}_\chi(C_1 + C_2(1 + \sqrt{(3d_2+2)d_2}))$. Exchanging the roles of μ_1 and μ_2 in the above reasonning, one obtains that $\mu_1 \otimes \mu_2$ also satisfies $\mathcal{T}_\chi(C_2 + C_1(1 + \sqrt{(3d_1+2)d_1}))$.

Lemma 4.1. *For $\beta \geq \alpha > 0$,*

$$\begin{aligned}\int_{\mathbb{R}^{d_1+d_2}} \left(\frac{\rho(x_1,x_2)}{\rho_1(x_1)} - 1\right)^2 1_{\{\rho_1(x_1)\geq \frac{1}{\alpha}\}} d\nu_1(x_1) d\mu_2(x_2) \\ + \beta \int_{\mathbb{R}^{d_1}} (\rho_1(x_1)-1)^2 1_{\{\rho_1(x_1)\geq \frac{1}{\alpha}\}} d\mu_1(x_1) \\ \leq \beta \int_{\mathbb{R}^{d_1+d_2}} (\rho(x_1,x_2)-1)^2 1_{\{\rho_1(x_1)\geq \frac{1}{\alpha}\}} d\mu_1(x_1) d\mu_2(x_2).\end{aligned}$$

Proof. Developping the squares and using the definition of ρ_1 and the equality $d\nu_1(x_1) = \rho_1(x_1)d\mu_1(x_1)$, one checks that the difference between the right-hand-side and the first term of the left-hand-side is equal to

$$\int_{\mathbb{R}^{d_1}} \left[\left(\beta - \frac{1}{\rho_1(x_1)}\right)\int_{\mathbb{R}^{d_2}} \rho^2(x_1,x_2)d\mu_2(x_2) + (1-2\beta)\rho_1(x_1) + \beta\right] 1_{\{\rho_1(x_1)\geq \frac{1}{\alpha}\}} d\mu_1(x_1).$$

One easily concludes by remarking that the first integral is retricted to the $x_1 \in \mathbb{R}^{d_1}$ such that $\frac{1}{\rho_1(x_1)} \leq \alpha \leq \beta$ and that

$$\int_{\mathbb{R}^{d_2}} \rho^2(x_1,x_2)d\mu_2(x_2) \geq \left(\int_{\mathbb{R}^{d_2}} \rho(x_1,x_2)d\mu_2(x_2)\right)^2 = \rho_1^2(x_1).$$

□

Lemma 4.2. *If a probability measure μ on $\mathbb{R}^d$ satisfies $\mathcal{T}(C)$, then*

$$\int_{\mathbb{R}^d} \left|x - \int_{\mathbb{R}^d} y d\mu(y)\right|^2 d\mu(x) \leq dC \text{ and } \int_{\mathbb{R}^d} \left|x - \int_{\mathbb{R}^d} y d\mu(y)\right|^4 d\mu(x) \leq (3d+2)dC^2.$$

Proof. According to Theorem 1.1, μ satisfies $\mathcal{P}(C)$. By spatial translation, one may assume that $\int_{\mathbb{R}^d} y d\mu(y) = 0$. Applying the Poincaré inequality $\mathcal{P}(C)$ to the functions $x = (x_1, \dots, x_d) \in \mathbb{R}^d \mapsto x_i$, $x \mapsto x_i^2$ and $x \mapsto x_i x_j$ with $1 \leq i \neq j \leq d$, yields,

$$\begin{aligned}\int_{\mathbb{R}^d} x_i^2 d\mu(x) &\leq C \\ \int_{\mathbb{R}^d} x_i^4 d\mu(x) &\leq 4C\int_{\mathbb{R}^d} x_i^2 d\mu(x) + \left(\int_{\mathbb{R}^d} x_i^2 d\mu(x)\right)^2 \leq 5C^2 \\ \int_{\mathbb{R}^d} (x_i x_j)^2 d\mu(x) &\leq C\int_{\mathbb{R}^d} x_i^2 + x_j^2 d\mu(x) + \left(\int_{\mathbb{R}^d} x_i x_j d\mu(x)\right)^2 \\ &\leq 2C^2 + \int_{\mathbb{R}^d} x_i^2 d\mu(x)\int_{\mathbb{R}^d} x_j^2 d\mu(x) \leq 3C^2.\end{aligned}$$

One easily concludes by summation of these inequalities. □

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