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Interacting particle models and the Pieri-type formulas : the symplectic case with non equal weights

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Abstract

This paper studies a particle model connected to a Pieri-type formula for the symplectic compact group. Particles evolve independently apart from a blocking and pushing interaction such that they remain in the set of symplectic Gelfand-Tsetlin patterns. We show that if only one row of the patterns is considered, one obtains a symplectic Schur process. The model leads in particular to the particle model with a wall defined by Jon Warren and Peter Windridge and to a matrix model previously studied by the author.

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1 introduction

The objective of this paper is to study an interacting particle model where particles can move to the left or to the right and are subject to blocking and pushing interactions. Moreover, particles are constrained to stay on the non-negative real axis. Such a model is referred to as an interacting particle model with a wall. We prove that our model is related to a Pieri-type formula for the symplectic compact group. One can find for instance in [2] and [7] other examples of models with a wall. In these last two references, models only differ by the behavior of the particles near the wall : in the first one, these particles are reflected by the wall, in the second one, they are blocked by the wall. Actually the first one, connected to models of [4], is strongly related to representations of the orthogonal group whereas the second one involves representations of the symplectic compact group. Pursuing the study of models with a wall in the same way as in [4] we construct here a new interacting particle model with a wall depending on parameters. Two particular values of the parameters lead on the one hand to a model studied in [7] and on the other hand to a random matrix model of [3].

Let us recall that Pieri's formula describes the product of a Schur polynomial by a complete symmetric function. It is a specific case of the Littlewood-Richardson rules for decomposing the tensor products of representations of the unitary group into irreducible components. In that specific case, irreducible components of the tensor products have a multiplicity equal to one. For the other compact Lie groups, similar formula exists but the decompositions aren't necessarily multiplicity-free. A fundamental difference between model considered here and models usually considered in the literature

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(see [2] and [7] for instance) is that it is related to tensor products of representations which aren't decomposed into irreducible components with multiplicity one. In addition, the case with non equal weights studied here is of particular interest since it exposes connections between symplectic Schur functions and the random particle models.

The paper is organized as follows. In the second section, I shall recall the definitions of the symplectic Gelfand-Tsetlin patterns and Schur functions. In the third one I will describe the interacting particle model studied in the paper. In the fourth section I'll recall some properties about tensor products of special representations of the symplectic compact group, which naturally leads to some Markov kernels involved in the interacting particle model. These Markov kernels are defined in the fifth section. Section six is devoted to a random matrix model related to the particle model. Results of the paper are stated in section seven. I will sketch the proofs in the last section.

2 Symplectic Gelfand-Tsetlin patterns, symplectic Schur functions

Here and elsewhere $\mathbb{N}$ stands for the set of nonnegative integers. For $n \in \mathbb{N}^*$ and $x, y \in \mathbb{R}^n$ such that $x_n \leq \cdots \leq x_1$ and $y_n \leq \cdots \leq y_1$, we write $x \preceq y$ if x and y are interlaced, i.e.

$$x_n \leq y_n \leq x_{n-1} \leq \cdots \leq x_1 \leq y_1.$$

When $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{n+1}$ we add the relation $y_{n+1} \leq x_n$. We denote by $|x|$ the sum of the coordinates $\sum_{i=1}^n x_i$.

Definition 2.1. *Let k be a positive integer.*

1. *We denote by GT_k the set of symplectic Gelfand-Tsetlin patterns defined by*

$$GT_k = \{(x^1, \cdots, x^k) : x^i \in \mathbb{N}^{[\frac{i+1}{2}]} \text{ and } x^{i-1} \preceq x^i, 1 \leq i \leq k\}.$$

2. *If $x = (x^1, \ldots, x^k)$ is a symplectic Gelfand-Tsetlin pattern, x^i is called the i^{th} row of x for $i \in \{1, \ldots, k\}$.*
3. *For $\lambda \in \mathbb{N}^{[\frac{k+1}{2}]}$ such that $\lambda_{[\frac{k+1}{2}]} \leq \cdots \leq \lambda_1$, the subset of GT_k of symplectic Gelfand-Tsetlin patterns having the k^{th} row equal to λ is denoted by $GT_k(\lambda)$.*

Note that the exponent in the coordinates of $x \in GT_k$ means floor. Usually, a symplectic Gelfand-Tsetlin pattern is represented by a triangular array as indicated at figure 1 for $k = 2r$. Actually the further to the right an entry is, the larger it is. Thus the interlacing property satisfied by the coordinates of a symplectic Gelfand-Tsetlin pattern is graphically represented. For x a symplectic Gelfand-Tsetlin pattern of GT_k and $(q_i)_{i \geq 1}$ a sequence of positive real numbers one defines $w_x^k(q_1, \ldots, q_{[\frac{k+1}{2}]})$ recursively by letting

$$w_x^1(q_1) = q_1^{|x^1|},$$

and

$$w_x^{2i}(q_1, \ldots, q_i) = w_x^{2i-1}(q_1, \ldots, q_i)\, q_i^{|x^{2i-1}|-|x^{2i}|},$$
$$w_x^{2i+1}(q_1, \ldots, q_i, q_{i+1}) = w_x^{2i}(q_1, \ldots, q_i)\, q_{i+1}^{|x^{2i+1}|-|x^{2i}|},$$

for $i \in \mathbb{N}^*$.

Definition 2.2. *For $\lambda \in \mathbb{N}^{[\frac{k+1}{2}]}$ such that $\lambda_1 \geq \cdots \geq \lambda_{[\frac{k+1}{2}]}$, we denote by s_λ^k the symplectic Schur function defined by*

$$s_\lambda^k(q) = \sum_{x \in GT_k(\lambda)} w_x^k(q),$$

for $q = (q_1, \dots, q_{[\frac{k+1}{2}]}) \in \mathbb{R}_+^{[\frac{k+1}{2}]}$.

Notice that the cardinality of $GT_k(\lambda)$ is equal to $s_\lambda^k(\mathbf{1})$, with $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{[\frac{k+1}{2}]}$.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & & & & & & & \\
 & & & & & -x_1^1 & & x_1^1 & & & & & & & \\
 & & & & -x_1^2 & & 0 & & x_1^2 & & & & & & \\
 & & & -x_1^3 & & -x_2^3 & & x_2^3 & & x_1^3 & & & & & \\
 & & -x_1^4 & & -x_2^4 & & 0 & & x_2^4 & & x_1^4 & & & & \\
 & & & & & \dots & & \dots & & & & & & & \\
 & -x_1^{2r-1} & & \dots & & & -x_r^{2r-1} & x_r^{2r-1} & & \dots & & & x_1^{2r-1} & & \\
-x_1^{2r} & & \dots & & & -x_r^{2r} & & 0 & & x_r^{2r} & & \dots & & & x_1^{2r}
\end{array}
$$

Figure 1: A symplectic Gelfand–Tsetlin pattern of GT_{2r}

3 Interacting particle models

In this section we construct two processes evolving on the set GT_k of symplectic Gelfand-Tsetlin patterns. These processes can be viewed as interacting particle models. For this, we associate to a symplectic Gelfand-Tsetlin pattern $x = (x^1, \dots, x^k)$, a configuration of particles on the integer lattice $\mathbb{Z}^2$ putting one particle labeled by (i, j) at point $(x_j^i, k - i)$ of $\mathbb{N}^2$ for $i \in \{1, \dots, k\}$, $j \in \{1, \dots, [\frac{i+1}{2}]\}$. Several particles can be located at the same point. In the sequel we will say "particle x_j^i" instead of saying "particle labeled by (i, j) located at point $(x_j^i, k - i)$".

3.1 Geometric jumps

Let $q = (q_1, \dots, q_r) \in \mathbb{R}_+^r$ and $\alpha \in (0, 1)$ such that $\alpha q_i \in (0, 1)$ and $\alpha q_i^{-1} \in (0, 1)$ for $i = 1, \dots, r$, with $r = [\frac{k+1}{2}]$. Consider two independent families

$$(\xi_j^i(n + \frac{1}{2}))_{i=1,\dots,k, j=1,\dots,[\frac{i+1}{2}]; n\geq 0}, \quad \text{and} \quad (\xi_j^i(n))_{i=1,\dots k, j=1,\dots,[\frac{i+1}{2}]; n\geq 1},$$

of independent geometric random variables such that

$$\mathbb{P}(\xi_j^{2i-1}(n + \frac{1}{2}) = y) = \mathbb{P}(\xi_j^{2i}(n) = y) = (\alpha q_i^{-1})^y (1 - \alpha q_i^{-1}), \quad y \in \mathbb{N},$$

and

$$\mathbb{P}(\xi_j^{2i-1}(n) = y) = \mathbb{P}(\xi_j^{2i}(n + \frac{1}{2}) = y) = (\alpha q_i)^y (1 - \alpha q_i), \quad y \in \mathbb{N}.$$

The evolution of the particles is given by a process $(X(t))_{t\geq 0}$ on the set GT_k of symplectic Gelfand-Tsetlin patterns. At each time $t \geq 0$, a particle labeled by (i, j) is at point $(X_j^i(t), k - i)$ of $\mathbb{N}^2$. Particles evolve as follows. At time 0 all particles are at zero, i.e. $X(0) = 0$. All particles try to jump to the left at times $n + \frac{1}{2}$ and to the right at times n, $n \in \mathbb{N}$. Suppose that at time n, after all particles have jumped, there is one particle at point $(X_j^i(n), k - i)$ of $\mathbb{N}^2$, for $i = 1, \dots, k$, $j = 1, \dots, [\frac{i+1}{2}]$. Positions of particles are updated recursively as follows (see figure 2).

At time $n + 1/2$: All particles try to jump to the left one after another in the lexicographic order pushing the other particles in order to stay in the set of symplectic

Gelfand-Tsetlin patterns and being blocked by the initial configuration $X(n)$ of the particles:

- Particle $X_1^1(n)$ tries to move to the left being blocked by 0, i.e.

$$X_1^1(n+\frac{1}{2}) = \max(X_1^1(n) - \xi_1^1(n+\frac{1}{2}), 0).$$

- Particle $X_1^2(n)$ tries to jump to the left. It is blocked by $X_1^1(n)$. If it is necessary it pushes $X_2^3(n)$ to an intermediate position denoted by $\tilde{X}_2^3(n)$, i.e.

$$X_1^2(n+\frac{1}{2}) = \max\big(X_1^1(n), X_1^2(n) - \xi_1^2(n+\frac{1}{2})\big)$$
$$\tilde{X}_2^3(n) = \min\big(X_2^3(n), X_1^2(n+\frac{1}{2})\big)$$

- Particle $X_1^3(n)$ tries to move to the left being blocked by $X_1^2(n)$:

$$X_1^3(n+\frac{1}{2}) = \max\big(X_1^2(n), X_1^3(n) - \xi_1^3(n+\frac{1}{2})\big).$$

  Particle $\tilde{X}_2^3(n)$ tries to move to the left being blocked by 0, i.e

$$X_2^3(n+\frac{1}{2}) = \max(\tilde{X}_2^3(n) - \xi_2^3(n+\frac{1}{2}), 0).$$

Suppose now that rows 1 through $l-1$ have been updated for some $l > 1$. Particles $X_2^l(n), \ldots, X_{[\frac{l+1}{2}]}^l(n)$ of row l are pushed to intermediate positions

$$\tilde{X}_j^l(n) = \min\big(X_j^l(n), X_{j-1}^{l-1}(n+\frac{1}{2})\big),\ j \in \{2, \ldots, [\frac{l+1}{2}]\}.$$

Then particles $X_1^l(n), \tilde{X}_2^l(n), \ldots, \tilde{X}_{[\frac{l+1}{2}]}^l(n)$ try to jump to the left being blocked as follows by the initial position $X(n)$ of the particles. For $j = 1, \ldots, [\frac{l+1}{2}]$,

$$X_j^l(n+\frac{1}{2}) = \max\big(X_j^{l-1}(n), \tilde{X}_j^l(n) - \xi_j^l(n+\frac{1}{2})\big),$$

with the convention that $X_{\frac{l+1}{2}}^{l-1}(n) = 0$ when l is odd.
At time $n+1$: All particles try to jump to the right one after another in the lexicographic order pushing particles in order to stay in the set of symplectic Gelfand-Tsetlin patterns and being blocked by the initial configuration $X(n+\frac{1}{2})$ of the particles. The first three rows are updated as follows.

- Particle $X_1^1(n+\frac{1}{2})$ moves to the right pushing $X_1^2(n+\frac{1}{2})$ to an intermediate position $\tilde{X}_1^2(n+\frac{1}{2})$:

$$X_1^1(n+1) = X_1^1(n+\frac{1}{2}) + \xi_1^1(n+1)$$
$$\tilde{X}_1^2(n+\frac{1}{2}) = \max\big(X_1^2(n+\frac{1}{2}), X_1^1(n+1)\big)$$

- Particle $\tilde{X}_1^2(n+\frac{1}{2})$ jumps to the right pushing $X_1^3(n+\frac{1}{2})$ to an intermediate position $\tilde{X}_1^3(n+\frac{1}{2})$, i.e.

$$X_1^2(n+1) = \tilde{X}_1^2(n+\frac{1}{2}) + \xi_1^2(n+1)$$
$$\tilde{X}_1^3(n+\frac{1}{2}) = \max\big(X_1^3(n+\frac{1}{2}), X_1^2(n+1)\big)$$

- Particle $X_2^3(n+\frac{1}{2})$ tries to move to the right being blocked by $X_1^2(n+\frac{1}{2})$. Particle $\tilde{X}_1^3(n+\frac{1}{2})$ moves to the right. That is

$$X_2^3(n+1) = \max(X_2^3(n+\frac{1}{2}) + \xi_2^3(n+1), X_1^2(n+\frac{1}{2}))$$
$$X_1^3(n+1) = \tilde{X}_1^3(n+\frac{1}{2}) + \xi_1^3(n+1)$$

Suppose rows 1 through $l-1$ have been updated for some $l > 1$. Then particles of row l are pushed to intermediate positions

$$\tilde{X}_j^l(n+\frac{1}{2}) = \max\left(X_j^{l-1}(n+1), X_j^l(n+\frac{1}{2})\right),\ j \in \{1,\dots,[\frac{l+1}{2}]\},$$

with the convention $X^{l-1}_{\frac{l+1}{2}}(n+1) = 0$ when l is odd. Then particles $\tilde{X}_1^l(n+\frac{1}{2}),\dots,\tilde{X}^l_{[\frac{l+1}{2}]}(n+\frac{1}{2})$ try to jump to the right being blocked by the initial position of the particles as follows. For $j = 1,\dots,[\frac{l+1}{2}]$,

$$X_j^l(n+1) = \min\left(X_{j-1}^{l-1}(n+\frac{1}{2}), \tilde{X}_j^l(n+\frac{1}{2}) + \xi_j^l(n+1)\right).$$

3.2 Exponential waiting times

The following interacting particle model has been introduced in [7]. In this model particles evolve on $\mathbb{N}^2$ and jump on their own volition by one rightwards or leftwards after an exponentially distributed waiting time. The evolution of the particles is described by a random process $(Y(t))_{t\geq 0}$ on GT_k. As in the previous model, at time $t \geq 0$ there is one particle labeled by (i,j) at point $(Y_j^i(t), k-i)$ of the integer lattice, for $i = 1,\dots,k$, $j = 1,\dots,[\frac{i+1}{2}]$. Particle labeled by $(2i,j)$ tries to jump to the left by one after an exponentially distributed waiting time with mean q_i or to the right by one after an exponentially distributed waiting time with mean q_i^{-1}. Particle labeled by $(2i-1,j)$ tries to jump to the left by one after an exponentially distributed waiting time with mean q_i^{-1} or to the right by one after an exponentially distributed waiting time with mean q_i. Waiting times are all independent. When a particle tries to jump, all particles are pushed and blocked according to the same rules as previously: particles above push and block particles below. That is if particle labeled by (i,j) wants to jump to the right at time $t \geq 0$ then

1. if $i,j \geq 2$ and $Y_j^i(t^-) = Y_{j-1}^{i-1}(t^-)$ then particles don't move and $Y(t) = Y(t^-)$.
2. else particles $(i,j),(i+1,j),\dots,(i+l,j)$ jump to the right by one for l the largest integer such that $Y_j^{i+l}(t^-) = Y_j^i(t^-)$ i.e.

$$Y_j^i(t) = Y_j^i(t^-)+1,\dots,Y_j^{i+l}(t) = Y_j^{i+l}(t^-)+1.$$

If particle labeled by (i,j) wants to jump to the left at time $t \geq 0$ then

1. if i is odd, $j = (i+1)/2$ and $Y_j^i(t^-) = 0$ then particle labeled by (i,j) doesn't move.
2. if i is odd, $j = (i+1)/2$ and $Y_j^i(t^-) \geq 1$ then $Y_j^i(t) = Y_j^i(t^-) - 1$.
3. if i is even or $j \neq (i+1)/2$, and $Y_j^i(t^-) = Y_j^{i-1}(t^-)$ then particles don't move.
4. if i is even or $j \neq (i+1)/2$, and $Y_j^i(t^-) > Y_j^{i-1}(t^-)$ then particles $(i,j),(i+1,j+1),\dots,(i+l,j+l)$ jump to the left by one for l the largest integer such that $Y_{j+l}^{i+l}(t^-) = Y_j^i(t^-)$. Thus

$$Y_j^i(t) = Y_j^i(t^-)-1,\dots,Y_{j+l}^{i+l}(t) = Y_{j+l}^{i+l}(t^-)-1.$$

Actually, process $(Y(t), t \geq 0)$ is obtained by letting α go to zero in the previous model. More precisely we get the following proposition.

Proposition 3.1. *The process $(X([\alpha^{-1}t]), t \geq 0)$ converges in the sense of finite-dimensional distributions towards the process $(Y(t), t \geq 0)$ as α goes to zero.*

Proof. The proposition is obtained by replacing q by α in Lemma 8.9 of [4]. □

4 A Pieri type formula for the symplectic group

Let r be a positive integer. One recalls some usual properties of the finite dimensional representations of the compact symplectic group Sp_{2r} (see for instance [5] for more details). The set of finite dimensional representations of Sp_{2r} is indexed by the set

$$\mathcal{W}_{2r} = \{\lambda \in \mathbb{R}^r : \lambda_r \in \mathbb{N}, \lambda_i - \lambda_{i+1} \in \mathbb{N}, i = 1, \ldots, r-1\}.$$

For $\lambda \in \mathcal{W}_{2r}$, using standard notations, we denote by V_λ the so called irreducible representation with highest weight λ of Sp_{2r}.

Let m be an integer and λ an element of $\mathcal{W}_{2r}$. Consider the irreducible representations V_λ and V_{γ_m} of Sp_{2r}, with $\gamma_m = (m, 0, \cdots, 0)$. The decomposition of the tensor product $V_\lambda \otimes V_{\gamma_m}$ into irreducible components is given by a Pieri-type formula for the symplectic group. It has been recalled in [3]. We have

$$V_\lambda \otimes V_{\gamma_m} = \oplus_\beta M_{\lambda,\gamma_m}(\beta) V_\beta, \tag{4.1}$$

where the direct sum is over all $\beta \in \mathcal{W}_{2r}$ such that there exists $c \in \mathcal{W}_{2r}$ which satisfies

$$\begin{cases} c \preceq \lambda, \quad c \preceq \beta \\ \\ \sum_{i=1}^r (\lambda_i - c_i + \beta_i - c_i) = m. \end{cases}$$

In addition, the multiplicity $M_{\lambda,\gamma_m}(\beta)$ of the irreducible representation with highest weight β is the number of $c \in \mathcal{W}_{2r}$ satisfying these relations. Note that $V_\lambda \otimes V_{\gamma_m}$ is not multiplicity-free if $m \notin \{0, 1\}$.

5 Markov kernels

Since for $\lambda \in \mathcal{W}_{2r}$ the Schur function s_λ^{2r} is the character of the irreducible representation V_λ, decomposition (4.1) implies

$$s_\lambda^{2r}(q) s_{\gamma_m}^{2r}(q) = \sum_{\beta \in \mathcal{W}_{2r}} M_{\lambda,\gamma_m}(\beta) s_\beta^{2r}(q).$$

Thus one defines a family $(\mu_m)_{m \geq 0}$ of Markov kernels on $\mathcal{W}_{2r}$ by letting

$$\mu_m(\lambda, \beta) = \frac{s_\beta^{2r}(q)}{s_\lambda^{2r}(q) s_{\gamma_m}^{2r}(q)} M_{\lambda,\gamma_m}(\beta),$$

for $\lambda, \beta \in \mathcal{W}_{2r}$ and $m \geq 0$. Let $\xi_1, \ldots, \xi_{2r}$ be independent random variable such that $\xi_1, \xi_2, \ldots, \xi_r$ are geometric random variables with respective parameters $\alpha q_1, \ldots, \alpha q_r$ and $\xi_r, \xi_{r+1}, \ldots, \xi_{2r}$ are geometric random variables with respective parameters $\alpha q_1^{-1}, \ldots, \alpha q_r^{-1}$. Consider a random variable T on $\mathbb{N}$ defined by

$$T = \sum_{i=1}^{2r} \xi_i.$$

Lemma 5.1. *The law of T is a measure ν on $\mathbb{N}$ defined by*

$$\nu(m) = \alpha^m a(q) s^{2r}_{\gamma_m}(q), \quad m \in \mathbb{N},$$

where

$$a(q) = \prod_{i=1}^{r} (1 - \alpha q_i)(1 - \alpha q_i^{-1}).$$

Proof. The lemma follows from straightforward computations. □

Lemma 5.1 implies in particular that the measure ν is a probability measure. Thus one defines a Markov kernel P_{2r} on $\mathcal{W}_{2r}$ by letting

$$P_{2r}(\lambda, \beta) = \sum_{m=0}^{+\infty} \mu_m(\lambda, \beta)\nu(m),$$

for $\lambda, \beta \in \mathcal{W}_{2r}$.

Proposition 5.2. *For $\lambda, \beta \in \mathcal{W}_{2r}$,*

$$P_{2r}(\lambda, \beta) = \sum_{c \in \mathcal{W}_{2r} : c \preceq \lambda, \beta} a(q) \frac{s^{2r}_{\beta}(q)}{s^{2r}_{\lambda}(q)} \alpha^{\sum_{i=1}^{r} (\lambda_i + \beta_i - 2c_i)}. \tag{5.1}$$

Proof. The proposition follows immediately from the tensor product rules recalled for the decomposition (4.1). □

We let $\mathcal{W}_{2r-1} = \mathcal{W}_{2r}$. For $c_0, \lambda, c, \beta \in \mathcal{W}_k$, we let

$$S_k((c_0, \lambda), (c, \beta)) = a(q) \frac{s^k_{\beta}(q)}{s^k_{\lambda}(q)} \alpha^{\sum_{i=1}^{r} (\lambda_i + \beta_i - 2c_i)} 1_{c \preceq \lambda, \beta}, \tag{5.2}$$

when $k = 2r$ and

$$S_k((c_0, \lambda), (c, \beta)) = \tilde{a}(q) \frac{s^k_{\beta}(q)}{s^k_{\lambda}(q)} \alpha^{\sum_{i=1}^{r} (\lambda_i + \beta_i - 2c_i)} ((1 - \alpha q_r^{-1}) 1_{c_r > 0} + 1_{c_r = 0}) 1_{c \preceq \lambda, \beta}, \tag{5.3}$$

when $k = 2r - 1$, with

$$\tilde{a}(q) = (1 - \alpha q_r) \prod_{i=1}^{r-1} (1 - \alpha q_i)(1 - \alpha q_i^{-1}).$$

The main purpose of this paper is to show that S_k describes the evolution of the k^{th} row of the random symplectic Gelfand-Tsetlin patterns $(X(t), t \geq 0)$. Note that Proposition 5.2 ensures that S_{2r} defines a Markov kernel on $\mathcal{W}_{2r} \times \mathcal{W}_{2r}$. There isn't such an argument for S_{2r-1}. Anyway as L_k and Q_k defined in section 8 are Markov kernels, Proposition 8.1 ensures that S_k is a Markov kernel in both the odd and the even cases. Thus one defines also a Markov kernel P_{2r-1} on $\mathcal{W}_{2r-1}$ by letting

$$P_{2r-1}(\lambda, \beta) = \sum_{c \in \mathcal{W}_{2r} :\, c \preceq \lambda, \beta} \tilde{a}(q) \frac{s^{2r-1}_{\beta}(q)}{s^{2r-1}_{\lambda}(q)} \alpha^{\sum_{i=1}^{r} (\lambda_i + \beta_i - 2c_i)} ((1 - \alpha q_r^{-1}) 1_{c_r > 0} + 1_{c_r = 0}). \tag{5.4}$$

Actually $P_{2r-1}(\lambda, .)$ is the image of the measure $S_{2r-1}((c_0, \lambda), (., .))$, for any arbitrary $c_0 \in \mathcal{W}_{2r}$, by the map

$$(x, y) \in \mathcal{W}_{2r} \times \mathcal{W}_{2r} \mapsto y \in \mathcal{W}_{2r}.$$

We'll see that the image measure P_k describes the evolution of the k^{th} row of the random symplectic Gelfand-Tsetlin patterns observed at integer times. This Markov kernel is relevant for the understanding of the relation between the particle model and the random matrix model of the next section.

6 Random matrices

We denote by $\mathbb{H}$ the set of quaternions. For us, a quaternion is just a 2×2 matrix Z with complex entries which can be written as

$$Z = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$. Its conjugate Z^* is the usual adjoint of the complex matrix Z. Let us denote by $\mathcal{M}_{r,m}$ the real vector space of $r \times m$ matrices with entries in $\mathbb{H}$ and by $\mathcal{P}_r$ the set of $r \times r$ Hermitian matrices with entries in $i\mathbb{H}$. Since a matrix in $\mathcal{P}_r$ is a $2r \times 2r$ Hermitian complex matrix, it has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2r}$. Moreover $\lambda_{2r-i+1} = -\lambda_i$, for $i = 1, \cdots, 2r$. We put on $\mathcal{M}_{r,m}$ the Euclidean structure defined by the scalar product,

$$\langle M, N \rangle = \mathrm{tr}(MN^*), \quad M, N \in \mathcal{M}_{r,m}.$$

Let $\mathcal{C}_r$ be the subset of $\mathbb{R}^r$ defined by

$$\mathcal{C}_r = \{x \in \mathbb{R}^r : x_1 > \cdots > x_r > 0\}.$$

Theorem 4.5 of [3] and Proposition 4.8 of [3] imply the following proposition.

Proposition 6.1. *Let r be a positive integer and $(M(n), n \geq 0)$, be a discrete process on $\mathcal{P}_r$ defined by*

$$M(n) = \sum_{l=1}^{n} Y_l \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y_l^*,$$

where the Y_l's are independent standard Gaussian variables in $\mathcal{M}_{r,1}$. For $n \in \mathbb{N}$, let $\Lambda_1(n), \cdots, \Lambda_r(n)$ be the r largest eigenvalues of $M(n)$ such that

$$\Lambda_1(n) \geq \cdots \geq \Lambda_r(n).$$

Then the process $(\Lambda(n), n \geq 0)$, is a Markov process with a transition densities p_r defined by

$$p_r(x, y) = \frac{d_r(y)}{d_r(x)} I(x, y), \quad x, y \in \mathcal{C}_r,$$

where

$$I(x, y) = \int_{\mathbb{R}_+^r} 1_{\{x, y \succ z\}} e^{-\sum_{i=1}^{r}(x_i + y_i - 2z_i)}\, dz,$$

and

$$d_r(x) = \prod_{1 \leq i < j \leq r} (x_i^2 - x_j^2) \prod_{i=1}^{r} x_i, \quad x, y \in \mathcal{C}_r.$$

7 results

We have introduced a Markov process $(X(t), t \geq 0)$ on the set of symplectic Gelfand-Tsetlin patterns. We will show that if only one row of the pattern is considered by itself, is found to be a Markov process too. Similar results have been proved in [7]. The main specificity of our model is that coordinates of particles are not updated in the same way at every times. We are mainly interested in the process $(X^k(n), n \geq 0)$ but intermediate states $(X^k(n + \frac{1}{2}), n \geq 0)$ are considered for the proofs. Actually these intermediate states come from the fact that the tensor product of (4.1) is not multiplicity-free. We let $Z^k(n) = (X^k(n - \frac{1}{2}), X^k(n))$, for $n \in \mathbb{N}$, with $X^k(-\frac{1}{2}) = 0$.

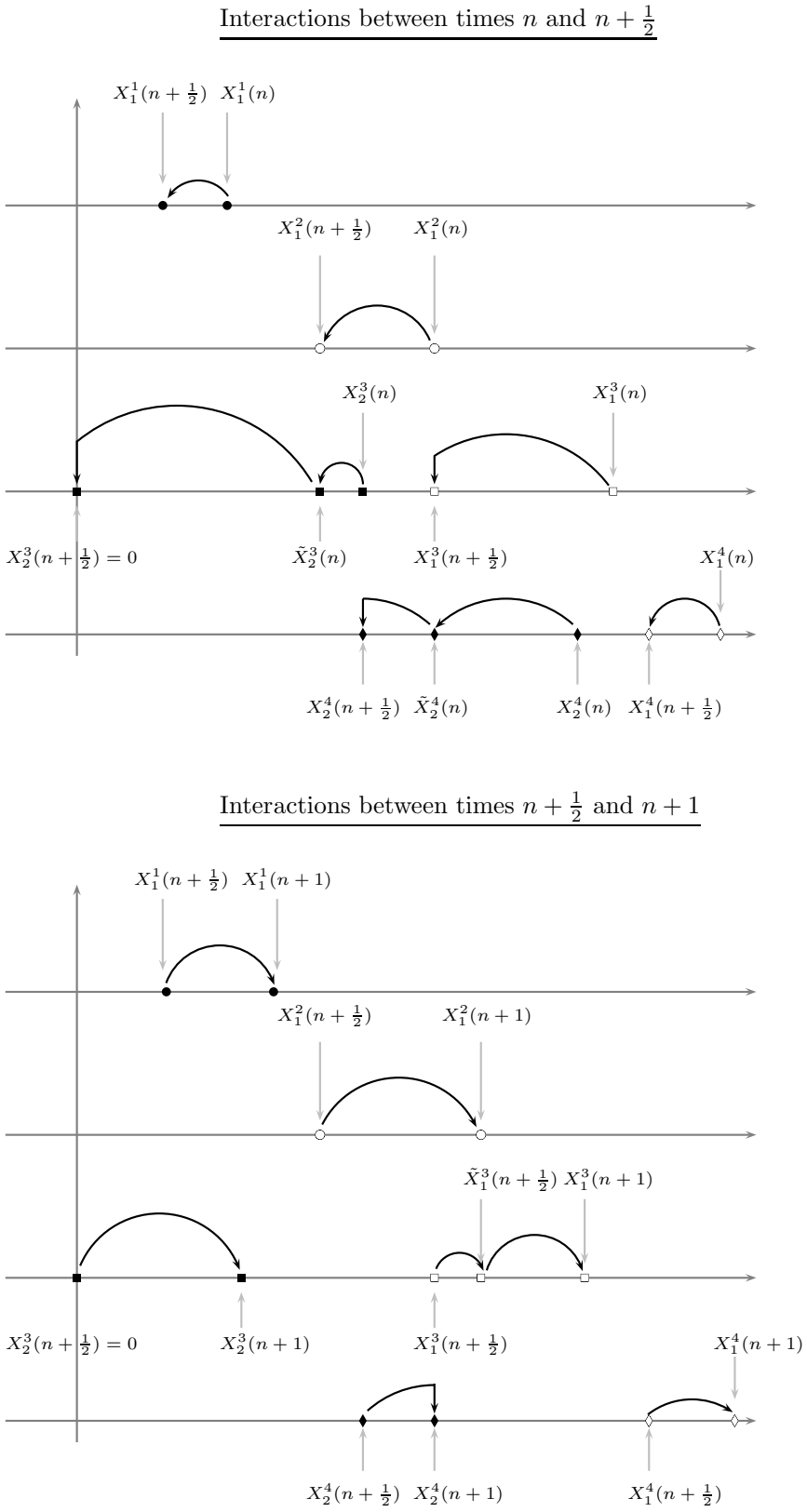

Figure 2: An example of blocking and pushing interactions between times n and $n+1$ for $k=4$. Different kinds of dots represent different particles.

Theorem 7.1. *The process $(Z^k(n), n \geq 0)$ is a Markov process on $\mathcal{W}_k \times \mathcal{W}_k$ with transition kernel S_k.*

If P_k is the Markov kernel defined in (5.1) and (5.4) then Theorem 7.1 implies immediately the following theorem which is our main result.

Theorem 7.2. *The process $(X^k(n))_{n\geq 0}$ is a Markov process on $\mathcal{W}_k$ with transition kernel P_k.*

Convergence stated in Proposition 3.1 and Theorem 7.2 lead to the following corollary, which is exactly Theorem 2.3 of [7]. Let us denote $e_1, \ldots, e_{[\frac{k+1}{2}]}$ the standard basis of $\mathbb{R}^{[\frac{k+1}{2}]}$.

Corollary 7.3. *The process $(Y^k(t), t \geq 0)$ is a Markov process with infinitesimal generator defined by*

$$A(x,y) = \frac{s_y^k(q)}{s_x^k(q)} 1_{y \in \mathcal{W}_k},$$

for $x \in \mathcal{W}_k, y \in \{x + e_1, \ldots, x + e_{[\frac{k+1}{2}]}, x - e_1, \ldots, x - e_{[\frac{k+1}{2}]}\}$.

Proof. Theorem 7.2 and lemma 2.21 of [1] imply that the process

$$(X^k([\alpha^{-1}t]), t \geq 0)$$

converges towards a Markov process with infinitesimal generator equal to A as α goes to zero. The convergence stated in Proposition 3.1 achieves the proof. □

If $(\Lambda(n), n \geq 0)$ is the process of eigenvalues considered in Proposition 6.1 then the following corollary holds.

Corollary 7.4. *Letting $q_i = 1$ for $i = 1, \ldots, r$, the process $((1-\alpha)X^{2r}(n), n \geq 1)$ converges in distribution towards the process of eigenvalues $(\Lambda(n), n \geq 1)$ as α goes to one.*

Proof. The Weyl dimension formula (see Knapp [5], Thm V.5.84) for the symplectic group gives

$$s_\lambda^{2r}(\mathbf{1}) = \prod_{1 \leq i < j \leq r} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n + 2 - j - i)}{(j-i)(2n+2-j-i)} \prod_{i=1}^{r} \frac{\lambda_i + n + 1 - i}{n + 1 - i}.$$

Thus the corollary follows immediately from Theorem 7.2 and Proposition 6.1. □

8 proofs

The proof of Theorem 7.1 rests on the same ingredients as the proof of Proposition 8.8 of [4]. It will follow from an intertwining property stated in Proposition 8.1 and an application of a criterion established in [6] by Pitman and Rogers who give a simple condition sufficient to ensure that a function of a Markov process is again a Markov process. For $\lambda \in \mathcal{W}_k$ we consider the measure M_λ on $GT_k(\lambda)$ defined by

$$M_\lambda = \sum_{x \in GT_k(\lambda)} \frac{w_x^k(q)}{s_\lambda^k(q)} \delta_x,$$

where δ_x is the dirac measure at x, and the measure m_λ defined as the image of the measure M_λ by the map $x \in GT_k(\lambda) \mapsto x^{k-1} \in \mathcal{W}_{k-1}$, i.e

$$m_\lambda = \sum_{\beta \in \mathcal{W}_{k-1} : \beta \preceq \lambda} q_r^{|\beta| - |\lambda|} \frac{s_\beta^{k-1}(q)}{s_\lambda^k(q)} \delta_\beta,$$

when $k = 2r$, and

$$m_\lambda = \sum_{\beta \in \mathcal{W}_{k-1}: \beta \preceq \lambda} q_r^{|\lambda|-|\beta|} \frac{s_\beta^{k-1}(\tilde{q})}{s_\lambda^k(q)} \delta_\beta,$$

when $k = 2r-1$, with $\tilde{q} = (q_1, \dots, q_{r-1})$. One defines a kernel L_k form $\mathcal{W}_k \times \mathcal{W}_k$ to $\mathcal{W}_{k-1} \times \mathcal{W}_k \times \mathcal{W}_k$ by letting

$$L_k((c,\lambda),(\beta,c',\lambda')) = m_\lambda(\beta) 1_{c \preceq \lambda} 1_{c=c', \lambda=\lambda'},$$

for $c, \lambda, c', \lambda' \in \mathcal{W}_k$, $\beta \in \mathcal{W}_{k-1}$.

As $(Z^k(n), n \geq 0)$ is conditionally independent of $(Z^i(t), t \geq 0, i = 1, \dots, k-2)$ given $(Z^{k-1}(t), t \geq 0)$, Theorem 7.1 may be proved by induction on k. The theorem is true for $k = 1$. Suppose that the process $(Z^{k-1}(n), n \geq 0)$ is Markovian with transition kernel S_{k-1}. The dynamic of the model implies that

$$(Z^{k-1}(n), Z^k(n), n \geq 0)$$

is also Markovian. As for any $\lambda \in \mathcal{W}_{k-1}$,

$$S_{k-1}((c_0,\lambda),(.,.))$$

doesn't depend on c_0, it implies that the process

$$((X^{k-1}(n), Z^k(n)), n \geq 0)$$

is also Markovian. Let us denote by Q_k its transition kernel. Proposition 8.1 claims that it satisfies an intertwining property, which implies, using the Rogers and Pitman criterion of [6], that the process $(Z^k(n), n \geq 0)$ is Markovian with transition kernel S_k. Thus the theorem is true for integer k.

Proposition 8.1.

$$L_k Q_k = S_k L_k$$

Proof. For $x, y, z \in \mathcal{W}_k$, we let

$$S_k(x,(y,z)) = S_k((c,x),(y,z)),$$

where c is any vector of $\mathcal{W}_k$ such that $c \preceq x$. Let ξ^+ and ξ^- be two geometric random variables with respective parameters αq_r and αq_r^{-1}, with $r = [\frac{k+1}{2}]$. For $a, b \in \mathbb{R}_+$ we denote by

$$\overset{b\leftarrow}{P}_k(a,.)$$

the law of $\max(a - \xi^+, b)$ when k is even and the law of $\max(a - \xi^-, b)$ when k is odd. We denote by

$$\overset{\rightarrow b}{P}_k(a,.)$$

the law of $\min(a + \xi^-, b)$ when k is even and the law of $\min(a + \xi^+, b)$ when k is odd. We have for $(u,z,y),(x,z',y') \in \mathcal{W}_{k-1} \times \mathcal{W}_k \times \mathcal{W}_k$ such that $u, z \preceq y$ and $x, z' \preceq y'$

$$Q_k((u,z,y),(x,z',y')) = \sum_{v \in \mathcal{W}_{k-1}} S_{k-1}(u,(v,x)) \prod_{i=1}^{[\frac{k+1}{2}]} \overset{u_i\leftarrow}{P}_k(y_i \wedge v_{i-1}, z_i') \times \prod_{i=0}^{[\frac{k+1}{2}]-1} \overset{\rightarrow v_i}{P}_k(z'_{i+1} \vee x_{i+1}, y'_{i+1}), \quad (8.1)$$

with the conventions that $x_{[\frac{k+1}{2}]} = u_{[\frac{k+1}{2}]} = 0$ when k is odd, $v_0 = +\infty$ in the odd and the even cases and the sum running over $v \in \mathcal{W}_{k-1}$ such that $v_i \in \{y'_{i+1}, \dots, x_i \wedge z'_i\}$, for $i \in \{1, \dots, [\frac{k+1}{2}] - 1\}$. Then we write

$$L_k Q_k((z,y),(x,z',y')) = \sum_{u \in \mathcal{W}_k, v \in \mathcal{W}_{k-1}} L_k((z,y),(u,z,y)) S_{k-1}(u,(v,x)) \times \prod_{i=1}^{[\frac{k+1}{2}]} \overleftarrow{P_k}^{u_i}(y_i \wedge v_{i-1}, z'_i) \times \prod_{i=0}^{[\frac{k+1}{2}]-1} \overrightarrow{P_k}^{v_i}(z'_{i+1} \vee x_{i+1}, y'_{i+1}).$$

We get the intertwining by first summing over u and then over v, using respectively identities (4) and (5) of Lemma 8.3 of [4].

□

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