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Optimal Novikov-type criteria for local martingales with jumps

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Abstract

We consider càdlàg local martingales M with initial value zero and jumps larger than a for some a larger than or equal to -1, and prove Novikov-type criteria for the exponential local martingale to be a uniformly integrable martingale. We obtain criteria using both the quadratic variation and the predictable quadratic variation. We prove optimality of the coefficients in the criteria. As a corollary, we obtain a verbatim extension of the classical Novikov criterion for continuous local martingales to the case of local martingales with initial value zero and nonnegative jumps.

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1 Introduction

The motivation of this paper is the question of when an exponential local martingale is a uniformly integrable martingale. Before introducing this problem, we fix our notation and recall some results from stochastic analysis.

Assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions, see [12] for the definition of this and other probabilistic concepts such as being a local martingale, locally integrable, locally square-integrable, and for the quadratic variation and quadratic covariation et cetera. All local martingales considered in the following are assumed to have càdlàg sample paths. For any local martingale M, we say that M has initial value zero if $M_0 = 0$. For any local martingale M with initial value zero, we denote by $[M]$ the quadratic variation of M, that is, the unique increasing càdlàg adapted process with initial value zero such that $\Delta[M] = (\Delta M)^2$ and $M^2 - [M]$ is a local martingale. If M furthermore is locally square integrable, we denote by $\langle M\rangle$ the predictable quadratic variation of M, which is the unique increasing càdlàg predictable process with initial value zero such that $[M] - \langle M\rangle$ is a local martingale.

For any local martingale with initial value zero, there exists by Theorem 7.25 of [2] a unique decomposition $M = M^c + M^d$, where M^c is a continuous local martingale and M^d is a purely discontinuous local martingale, both with initial value zero. Here, we say that a local martingale with initial value zero is purely discontinuous if it has zero quadratic covariation with any continuous local martingale with initial value zero.

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We refer to M^c as the continuous martingale part of M, and refer to M^d as the purely discontinuous martingale part of M.

Let M be a local martingale with initial value zero and $\Delta M \geq -1$. The exponential martingale of M, also known as the Doléans-Dade exponential of M, is the unique càdlàg solution in Z to the stochastic differential equation $Z_t = 1 + \int_0^t Z_{s-} \, \mathrm{d}M_s$, given explicitly as

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M^c]_t\right) \prod_{0<s\leq t} (1 + \Delta M_s) \exp(-\Delta M_s), \tag{1.1}$$

see Theorem II.37 of [12]. Applying Theorem 9.2 of [2], we find that Z always is a local martingale with initial value one. Also, $\mathcal{E}(M)$ is always nonnegative. We wish to understand when $\mathcal{E}(M)$ is a uniformly integrable martingale.

The question of when $\mathcal{E}(M)$ is a uniformly integrable martingale has been considered many times in the litterature, and is not only of theoretical interest, but has several applications in connection with other topics. In particular, exponential martingales are of use in mathematical finance, where checking uniform integrability of a particular exponential martingale can be used to prove absence of arbitrage and obtain equivalent martingale measures for option pricing. For more on this, see [11] or chapters 10 and 11 of [1]. Also, exponential martingales arise naturally in connection with maximum likelihood estimation for stochastic processes, where the likelihood viewed as a stochastic process often is an exponential martingale which is a true martingale, see for example the likelihood for parameter estimation for Poisson processes given in (3.43) of [6] or the likelihood for parameter estimation for diffusion processes given in Theorem 1.12 of [7]. Finally, exponential martingales which are true martingales can be used in the explicit construction of various probabilistic objects, for example solutions to stochastic differential equations, as in Section 5.3.B of [5].

Several sufficient criteria for $\mathcal{E}(M)$ to be a uniformly integrable martingale are known. First results in this regard were obtained by [9] for the case of continuous local martingales. Here, we are interested in the case where the local martingale M is not necessarily continuous. Sufficient criteria for $\mathcal{E}(M)$ to be a uniformly integrable martingale in this case have been obtained by [8], [3], [10], [15] and [4].

We now explain the particular result to be obtained in this paper. In [9], the following result was obtained: If M is a continuous local martingale with initial value zero and $\exp(\frac{1}{2}[M]_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. This criterion is known as Novikov's criterion. We wish to understand whether this result can be extended to local martingales which are not continuous.

In the case with jumps, another process in addition to the quadratic variation process is relevant: the predictable quadratic variation. As noted earlier, the predictable quadratic variation is defined for any locally square-integrable local martingale M with initial value zero, is denoted $\langle M \rangle$, and is the unique càdlàg predictable, increasing and locally integrable process with initial value zero such that $[M] - \langle M \rangle$ is a local martingale, see p. 124 of [12]. For a continuous local martingale M with initial value zero, we have that M always is locally square integrable and $\langle M \rangle = [M]$.

Using the predictable quadratic variation, the following result is demonstrated in Theorem 9 of [11]. Let M be a locally square integrable local martingale with initial value zero and $\Delta M \geq -1$. It then holds that if $\exp(\frac{1}{2}\langle M^c \rangle_\infty + \langle M^d \rangle_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. This is an extension of the classical Novikov criterion of [9] to the case with jumps. [11] also argue in Example 10 that the constants in front of $\langle M^c \rangle$ and $\langle M^d \rangle$ are optimal, although their argument contains a flaw, namely that the formula (28) in that paper does not hold.

In this paper, we specialize our efforts to the case where M has jumps larger than or equal to a for some $a \geq -1$ and prove results of the same type, requiring either that M is a locally square integrable local martingale with initial value zero and that $\exp(\frac{1}{2}\langle M^c\rangle_\infty + \alpha(a)\langle M^d\rangle_\infty)$ is integrable for some $\alpha(a)$, or that M is a local martingale with initial value zero and that $\exp(\frac{1}{2}[M^c]_\infty + \beta(a)[M^d]_\infty)$ is integrable for some $\beta(a)$. For all $a \geq -1$, we identify the optimal value of $\alpha(a)$ and $\beta(a)$, in particular giving an argument circumventing the problems of Example 10 in [11]. Our results are stated as Theorem 2.4 and Theorem 2.5. In particular, we obtain that for local martingales M with initial value zero and $\Delta M \geq 0$, $\mathcal{E}(M)$ is a uniformly integrable martingale if $\exp(\frac{1}{2}[M]_\infty)$ is integrable or if M is locally square integrable and $\exp(\frac{1}{2}\langle M\rangle_\infty)$ is integrable, and we obtain that both the constants in the exponents and the requirement on the jumps of M are optimal. This result is stated as Corollary 2.6 and yields a verbatim extension of the Novikov criterion to local martingales M with initial value zero and $\Delta M \geq 0$.

2 Main results and proofs

For $a > -1$ with $a \neq 0$, we define

$$\alpha(a) = \frac{(1+a)\log(1+a) - a}{a^2} \quad \text{and} \tag{2.1}$$

$$\beta(a) = \frac{(1+a)\log(1+a) - a}{a^2(1+a)}, \tag{2.2}$$

and put $\alpha(0) = \beta(0) = \frac{1}{2}$ and $\alpha(-1) = 1$. The functions α and β will yield the optimal constants in the criteria we will be demonstrating.

Before proving our main results, Theorem 2.4 and Theorem 2.5, we state three lemmas. Lemma 2.1 follows by simple calculus, Lemma 2.2 by the properties of Poisson processes, and Lemma 2.3 is a consequence of the optional sampling theorem for nonnegative supermartingales. Also, in the proof of Theorem 2.4, note that for a standard Poisson process N, it holds that with $M_t = N_t - t$, we have $\langle M\rangle_t = t$, since $[M]_t = N_t$ by Definition VI.37.6 of [14] and since $\langle M\rangle$ is the unique predictable and locally integrable increasing process making $[M] - \langle M\rangle$ a local martingale.

Lemma 2.1. *The functions α and β are continuous, positive and strictly decreasing. Furthermore, $\beta(a)$ tends to infinity as a tends to minus one.*

Lemma 2.2. *Let N be a standard Poisson process, let b and λ be in $\mathbb{R}$, and define $f_b(\lambda) = \exp(-\lambda) + \lambda(1+b) - 1$. With $L^b_t = \exp(-\lambda(N_t - (1+b)t) - tf_b(\lambda))$, L^b is a nonnegative martingale with respect to the filtration induced by N.*

Lemma 2.3. *Let M be a local martingale with initial value zero and $\Delta M \geq -1$. Then $\mathcal{E}(M)$ is almost surely convergent, $E\mathcal{E}(M)_\infty \leq 1$, and $\mathcal{E}(M)$ is a uniformly integrable martingale if and only if $E\mathcal{E}(M)_\infty = 1$.*

Theorem 2.4. *Fix $a \geq -1$. Let M be a locally square integrable local martingale with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$. If $\exp(\frac{1}{2}\langle M^c\rangle_\infty + \alpha(a)\langle M^d\rangle_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. Furthermore, for all $a \geq -1$, the coefficients $\frac{1}{2}$ and $\alpha(a)$ in front of $\langle M^c\rangle_\infty$ and $\langle M^d\rangle_\infty$ are optimal in the sense that the criterion is false if any of the coefficients are reduced.*

Proof. Sufficiency. With $h(x) = (1+x)\log(1+x) - x$, we find by Lemma 2.1 that for $-1 \leq a \leq x$, $\alpha(a) \geq \alpha(x)$, which implies $h(x) \leq \alpha(a)x^2$. Letting $a \geq -1$ and letting M be a locally square integrable local martingale with initial value zero, $\Delta M 1_{(\Delta M \neq 0)} \geq a$ and $\exp(\frac{1}{2}\langle M^c\rangle_\infty + \alpha(a)\langle M^d\rangle_\infty)$ integrable, we obtain that for all $t \geq 0$, the inequality

$(1+\Delta M_t)\log(1+\Delta M_t)-\Delta M_t \le \alpha(a)(\Delta M_t)^2$ holds, and so Theorem III.1 of [8] shows that $\mathcal{E}(M)$ is a uniformly integrable martingale. Thus, the condition is sufficient.

As regards optimality of the coefficients, optimality of the coefficient $\frac{1}{2}$ in front of $\langle M^c\rangle$ is well-known, see [9]. It therefore suffices to to prove optimality of the coefficient $\alpha(a)$ in front of $\langle M^d\rangle$. To do so, we need to show the following: That for each positive ε, there exists a locally square integrable local martingale with initial value zero and $\Delta M 1_{(\Delta M\neq 0)} \ge a$ such that $\exp(\frac{1}{2}\langle M^c\rangle_\infty + (1-\varepsilon)\alpha(a)\langle M^d\rangle_\infty)$ is integrable, while $\mathcal{E}(M)$ is not a uniformly integrable martingale.

The case $a>0$. Let $\varepsilon, b>0$, put $T_b=\inf\{t\ge 0 \mid N_t-(1+b)t=-1\}$ and define M by $M_t = a(N_t^{T_b} - t\wedge T_b)$. We claim that we may choose $b>0$ such that M satisfies the requirements stated above. It holds that M is a locally square integrable local martingale with initial value zero and $\Delta M 1_{(\Delta M\neq 0)}\ge a$, and M is purely discontinuous by Definition 7.21 of [2] since it is of locally integrable variation. In particular, we have $M^c=0$, so it suffices to show that $\exp((1-\varepsilon)\alpha(a)\langle M\rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale.

To show this, we first argue that T_b is almost surely finite. To this end, note that since $t\mapsto N_t-(1+b)t$ only has nonnegative jumps, has initial value zero and decreases between jumps, the process hits -1 if and only if it is less than or equal to -1 immediately before one of its jumps. Therefore, with U_n denoting the n'th jump time of N, we have

$$\begin{aligned} P(T_b=\infty) &= P(\cap_{n=1}^\infty (N_{U_n -} - (1+b)U_n > -1)) \\ &\le P(\limsup_{n\to\infty} U_n/n \le (1+b)^{-1}), \end{aligned} \tag{2.3}$$

which is zero, as $\lim_{n\to\infty} U_n/n = 1$ almost surely by the law of large numbers, and $(1+b)^{-1}<1$ as $b>0$. Thus, T_b is almost surely finite, and by the path properties of N, $N_{T_b}=(1+b)T_b-1$ almost surely. We then obtain

$$\begin{aligned} \mathcal{E}(M)_\infty &= \exp(a(N_{T_b}-T_b) + N_{T_b}(\log(1+a)-a)) \\ &= (1+a)^{-1}\exp(T_b((1+b)\log(1+a)-a)). \end{aligned} \tag{2.4}$$

Recalling Lemma 2.3, we wish to choose $b>0$ with the property that $E\mathcal{E}(M)_\infty<1$ and $E\exp((1-\varepsilon)\alpha\langle M\rangle_\infty)<\infty$ holds simultaneously. Note that $\langle M\rangle_\infty = a^2T_b$. Therefore, we need to select a positive b with the properties that

$$E\exp(T_b((1+b)\log(1+a)-a)) < 1+a \text{ and} \tag{2.5}$$

$$E\exp(T_b a^2(1-\varepsilon)\alpha(a)) < \infty. \tag{2.6}$$

Consider some $b>0$ and let f_b be as in Lemma 2.2. By that same lemma, the process L^b defined by putting $L^b_t = \exp(-\lambda(N_t-(1+b)t)-tf_b(\lambda))$ is a martingale. In particular, it is a nonnegative supermartingale with initial value one, so Theorem II.77.5 of [13] yields $1\ge EL^b_{T_b} = E\exp(\lambda - T_b f_b(\lambda))$, and so $E\exp(-T_bf_b(\lambda))\le\exp(-\lambda)$. Note that $f_b'(\lambda) = -\exp(-\lambda)+1+b$, such that $-f_b$ takes its maximum at $-\log(1+b)$, and the maximum is $h(b)$. In particular, $E\exp(T_bh(b))$ is finite. Next, define a function λ by putting $\lambda(b) = -\log((1+a)\frac{b}{a})$, we then have $E\exp(-T_bf_b(\lambda(b)))\le(1+a)\frac{b}{a}$, which is strictly less than $1+a$ whenever $b<a$. Thus, if we can choose $b\in(0,a)$ such that

$$(1+b)\log(1+a)-a \le -f_b(\lambda(b)) \text{ and} \tag{2.7}$$

$$a^2(1-\varepsilon)\alpha(a)\le h(b), \tag{2.8}$$

we will have achieved our end, since (2.7) implies (2.5) and (2.8) implies (2.6). By elementary rearrangements, we find that (2.7) is satisfied for all $b\in(0,a)$. Thus, it suffices to choose $b \in (0,a)$ such that (2.8) is satisfied, corresponding to the choice of a $b \in (0,a)$ such that $(1-\varepsilon)h(a) \le h(b)$. As h is positive and continuous on $(0,\infty)$, this is possible by choosing b close enough to a. With this choice of b, we now obtain M yielding an example proving that the coefficient $\alpha(a)$ is optimal.

The case $a = 0$. Let $\varepsilon > 0$. To prove optimality, we wish to identify a locally square integrable local martingale M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \ge 0$ such that $\exp((1-\varepsilon)\alpha(0)\langle M\rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale. Recalling that α is positive and continuous, we may pick $a > 0$ so close to zero that $(1-\varepsilon)\alpha(0) \le (1-\frac{1}{2}\varepsilon)\alpha(a)$. We may then use what already was shown for $a > 0$ to obtain that $\alpha(0)$ is optimal.

The case $-1 < a < 0$. Let $\varepsilon > 0$, let $-1 < b < 0$, let $c > 0$ and define a stopping time T_{bc} by putting $T_{bc} = \inf\{t \ge 0 \mid N_t - (1+b)t \ge c\}$. Also define M by $M_t = a(N_t^{T_{bc}} - t \wedge T_{bc})$. We claim that we can choose $b \in (-1,0)$ and $c > 0$ such that M satisfies the requirements to show optimality. Similarly to the case $a > 0$, M is a purely discontinuous locally square integrable local martingale with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \ge a$, so it suffices to show that $\exp((1-\varepsilon)\alpha(a)\langle M\rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale. We first investigate some properties of T_{bc}. As $t \mapsto N_t - (1+b)t$ only has nonnegative jumps, has initial value zero and decreases between jumps, the process advances beyond c at some point if and only it advances beyond c at one of its jump times. Therefore, with U_n denoting the n'th jump time of N,

$$\begin{aligned} P(T_{bc} = \infty) &= P(\cap_{n=1}^\infty (N_{U_n} - (1+b)U_n < c)) \\ &\le P(\liminf_{n\to\infty} U_n/n \ge (1+b)^{-1}), \end{aligned} \tag{2.9}$$

which is zero, as U_n/n tends to one almost surely and $(1+b)^{-1} > 1$. Thus, T_{bc} is almost surely finite. Furthermore, by the path properties of N, $N_{T_{bc}} \ge (1+b)T_{bc} + c$ and $N_{T_{bc}} \le (1+b)T_{bc} + c + 1$ almost surely. Since $\log(1+a) \le 0$, we in particular obtain $N_{T_{bc}} \log(1+a) \le ((1+b)T_{bc} + c)\log(1+a)$ almost surely. From this, we conclude that

$$\begin{aligned} \mathcal{E}(M)_\infty &= \exp(a(N_{T_{bc}} - T_{bc}) + N_{T_{bc}}(\log(1+a) - a)) \\ &\le \exp(((1+b)T_{bc} + c)\log(1+a) - aT_{bc}) \\ &= (1+a)^c \exp(T_{bc}((1+b)\log(1+a) - a)). \end{aligned} \tag{2.10}$$

We wish to choose $-1 < b < 0$ and $c > 0$ such that $E\exp((1-\varepsilon)\alpha(a)\langle M\rangle_\infty) < \infty$ and $E\mathcal{E}(M)_\infty < 1$ holds simultaneously. As $\langle M\rangle_\infty = a^2 T_{bc}$, this is equivalent to choosing $-1 < b < 0$ and $c > 0$ such that

$$E\exp(T_{bc}((1+b)\log(1+a) - a)) < (1+a)^{-c} \text{ and} \tag{2.11}$$

$$E\exp(T_{bc}a^2(1-\varepsilon)\alpha(a)) < \infty. \tag{2.12}$$

Let f_b and L^b be as in Lemma 2.2. The process L^b is then a nonnegative supermartingale. As $N_{T_{bc}} \le (1+b)T_{bc} + c + 1$, the optional stopping theorem allows us to conclude that for $\lambda \ge 0$,

$$\begin{aligned} 1 \ge EL^b_{T_{bc}} &= E\exp(-\lambda(N_{T_{bc}} - (1+b)T_{bc}) - T_{bc}f_b(\lambda)) \\ &\ge E\exp(-(c+1)\lambda - T_{bc}f_b(\lambda)), \end{aligned} \tag{2.13}$$

so that $E\exp(-T_{bc}f_b(\lambda)) \le \exp((c+1)\lambda)$. As in the case $a > 0$, $-f_b$ takes its maximum at $-\log(1+b)$, and the maximum is $h(b)$, leading us to conclude that $E\exp(T_{bc}h(b))$ is finite. Put $\lambda(b,c) = (c+1)^{-1}\log((1+a)^{-c}\frac{b}{a})$. For all $b \in (a,0)$, $\frac{b}{a} < 1$, leading to $E\exp(-T_{bc}f_b(\lambda(b,c))) \le (1+a)^{-c}\frac{b}{a} < (1+a)^{-c}$. Therefore, if we can choose $b \in (a,0)$

and $c > 0$ such that

$$(1+b)\log(1+a) - a \leq -f_b(\lambda(b,c)) \text{ and} \tag{2.14}$$

$$a^2(1-\varepsilon)\alpha(a) \leq h(b), \tag{2.15}$$

we will have obtained existence of a local maringale yielding the desired optimality of $\alpha(a)$. We first note that $a^2(1-\varepsilon)\alpha(a) \leq h(b)$ is equivalent to $(1-\varepsilon)h(a) \leq h(b)$. As h is continuous and positive on $(-1,0)$, we find that (2.15) is satisfied for $a < b < 0$ with b close enough to a. Next, we turn our attention to (2.14). Elementary rearrangements show that (2.14) is equivalent to

$$0 \leq 1 + a - (1+a)^{\frac{c}{c+1}} \left(\frac{a}{b}\right)^{\frac{1}{c+1}} + \frac{1+b}{c+1}\left(\log\frac{a}{b} - \log(1+a)\right). \tag{2.16}$$

Fixing $a < b < 0$, we wish to argue that for b close enough to a, (2.16) holds for c large enough. To this end, let $\rho_b(c)$ denote the right-hand side of (2.16). It then holds that $\lim_{c\to\infty}\rho_b(c) = 0$. Also, we have

$$\rho_b'(c) = \frac{\log(1+a) - \log\frac{a}{b}}{(c+1)^2}\left(1 + b - \exp\left(\frac{c}{c+1}\log(1+a) + \frac{1}{c+1}\log\frac{a}{b}\right)\right). \tag{2.17}$$

Now note that for $a < b$, we obtain

$$\lim_{c\to\infty} 1 + b - \exp\left(\frac{c}{c+1}\log(1+a) + \frac{1}{c+1}\log\frac{a}{b}\right) = 1 + b - (1+a) > 0, \tag{2.18}$$

and for b close enough to a, $\log(1+a) - \log\frac{a}{b} < 0$, since $a < 0$. Therefore, for all c large enough, $\rho_b'(c) < 0$. For any such c, we obtain $\rho_b(c) > 0$. Thus, we conclude that for b close enough to a, it holds that $\rho_b(c) > 0$ for c large enough.

We now collect our conclusions in order to obtain $b \in (a,0)$ and $c > 0$ satisfying (2.14) and (2.15). First choose $b \in (a,0)$ so close to a that the inequalities $(1-\varepsilon)h(a) \leq h(b)$ and $\log(1+a) - \log\frac{a}{b} < 0$ hold. Pick c so large that $\rho_b(c) > 0$. By our deliberations, (2.14) and (2.15) then both hold, demonstrating the existence of a locally square integrable local martingale M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$ such that integrability of $\exp((1-\varepsilon)\alpha(a)\langle M\rangle_\infty)$ holds while $\mathcal{E}(M)$ is not a uniformly integrable martingale.

The case $a = -1$. This follows similarly to the case $a = 0$. □

Theorem 2.5. *Fix $a > -1$. Let M be a local martingale with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$. If $\exp(\frac{1}{2}[M^c]_\infty + \beta(a)[M^d]_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. Furthermore, for all $a > -1$, the coefficients $\frac{1}{2}$ and $\beta(a)$ in front of $[M^c]_\infty$ and $[M^d]_\infty$ are optimal in the sense that the criterion is false if any of the coefficients are reduced. Also, there exists no $\beta(-1)$ such that for M with $\Delta M 1_{(\Delta M \neq 0)} \geq -1$, integrability of $\exp(\frac{1}{2}[M^c]_\infty + \beta(-1)[M^d]_\infty)$ suffices to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale.*

Proof. Sufficiency. We proceed in a manner closely related to the proof of Theorem 2.4. Defining g by putting $g(x) = \log(1+x) - x/(1+x)$, Lemma 2.1 yields that for $-1 < a \leq x$, $\beta(a) \geq \beta(x)$, yielding $g(x) \leq \beta(a)x^2$. Letting $a > -1$ and letting M be a local martingale with initial value zero, $\Delta M 1_{(\Delta M \neq 0)} \geq a$ and $\exp(\frac{1}{2}[M^c]_\infty + \beta(a)[M^d]_\infty)$ integrable, we obtain for all $t \geq 0$ the inequality $\log(1+\Delta M_t) - \Delta M_t/(1+\Delta M_t) \leq \beta(a)(\Delta M_t)^2$, and so Theorem III.7 of [8] shows that $\mathcal{E}(M)$ is a uniformly integrable martingale. Thus, the condition is sufficient. As regards optimality, as in Theorem 2.4, optimality of the $\frac{1}{2}$ in front of $[M^c]$ follows from [9], so it suffices to consider the coefficient $\beta(a)$ in front of $[M^d]$, which we will do with methods similar to those employed in the proof of Theorem 2.4.

The case $a > 0$. Let $\varepsilon, b > 0$, put $T_b = \inf\{t \ge 0 \mid N_t - (1+b)t = -1\}$ and define $M_t = a(N_t^{T_b} - t \wedge T_b)$. Noting that $[M]_\infty = a^2 N_{T_b}$, we may argue as in the proof of Theorem 2.4 and obtain that it suffices to identify $b > 0$ such that

$$E\exp(T_b((1+b)\log(1+a) - a)) < 1 + a \text{ and} \tag{2.19}$$

$$E\exp(N_{T_b} a^2 (1-\varepsilon)\beta(a)) < \infty. \tag{2.20}$$

Let f_b be as in Lemma 2.2. As in the proof of Theorem 2.4, we obtain that $E\exp(T_b h(b))$ is finite, where $h(x) = (1+x)\log(1+x) - x$, and with $\lambda(b) = -\log((1+a)\frac{b}{a})$, it holds that $E\exp(-T_b f_b(\lambda(b))) < 1 + a$ for $b < a$. As $N_{T_b} = (1+b)T_b - 1$ almost surely and $g(b) = h(b)/(1+b)$, we then also obtain that $E\exp(N_{T_b} g(b))$ is finite. Thus, if we can choose $b \in (0, a)$ such that

$$(1+b)\log(1+a) - a \le -f_b(\lambda(b)) \text{ and} \tag{2.21}$$

$$a^2(1-\varepsilon)\beta(a) \le g(b), \tag{2.22}$$

we will obtain the desired result, as (2.21) implies (2.19) and (2.22) implies (2.20). As earlier noted, (2.21) always holds for $0 < b < a$. As for (2.22), this requirement is equivalent to having that $(1-\varepsilon)g(a) \le g(b)$ for some $b \in (0, a)$, which by continuity of g can be obtained by choosing b close enough to a. Choosing b in this manner yields the result.

The case $a = 0$. This follows as in the proof of Theorem 2.4.

The case $-1 < a < 0$. Let $\varepsilon > 0$, let $-1 < b < 0$, let $c > 0$ and define a stopping time T_{bc} by putting $T_{bc} = \inf\{t \ge 0 \mid N_t - (1+b)t \ge c\}$. Further define $M_t = a(N_t^{T_{bc}} - t \wedge T_{bc})$. As in the case $a > 0$, we may apply the same methods as in the proof of Theorem 2.4 and obtain that in order to obtain the desired result, it suffices to identify $b \in (a, 0)$ and $c > 0$ such that

$$(1+b)\log(1+a) - a \le -f_b(\lambda(b, c)) \text{ and} \tag{2.23}$$

$$a^2(1-\varepsilon)\beta(a) \le g(b), \tag{2.24}$$

By arguments as in the proof of the corresponding case of Theorem 2.4, we find that by first picking b close enough to a and then c large enough, we can ensure that both (2.23) and (2.24) hold, yielding optimality for this case.

The case $a = -1$. For this case, we need to show that for any $\gamma \ge 0$, it does not hold that finiteness of $E\exp(\gamma[M^d]_\infty)$ implies that $\mathcal{E}(M)$ is a uniformly integrable martingale. Let $\gamma \ge 0$. By Lemma 2.1, $\beta(a)$ tends to infinity as a tends to -1. Therefore, we may pick $a > -1$ so small that $\beta(a) \ge \gamma$. By what we already have shown, there exists M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \ge -1$ such that $E\exp(\beta(a)[M^d]_\infty)$ and thus $E\exp(\gamma[M^d]_\infty)$ is finite, while $\mathcal{E}(M)$ is not a uniformly integrable martingale. □

Corollary 2.6. *Let M be a local martingale with initial value zero and $\Delta M \ge 0$. If $\exp(\frac{1}{2}[M]_\infty)$ is integrable or if M is locally square integrable and $\exp(\frac{1}{2}\langle M\rangle_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. Furthermore, this criterion is optimal in the sense that if either the constant $\frac{1}{2}$ is reduced, or the requirement on the jumps is weakened to $\Delta M \ge -\varepsilon$ for some $\varepsilon > 0$, the criterion ceases to be sufficient.*

Proof. This follows by combining Theorem 2.4 and Theorem 2.5 with the fact that α and β both are strictly decreasing by Lemma 2.1. □

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