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Lower bounds for the probability of a union via chordal graphs

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Abstract

We establish new Bonferroni-type lower bounds for the probability of a union of finitely many events where the selection of intersections in the estimates is determined by the clique complex of a chordal graph.

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1 Introduction

The classical Bonferroni inequalities state that for any finite collection of events $\{A_v\}_{v\in V}$,

$$\begin{aligned} \Pr\left(\bigcup_{v\in V} A_v\right) &\le \sum_{\substack{I\in\mathscr{P}^*(V)\\ |I|\le 2r-1}} (-1)^{|I|-1}\Pr\left(\bigcap_{i\in I} A_i\right) \\ \Pr\left(\bigcup_{v\in V} A_v\right) &\ge \sum_{\substack{I\in\mathscr{P}^*(V)\\ |I|\le 2r}} (-1)^{|I|-1}\Pr\left(\bigcap_{i\in I} A_i\right) \end{aligned} \qquad (r=1,2,3,\dots), \tag{1.1}$$

where $\mathscr{P}^*(V)$ denotes the set of non-empty subsets of V. Numerous variants of these inequalities are known; see, e.g., Galambos and Simonelli [6] for a survey on *Bonferroni-type* inequalities, which are variants of (1.1) that are applicable to any finite family of events. More recent variants arising from abstract tubes [3, 8] and monomial ideals [11] take advantage of the underlying structure of events, thus reducing the sum in (1.1) to a subcomplex of $\mathscr{P}^*(V)$, while providing provably tighter bounds at all truncations.

This short note is inspired by both lines of research. Our main result is a variant of the lower bound in (1.1), which is applicable to any finite family of events, and where the selection of intersections in the estimates is determined by the clique complex of a chordal graph.

We refer to [2] for terminology on graphs. We write $G=(V,E)$ to denote that G is a graph having vertex-set V, which we assume to be finite, and edge-set E, consisting of two-element subsets of V. We use $\mathscr{C}(G)$ to denote the *clique complex* of G, that is, the abstract simplicial complex consisting of all non-empty cliques of G. A graph is

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referred to as *chordal* if it contains no cycles of length four or any higher length as induced subgraphs.

Our main result in Section 2 complements the upper bound in Proposition 1.1 below, which interpolates between Boole's inequality (G edgeless or $r = 1$), Hunter's inequality [7] (G a tree and $r \geq 2$), and the sieve formula (G complete and $r \geq (|V| + 1)/2$). For $r = 2$, the bound in Proposition 1.1 was independently found by Boros and Veneziani [1] by linear programming techniques; see also [10] for a discussion on this particular case.

Proposition 1.1 ([3, 4])**.** *Let $\{A_v\}_{v\in V}$ be a finite collection of events, where the indices form the vertices of a chordal graph G. Then,*

$$\Pr\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in\mathscr{C}(G)\\ |I|\leq 2r-1}} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right) \quad (r = 1, 2, 3, \ldots). \tag{1.2}$$

2 Main result

We use $c(G)$ resp. $\alpha(G)$ to denote the number of connected components resp. the independence number of G. Our main result is a lower bound analogue of (1.2).

Theorem 2.1. *Let $\{A_v\}_{v\in V}$ be a non-empty finite collection of events, where the indices form the vertices of a chordal graph G. Then,*

$$\Pr\left(\bigcup_{v\in V} A_v\right) \geq \frac{1}{\alpha(G)} \sum_{\substack{I\in\mathscr{C}(G)\\ |I|\leq 2r}} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right) \quad (r = 1, 2, 3, \ldots). \tag{2.1}$$

Remark 2.2. *Due to (1.2) the upper bound in (2.1) is non-negative for $r \geq |V|/2$. If G is complete, (2.1) coincides with the classical Bonferroni lower bounds.*

The proof of Theorem 2.1 is facilitated by the following proposition.

Proposition 2.3. *For any chordal graph $G = (V, E)$,*

$$\sum_{\substack{I\in\mathscr{C}(G)\\ |I|\leq 2r}} (-1)^{|I|-1} \leq c(G) \quad (r = 1, 2, 3, \ldots), \tag{2.2}$$

with equality if $r \geq |V|/2$.

Proof. If G is connected, the result follows from the following topological results:

i) The clique complex of any connected chordal graph is contractible ([5]).

ii) For any contractible abstract simplicial complex, the Euler characteristic of its $(2r-1)$-skeleton is at most 1, with equality if $r \geq |V|/2$ ([8]).

If G is disconnected, apply the inequality to each of its connected components. □

Proof of Theorem 2.1. For any $J \subseteq V$, $J \neq \emptyset$, define

$$B_J := \bigcap_{i\in J} A_i \cap \bigcap_{i\notin J} \overline{A_i}.$$

Note that the B_J's form a partition of $\bigcup_{v\in V} A_v$. Clearly, $G[J]$ is chordal and $c(G[J]) \leq \alpha(G)$ for any $J \subseteq V$, $J \neq \emptyset$. Hence, by applying Proposition 2.3 to $G[J]$ we obtain

$$\begin{aligned}\Pr\left(\bigcup_{v\in V} A_v\right) = \Pr\left(\bigcup_{\substack{J\subseteq V\\ J\neq\emptyset}} B_J\right) &= \sum_{\substack{J\subseteq V\\ J\neq\emptyset}} \Pr(B_J) \geq \frac{1}{\alpha(G)} \sum_{\substack{J\subseteq V\\ J\neq\emptyset}} c(G[J]) \Pr(B_J) \\ &\geq \frac{1}{\alpha(G)} \sum_{\substack{J\subseteq V\\ J\neq\emptyset}} \sum_{\substack{I\in\mathscr{C}(G[J])\\ |I|\leq 2r}} (-1)^{|I|-1} \Pr(B_J) = \frac{1}{\alpha(G)} \sum_{\substack{I\in\mathscr{C}(G)\\ |I|\leq 2r}} (-1)^{|I|-1} \sum_{J\supseteq I} \Pr(B_J) \\ &= \frac{1}{\alpha(G)} \sum_{\substack{I\in\mathscr{C}(G)\\ |I|\leq 2r}} (-1)^{|I|-1} \Pr\left(\bigcup_{J\supseteq I} B_J\right) = \frac{1}{\alpha(G)} \sum_{\substack{I\in\mathscr{C}(G)\\ |I|\leq 2r}} (-1)^{|I|-1} \Pr\left(\bigcap_{i\in I} A_i\right). \square\end{aligned}$$

Remark 2.4. *In view of the preceding proof and Proposition 2.3 the best bound in (2.1) is obtained for* $r \geq |V|/2$. *Tighter bounds can be obtained by replacing* $\alpha(G)$ *by* $\alpha'(G) := \max\{c(G[J]) \mid \emptyset \neq J \subseteq V, B_J \neq \emptyset\}$. *In particular, if* $G[J]$ *is connected whenever* $B_J \neq \emptyset$ *for any choice of* $J \subseteq V$, $J \neq \emptyset$, *then* $\alpha'(G) = 1$. *In this case, for* $r \geq \frac{|V|+1}{2}$ *the inequalities in (1.2) and (2.1) turn into an identity, which is well-known in abstract tube theory [3].*

Remark 2.5. *The requirement that* G *is chordal cannot be omitted from Theorem 2.1. Consider the non-chordal graph* $G = (V, E)$ *depicted in Figure 1, and consider events* A_v, $v \in V$, *with* $\Pr(A_v) = 1$ *for any* $v \in V$. *The clique complex of this graph consists of 8 cliques of size 1, 20 cliques of size 2, and 16 cliques of size 3. Since* $\alpha(G) = 3$, *the first inequality in Theorem 2.1 gives the non-valid bound* $1 \geq \frac{1}{3}(8 - 20 + 16) = \frac{4}{3}$ *for* $r \geq 2$.

This graph G *is a subgraph of* G_3 *— the first graph in an infinite sequence* $(G_k)_{k=3,5,\ldots}$ *of non-chordal graphs for which (2.1) does not hold. Each* G_k *is the join of* k *disjoint copies of the edgeless graph on three vertices. By letting* $r \geq \frac{3k}{2}$ *and* $\Pr(A_v) = 1$ *for any vertex* v *of* G_k, *(2.1) specializes to* $1 \geq \frac{1}{3}(1 + 2^k)$, *which does not hold for any* $k \geq 3$.

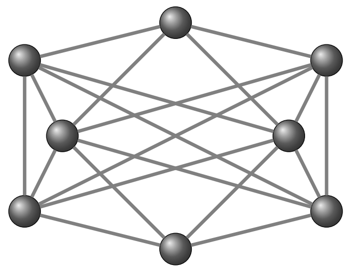

Figure 1: A non-chordal graph for which (2.1) does not hold.

3 Particular cases

Similar to [3], where upper bounds are derived from (1.2) for different choices of G, lower bound analogues of these bounds can be derived from (2.1). As examples we consider lower bound analogues of Hunter's inequality [7] and Seneta's inequality [9].

Corollary 3.1. *Let* $\{A_v\}_{v\in V}$ *be a non-empty finite collection of events, where the indices form the vertices of a tree* $G=(V,E)$*. Then,*

$$\Pr\left(\bigcup_{v\in V} A_v\right) \geq \frac{1}{\alpha(G)}\left(\sum_{v\in V}\Pr(A_v) - \sum_{\{v,w\}\in E}\Pr(A_v\cap A_w)\right).$$

Proof. Since any tree is chordal, the result follows from Theorem 2.1 with $r\geq 2$. □

Corollary 3.2. *Let* $A_1,\dots,A_n$ *be a finite collection of events, and* $j,k\in\{1,\dots,n\}$*. Then,*

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \geq \frac{1}{n-2+\delta_{jk}}\left(\sum_{i=1}^{n}\Pr(A_i) - \sum_{\substack{i=1\\ i\neq j}}^{n}\Pr(A_i\cap A_j) - \sum_{\substack{i=1\\ i\neq j,k}}^{n}\Pr(A_i\cap A_k) + \sum_{\substack{i=1\\ i\neq j,k}}^{n}\Pr(A_i\cap A_j\cap A_k)\right),$$

provided $n>2-\delta_{jk}$ *where* δ_{jk} *denotes the Kronecker delta.*

Proof. Define $G=K_{2-\delta_{jk}}*L_{n-2+\delta_{jk}}$ (the join of a complete graph on $2-\delta_{jk}$ vertices and an edgeless graph on $n-2+\delta_{jk}$ vertices), and apply Theorem 2.1 with $r\geq|V|/2$. □

References

[1] Boros E. and Veneziani P.: Bounds of degree 3 for the probability of the union of events. *Rutcor Research Report* 3-02, 2002.

[2] Diestel R.: Graph Theory. Graduate Texts in Mathematics, 3rd edition. *Springer-Verlag*, New York, 2005. MR-2159259

[3] Dohmen K.: Improved Bonferroni Inequalities via Abstract Tubes. Lecture Notes in Mathematics, No. 1826. *Springer-Verlag*, Berlin-Heidelberg, 2003. MR-2019293

[4] Dohmen K.: Bonferroni-type inequalities via chordal graphs. *Combin. Probab. Comput.* **11**, (2002), 349–351. MR-1918721

[5] Edelman P.H. and Reiner V.: Counting the interior points of a point configuration. *Discrete Comput. Geom.* **23**, (2000), 1–13. MR-1727120

[6] Galambos J. and Simonelli I.: Bonferroni-type Inequalities with Applications. *Springer-Verlag*, New York, 1996. MR-1402242

[7] Hunter D.: An upper bound for the probability of a union. *J. Appl. Prob.* **13**, (1976), 597–603. MR-0415722

[8] Naiman D.Q. and Wynn H.P.: Abstract tubes, improved inclusion-exclusion identities and inequalities and importance sampling. *Ann. Statist.* **25**, (1997), 1954–1983. MR-1474076

[9] Seneta E.: Degree, iteration and permutation in improving Bonferroni-type bounds. *Austral. J. Statist.* **30A**, (1988), 27–38.

[10] Veneziani P.: Upper bounds of degree 3 for the probability of the union of events via linear programming. *Discrete Appl. Math.* **157**, (2009), 858–863. MR-2499501

[11] Sáenz-de-Cabezón E. and Wynn H.P.: Betti numbers and minimal free resolutions for multi-state system reliability bounds. *J. Symb. Comp.* **44**, (2009), 1311–1325. MR-2532174

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