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# An approximation scheme of stochastic Stokes equations*

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**Abstract**

This work is concerned with the approximation to the solutions of the stochastic Stokes equations by the splitting up method. We apply the resolvent operator to evaluate the solution of the deterministic equations at the endpoints of every small interval, and the error is estimated.

**Keywords:** stochastic Stokes equations; splitting up; resolvent; semigroup.
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## 1 Introduction

Let $D$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial D$, where $d = 2$ or $d = 3$. Denote by $v : D \to \mathbb{R}^d$ the velocity vector field and $\pi : D \to \mathbb{R}$ the scalar pressure. The stochastic Stokes equations describe the time evolution of incompressible fluid flow and are given as follows

$$\frac{\partial v}{\partial t} - \Delta v + \nabla \pi = \tilde{\sigma}(v)\dot{W}_t \quad \text{in } D \times (0,T); \tag{1.1}$$

$$\nabla \cdot v = 0 \quad \text{in } D \times (0,T) \tag{1.2}$$

with no-slip boundary condition

$$v = 0 \quad \text{on } \partial D \times [0,T], \tag{1.3}$$

where $W(t)$ is a $U$-valued Wiener process in a given real separable Hilbert space $(U, |\cdot|_U, <\cdot, \cdot>_U)$. Here the viscosity coefficient is assumed to be $1$ since no information is obtained on the dependence of the error on the viscosity coefficient.

To formulate the stochastic Stokes equations, we need the usual Sobolev space $H^{m,p}(D)$ ($m$ is an integer) which is space of all functions whose derivatives up to order $m$ belong to space $L^p(D)$. Denoted by $H_0^{1,2}(D)$ the completion of $C_0^\infty(\Omega)$ (the set of smooth functions with compact supports) with respect to the norm of $H^{1,2}(D)$. We introduce the space $H^0$ for stochastic Stokes equations:

$$H^0 = \{u \in (L^2(D))^d : \ div\ u = 0\ in\ D\ \ and\ u \cdot N = 0\ on\ \partial D\},$$

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where $N$ is the exterior normal vector field. $H^0$ is a closed subspace in $(L^2(D))^d$.

Define the Stokes operator $A\ :\ D(A) \to H^0$ by

$$A = -P\Delta,$$

where $P\ :\ (L^2(D))^d \to H^0$ is Leray projection and $D(A) = (H^{2,2}(D))^d \cap (H_0^{1,2}(D))^d \cap H^0$.

Applying the operator $P$ on each term of equation (1.1), we can rewrite equation (1.1)-(1.3) to be the following infinite dimensional stochastic Stokes equation:

$$v' + Av = \sigma(v)\dot{W} \ \text{ in } (0,T), \tag{1.4}$$

Where $\sigma = P\tilde{\sigma}$. We consider a fixed complete stochastic basis $(\Omega, P, \mathcal{F};\ \{\mathcal{F}_t; t \geq 0\})$. Let $Q$ be a symmetric nonnegative trace class operator on $U$. Then there exists a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in $U$ and a real numbers $\alpha_i$ such that

$$Qe_i = \alpha_i e_i,$$

where $\alpha_i \geq 0$ and $\sum\limits_i \alpha_i < \infty$. For arbitrary $t$, $W$ has the expansion

$$W(t) = \sum_i \sqrt{\alpha_i}\beta_i(t)e_i,$$

where $\beta_i,\ i = 1, 2, \cdots$ are independent real-valued standard Wiener processes. For convenience, we denote $(H^{2,2}(D) \cap H^0(D))^d$ by $H^2$. Define

$$|G|^2_{Q,H^s} = Tr(GQG^*) = \sum_i \alpha_i |Ge_i|^2_{H^s} \ \text{ for } \mathrm{G} \in \mathcal{L}(\mathrm{U}, \mathrm{H^s}), \ \ \mathrm{s} = 0, 2.$$

We assume that $\sigma(\cdot):\ H^s \to \mathcal{L}(U, H^s)$, $s = 0, 2$ satisfies

(H7) $|\sigma(u) - \sigma(v)|^2_{Q,H^s} \leq L|u - v|^2_{H^s}$.

The mathematical theory and numerical techniques of finding solutions of the deterministic differential equations and stochastic differential equations have been considered in a large amount of literatures. We cite here Beale and Greengard [5], Grecksch and Kloeden [11], Gyoengy and Nualart [12], Germani and Piccioni [13], Kloeden and Platen [17], where the finite difference method, finite element method, Galerkin's approximation, Wiener chaos decomposition, and the combination of different numerical methods are applied respectively. Also, splitting-up method has been the subject of intense investigation which first appears as Trotter's formula in Trotter [23]. For more guidance we refers to the book and articles Marchuk [18], Barbu [2] and Teman [24]. We concentrate on the splitting up method to approximate the solutions of stochastic Stokes equations. Denoted by $v(t) = T_1(t,s)\xi_1,\ u(t) = T_2(t,s)\xi_2,\ y(t) = T_3(t,s)\xi_3$ the respective solutions of equations

$$\begin{cases} v'(t) + Av(t) = \sigma(v(t))\dot{W}, & t \in (s, T]; \\ v(s) = \xi_1, \end{cases}$$

$$\begin{cases} u'(t) + Au(t) = 0, & t \in (s, T]; \\ u(s) = \xi_2, \end{cases}$$

and

$$\begin{cases} y'(t) = \sigma(y(t))\dot{W}, & t \in (s, T]; \\ y(s) = \xi_3. \end{cases}$$

Then the convergence result

$$\lim_{\varepsilon\to 0} y_\varepsilon(t) = v(t),$$

where $y_\varepsilon$ is the approximate solution which is defined by iteration. It will be specified in equation (2.5) in Section 2, is equivalent to Lie-Trotter type formula:

$$\lim_{n\to\infty}\prod_{i=0}^{n-1} T_3(\frac{n-i}{n}t, \frac{n-1-i}{n}t)T_2(\frac{n-i}{n}t, \frac{n-1-i}{n}t)\xi_1 = T_1(t,0)\xi_1.$$

The framework of the scheme can be roughly explained as following. One can decompose a complicated stochastic differential equation to a deterministic equation and a stochastic equation which are simpler to handle than the original problem. Let an initial value $v_0$ be given. We divide firstly the time interval $[0,T]$ into $n$ subintervals and each of size $\varepsilon = \frac{T}{n}$. The splitting scheme defines an approximate solution of the SDE. Let $u_\varepsilon(\varepsilon)$ be the solution of the deterministic equations at the time $\varepsilon$, with initial condition $y_\varepsilon(0)$. Then $y_\varepsilon(\varepsilon)$ is defined to be the solution of the SDE at the time $\varepsilon$ with initial condition $u_\varepsilon(\varepsilon)$. Recursively, one can define the approximate solution $y_\varepsilon(m\varepsilon)$, for integers $m$, $0 \le m \le n$.

The aim of this paper is to adopt splitting up method to approximate the solutions of stochastic Stokes equations. We replace $u_\varepsilon((m+1)\varepsilon) = e^{-\varepsilon A}y_\varepsilon(m\varepsilon)$ with $u_\varepsilon((m+1)\varepsilon) = (I+\varepsilon A)^{-1}y_\varepsilon(m\varepsilon)$. Regarding to effective and practical computations, resolvent method is more convenient.

In Beale and Greengard [5] and Popa [21], the authors dealt with the approximation of the solutions of deterministic Navier-Stokes equations by splitting up them into two partial differential equations: the Euler equations with the tangential boundary condition and the Stokes equations with the no-slip boundary condition on sufficiently small time intervals. As a preparation to approximate the solutions of stochastic Navier-Stokes equations, we use splitting up methods to approximate the solutions of stochastic Stokes equations. Then we will try to extend the result of Beale and Greengard [5] to stochastic Navier-Stokes equations.

Let us compare our results to recent results on splitting schemes and approximation methods for stochastic (partial) differential equations. To the best of our knowledge, it is the first time to use resolvent for approximating the endpoints of the solution of the deterministic equations at every small interval. This not only allow us get the similar results of Asiminoaei and Rascanu [1], Bensoussan, Glowinski and Rascanu [6] and [7], Gyoengy and Krylov [10], but also introduce a easier numerical computation theoretically. For high order method using in SPDEs, there are some works such as Doersek and Teichmann [8] with nice operator, Jentzen and Kloeden [15] with smooth drift and additive noise. For other numerical method like Euler's method applied in stochastic ordinary differential equations, see recent papers Hutzenthaler, Jentzen and Kloeden [14], Kloeden and Neuenkirch [16] and reference in. We also mention that the splitting up method for solving Hamitton-Jacobi equations and, implicitly, for calculating the value function was initiated by Barbu in [2], [3], and developed by him and Popa, separately, in [4], [20], [22].

This paper will be arranged in the following way. In section 2, we shall introduce the approximation scheme and give the main convergence result. Section 3 is devoted to the proof of error between approximation solution and the exact solution.

## 2 The Scheme and Main Result

In this section, we consider stochastic Stokes equations on $H^0$:

$$\begin{cases} v'(t) + Av(t) = \sigma(v(t))\dot{W}, & t \in (0,T]; \\ v(0) = v_0. \end{cases} \tag{2.1}$$

Let us denote by $M_W^2(0,T;H)$ the space of all adapted processes in $L^2(0,T;\ L^2(\Omega;H))$, where $H$ is any separable Hilbert space, and denote by $C_W([0,T];H)$ its subspace of processes in $C([0,T];\ L^2(\Omega;H))$. As the similar proof in Theorem 3.1 in Flandoli [9], we have the following existence and uniqueness of weak and strong solutions.

**Proposition 2.1.** *Let $v_0 \in L^2(\Omega;H^s)$, $s=0,2$, under corresponding assumptions (H7), then there exists an unique solution to equation (2.1) which means $v \in M_W^2(0,T;\ H^{s+1}) \cap C_W([0,T];\ H^s)$.*

To approximate the solution of equation (2.1), firstly we split up the stochastic Stokes equations to be a deterministic Stokes equation

$$\bar{u}'(t) + A\bar{u}(t) = 0, \tag{2.2}$$

and a stochastic equation

$$y'(t) = \sigma(y(t))\dot{W}. \tag{2.3}$$

Then we apply Trotter scheme to get the approximation of the solution of equation (2.1).

Let $n \in \mathbb{N}$, $\varepsilon = \frac{T}{n}$. Denote $t_m = m\varepsilon$, $m = 1,2,\cdots,n-1$. Consider firstly the deterministic equation in the interval $(0,\varepsilon]$ with initial value $v_0$:

$$\begin{cases} \bar{u}_\varepsilon'(t) + A\bar{u}_\varepsilon(t) = 0, & t \in (0,\varepsilon]; \\ \bar{u}_\varepsilon(0+) = v_0. \end{cases}$$

The mild form of solution at point $\varepsilon$ is $\bar{u}_\varepsilon(\varepsilon) = e^{-\varepsilon A}v_0$. Suppose $\varepsilon$ is quite small. In view of the exponential formula (see Theorem 8.3 in Chapter 1 in Pazy [19]), we replace $e^{-\varepsilon A}$ with operator $(I+\varepsilon A)^{-1}$. So we shall take $u_1 := u_\varepsilon(\varepsilon) := (I+\varepsilon A)^{-1}v_0$ instead of $\bar{u}_\varepsilon(\varepsilon) = e^{-\varepsilon A}v_0$.

Then consider the stochastic equation:

$$\begin{cases} y_\varepsilon'(t) = \sigma(y_\varepsilon(t))\dot{W}, & t \in (0,\varepsilon]; \\ y_\varepsilon(0+) = u_1. \end{cases}$$

The solution of the equation can be written as

$$y_\varepsilon(t) = (I+\varepsilon A)^{-1}v_0 + \int_0^t \sigma(y_\varepsilon(s))dW_s,\ t \in (0,\varepsilon].$$

In the interval $(\varepsilon, 2\varepsilon]$, we consider firstly the deterministic Stokes equation (2.2) with the initial value $y_\varepsilon(\varepsilon)$, and we take

$$u_2 := u_\varepsilon(2\varepsilon) := (I+\varepsilon A)^{-1}y_\varepsilon(\varepsilon);$$

Then, the solution to equation (2.3) with initial value $u_2$ can be written as

$$\begin{aligned} y_\varepsilon(t) &= u_2 + \int_\varepsilon^t \sigma(y_\varepsilon(s))dW_s \\ &= (I+\varepsilon A)^{-2}v_0 + \int_0^\varepsilon (I+\varepsilon A)^{-1}\sigma(y_\varepsilon(s))dW_s + \int_\varepsilon^t \sigma(y_\varepsilon(s))dW_s. \end{aligned}$$

By induction, we have the scheme in the interval $(m\varepsilon,(m+1)\varepsilon]$, $m=0,1,\cdots,n-1$, with the initial condition $y_\varepsilon(m\varepsilon)$:

$$u_{m+1} := u_\varepsilon((m+1)\varepsilon) \quad := \quad (I+\varepsilon A)^{-1}y_\varepsilon(m\varepsilon); \tag{2.4}$$

$$\begin{aligned} y_\varepsilon(t) &= u_{m+1} + \int_{m\varepsilon}^t \sigma(y_\varepsilon(s))dW_s \\ &= (I+\varepsilon A)^{-(m+1)}v_0 + \sum_{i=1}^m \int_{(i-1)\varepsilon}^{i\varepsilon} (I+\varepsilon A)^{-(m+1-i)}\sigma(y_\varepsilon(s))dW_s \\ &\quad + \int_{m\varepsilon}^t \sigma(y_\varepsilon(s))dW_t. \end{aligned} \tag{2.5}$$

Introduce the notation

$$d(n,t) = [\frac{t}{\varepsilon}]\varepsilon, \quad t \in [0,T],$$

and $d^*(n,t) = d(n,t) + \varepsilon$. Then $d(n,t) \le t < d^*(n,t)$ for every $0 \le t \le T$. Thus,

$$\begin{aligned} y_\varepsilon(t) &= (I+\varepsilon A)^{-(\frac{d(n,t)}{\varepsilon}+1)}v_0 + \int_0^{d(n,t)} (I+\varepsilon A)^{-\frac{d(n,t)-d(n,s)}{\varepsilon}}\sigma(y_\varepsilon(s))dW_s \\ &\quad + \int_{d(n,t)}^t \sigma(y_\varepsilon(s))dW_t,\ t \in (0,T]. \end{aligned} \tag{2.6}$$

**Remark 2.2.** *We see that $u_{m+1}$, $m = 0, 1, \cdots, n-1$, are $\mathcal{F}_{m\varepsilon}$-measurable. $y_\varepsilon(t)$ are left continuous and with limit to right. Their discontinuity points are $0,\ \varepsilon,\ 2\varepsilon, \cdots, (n-1)\varepsilon$.*

Now we present our main convergence results.

**Theorem 2.3.** *Assume that (H7) holds. $v(t)$ is the solution of equation (2.1). We have the following convergence results:*
*(i) If the initial value $v_0 \in L^2(\Omega;\ H^0)$,*

$$y_\varepsilon(t) \to v(t) \text{ in } L^2(\Omega;\ H^0), \text{ for } t \in [0,T]; \tag{2.7}$$

*(ii) If $v_0 \in L^2(\Omega;\ H^2)$,*

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{H^0}] \le Ce^{4Ct\varepsilon} \cdot \varepsilon, \tag{2.8}$$

*where $C$ is a constant independent of $\varepsilon$.*

## 3 The Proof of Main Convergence Result

In this section, we prove our convergence result Theorem 2.3.

**Lemma 3.1.** *Assume that (H7) holds and the initial value $v_0 \in L^2(\Omega;\ H^s)$, $s = 0, 2$. Then*

$$\mathbb{E}[|y_\varepsilon(t)|^2_H] \le 2e^{2Lt}(\mathbb{E}[|v_0|^2_H] + \sigma(0)t), \tag{3.1}$$

*where $H$ denotes, if there is not confusion, $H^s$, $s = 0, 2$.*

*Proof.* Before proving the boundedness of $y_\varepsilon(t)$, we present an estimation from [5]: there exist a constant $\varepsilon_0 > 0$ such that for all complex $\varepsilon$ with $Re(\varepsilon) \ge -\varepsilon_0$,

$$\|(A+\varepsilon I)^{-1}\| \le \frac{C}{|\varepsilon|+1}, \tag{3.2}$$

where $\|\cdot\|$ denotes the norm of the space of linear continuous operators on space $H^0$. Let $\varepsilon_1 = \frac{1}{\varepsilon}$ and we still use the notation $\varepsilon$ instead of $\varepsilon_1$, then

$$\|(I+\varepsilon A)^{-1}\| \le \frac{1}{|\varepsilon|+1}. \tag{3.3}$$

From the approximation (2.5), we get, for $t \in (m\varepsilon, (m+1)\varepsilon]$,

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] &\leq 2\mathbb{E}[|(I+\varepsilon A)^{-(m+1)}v_0|_H^2] \\ &\quad +2\sum_{i=1}^m \mathbb{E}[|\int_{t_{i-1}}^{t_i}(I+\varepsilon A)^{-(m+1-i)}\sigma(y_\varepsilon(s))dW_s|_H^2] \\ &\quad +2\mathbb{E}[|\int_{t_m}^t \sigma(y_\varepsilon(s))dW_s|_H^2].\end{aligned}$$

Using inequality (3.3) and the following property of martingale

$$\mathbb{E}[|\int_0^t \sigma(y_\varepsilon(s))dW_s|_H^2] = \mathbb{E}[\int_0^t |\sigma(y_\varepsilon(s))|_{Q,H}^2 ds],$$

we infer that

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] &\leq 2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2\sum_{i=1}^m \mathbb{E}[\int_{t_{i-1}}^{t_i}|(I+\varepsilon A)^{-(m+1-i)}\sigma(y_\varepsilon(s))|_{Q,H}^2 ds] \\ &\quad +2\mathbb{E}[\int_{t_m}^t |\sigma(y_\varepsilon(s))|_{Q,H}^2] \\ &\leq 2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2\sum_{i=1}^m (\frac{1}{1+\varepsilon})^{2(m+1-i)}\mathbb{E}[\int_{t_{i-1}}^{t_i}|\sigma(y_\varepsilon(s))|_{Q,H}^2 ds] \\ &\quad +2\mathbb{E}[\int_{t_m}^t |\sigma(y_\varepsilon(s))|_{Q,H}^2 ds].\end{aligned}$$

Then applying (H7), one obtains

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] &\leq 2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2\sigma(0)\varepsilon\sum_{i=1}^m (\frac{1}{1+\varepsilon})^{2i} + 2\sigma(0)(t-t_m) \\ &\quad +2L\int_0^t (\frac{1}{1+\varepsilon})^{2(m-\frac{d(n,s)}{\varepsilon})} E[|y_\varepsilon(s)|_H^2]ds.\end{aligned}$$

It follows by Gronwall's inequality that

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] &\leq (2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2\sigma(0)\varepsilon\sum_{i=1}^m (\frac{1}{1+\varepsilon})^{2i} + 2\sigma(0)(t-t_m)) \\ &\quad \times \exp\{2L\varepsilon\sum_{i=1}^m (\frac{1}{1+\varepsilon})^{2i} + 2L(t-t_m))\}.\end{aligned}$$

Therefore, we get the result by using inequality (3.3). □

Applying Ito formula to $|y_\varepsilon(t)|_H^2$, one can improve a little bit the estimate from equations (2.4) and (2.5). Although this is not essential in the following estimate, but since the improved estimate can bring us better error estimate, we shall give the details here. For $t \in (m\varepsilon, (m+1)\varepsilon]$,

$$|y_\varepsilon(t)|_H^2 = |u_\varepsilon((m+1)\varepsilon)|_H^2 + 2\int_{m\varepsilon}^t (y_\varepsilon(s),\ \sigma(y_\varepsilon(s))dW_s)_H + \int_{m\varepsilon}^t |\sigma(y_\varepsilon(s))|_{Q,H}^2.$$

Then,

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] &= \mathbb{E}[|u_{m+1}|_H^2] + \mathbb{E}[\int_{m\varepsilon}^t |\sigma(y_\varepsilon(s))|_{Q,H}^2 ds] \\ &= \mathbb{E}[|(I+\varepsilon A)^{-1}y_\varepsilon(m\varepsilon)|_H^2] + \int_{m\varepsilon}^t \mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds \\ &\leq (1+\varepsilon)^{-2}\mathbb{E}[|y_\varepsilon(m\varepsilon)|_H^2] + \int_{m\varepsilon}^t \mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds.\end{aligned}$$

By induction,

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] &\le (1+\varepsilon)^{-2(m+1)}\mathbb{E}[|v_0|_H^2] + \sum_{i=1}^m (1+\varepsilon)^{-2(m+1-i)} \int_{t_{i-1}}^{t_i} \mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds \\ &\quad + \int_{t_m}^t \mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds.\end{aligned}$$

Processing as in Lemma 3.1, we have

$$\mathbb{E}[|y_\varepsilon(t)|_H^2] \le e^{Lt}(\mathbb{E}[|v_0|_H^2] + \sigma(0)t). \tag{3.4}$$

The following inequality plays an important role in proving the convergence with rate.

**Proposition 3.2.** *For $t,\ r \in (0,T]$. Let $t = j\varepsilon$, and $r \in ((j-1)\varepsilon, j\varepsilon)$, $j(\ge 2)$ is an integer. Then,*

$$|(I+\varepsilon A)^{-j}g - e^{-rA}g|_{H^0} \le C \cdot \varepsilon^{\frac{1}{2}} \cdot |g|_{H^2}, \text{ for } g \in D(A), \tag{3.5}$$

*where $C > 0$ is a constant independent of $\varepsilon$.*

To prove this Proposition we need the result from Lemma 5.1 in Chapter 3 in [19].

**Lemma 3.3.** *Let $S$ be a bounded linear operator satisfying*

$$\|S^k\| \le MN^k, \text{ for } k = 1,2,\cdots,\ N \ge 1,$$

*Then for every $n \ge 0$, we have*

$$|e^{(S-I)n}g - S^n g| \le MN^{n-1}e^{(N-1)n}[n^2(N-1)^2 + nN]^{\frac{1}{2}}|g - Sg|.$$

*Proof of Proposition 3.2.* The difference can be separated in the following way:

$$\begin{aligned}(I+\varepsilon A)^{-j}g - e^{-rA}g &= [(I+\varepsilon A)^{-j}g - e^{-tA}g] + [e^{-tA}g - e^{-rA}g] \\ &= [(I+\varepsilon A)^{-j}g - e^{-tA}g] + e^{-rA}[e^{-(t-r)A} - I]g.\end{aligned}$$

We have

$$\begin{aligned}|e^{-rA}[e^{-(t-r)A} - I]g|_{H^0} &= |\int_0^{t-r} e^{-rA}e^{-sA}Ag|_{H^0} \\ &\le C_1 \cdot \varepsilon \cdot |g|_{H^2}, \text{ for } g \in D(A),\end{aligned} \tag{3.6}$$

where $C_1$ is a constant independent of $\varepsilon$.

So we only have to evaluate the term $(I+\varepsilon A)^{-j}g - e^{-tA}g$. Denoted by $J_\varepsilon = (I+\varepsilon A)^{-1}$. It satisfies $\|J_\varepsilon\| \le \frac{1}{1+\varepsilon}$ and

$$\|J_\varepsilon^k\| \le (\frac{1}{1+\varepsilon})^k < 1, \text{ for } k = 1,2,\cdots.$$

Because, for $g \in D(A)$,

$$|J_\varepsilon g - g|_{H^0} = |\varepsilon A J_\varepsilon g|_{H^0} = |\varepsilon J_\varepsilon A g|_{H^0} \le \varepsilon |Ag|_{H^0}, \tag{3.7}$$

the operators $-A_\varepsilon := -\varepsilon^{-1}(I - J_\varepsilon)$ are bounded and they are the infinitesimal generators of uniformly continuous semigroups $S_\varepsilon(t)$ which satisfy

$$\|S_\varepsilon(t)\| = \|e^{-tA_\varepsilon}\| = \|e^{-\frac{t}{\varepsilon}} \sum_{k=0}^\infty (-\frac{t}{\varepsilon})^k \frac{1}{k!} J_\varepsilon^k\| \le e^{-\frac{t}{\varepsilon}} \sum_{k=0}^\infty (-\frac{t}{\varepsilon})^k \frac{1}{k!} \|J_\varepsilon^k\| \le 1.$$

We infer from Lemma 3.2 that,

$$\begin{aligned} |S_\varepsilon(t)g - J_\varepsilon^j g|_{H^0} &\le j^{\frac{1}{2}} \cdot |g - J_\varepsilon g|_{H^0} \\ &= j^{\frac{1}{2}} \cdot \frac{t}{j} \cdot |\frac{g - J_\varepsilon g}{\varepsilon}|_{H^0}, \ \text{ for } g \in D(A). \end{aligned}$$

It follows from (3.7) that,

$$|S_\varepsilon(t)g - J_\varepsilon^j g|_{H^0} \le C_2 \varepsilon^{\frac{1}{2}} \cdot |Ag|_{H^0}, \ \text{ for } g \in D(A), \tag{3.8}$$

where $C_2 > 0$ is a constant independent of $\varepsilon$.

Next, consider the estimation of $|S_\varepsilon(t)g - e^{-tA}g|_{H^0}$. $S_\varepsilon(t)$ can be written formally $e^{-tA(I+\varepsilon A)^{-1}}$. Introduce the functional

$$f(s) = e^{-tA(I+sA)^{-1}}g, \ \text{for fixed } g \in D(A).$$

We see that $f(0) = e^{-tA}g$. Moreover, $f: [0,1] \to H^0$ is continuous on $[0,1]$, differentiable on $(0,1)$. So Lagrange mean value theorem implies that

$$|f(s) - f(0)|_{H^0} \le s \cdot \sup_{r\in(0,1)} |f'(r)|_{H^0}.$$

That means

$$|S_\varepsilon(t)g - e^{-tA}g|_{H^0} \le C_3 \varepsilon \cdot |g|_{H^2}, \ \text{ for } g \in D(A). \tag{3.9}$$

Combining inequalities (3.6), (3.8) and (3.9), we get our result (3.5). □

*Proof of Theorem 2.3.* We know that

$$\begin{aligned} & \mathbb{E}[|y_\varepsilon(t) - v(t)|_{H^0}^2] \\ \le\ & 2\mathbb{E}[|[(I+\varepsilon A)^{-(m+1)} - e^{-tA}]v_0|_{H^0}^2] \\ & +4\mathbb{E}[|\int_0^{t_m} [(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))dW_s|_{H^0}^2] \\ & +4\mathbb{E}[|\int_{t_m}^t [I - e^{-(t-s)A}]\sigma(y_\varepsilon(s))dW_s|_{H^0}^2] \\ & +4\mathbb{E}[|\int_0^t e^{-(t-s)A}[\sigma(y_\varepsilon(s)) - \sigma(v(s))]dW_s|_{H^0}^2] \\ :=\ & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.10}$$

For $v_0 \in D(A)$, from Proposition 3.2, we have the following two inequalities

$$|[(I+\varepsilon A)^{-(m+1)} - e^{-tA}]v_0|_{H^0}^2 \le C \cdot \varepsilon \cdot |v_0|_{H^2};$$

$$\begin{aligned} & |[(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))|_{Q,H^0}^2 \\ =\ & \sum_i \alpha_i |[(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))e_i|_{H^0} \\ =\ & \sum_i \alpha_i |[(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))e_i|_{H^0} \\ \le\ & \sum_i \alpha_i [C\varepsilon |\sigma(y_\varepsilon(s))e_i|_{H^2}^2] \\ =\ & C\varepsilon \sum_i \alpha_i |\sigma(y_\varepsilon(s))e_i|_{H^2}^2, \ s \in [t_m, t). \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{H^0}] \\ \leq \quad & 2C\varepsilon \mathbb{E}[|v_0|^2_{H^2}] + 4C\varepsilon \int_0^t \sigma(0) + L\mathbb{E}[|y_\varepsilon(s)|^2_{H^2})ds \\ & +4CL \int_0^t \mathbb{E}[|y_\varepsilon(s) - v(s)|_{H^0}]ds. \end{aligned}$$

It follows from Gronwall's inequality that

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{H^0}] \leq \varepsilon \cdot Ce^{4C\varepsilon t},$$

which implies the estimate (ii).

For $v_0 \in H^0$, using inequality (3.10) and denoting by $\tilde{C}(\varepsilon) = I_1 + I_2 + I_3$, we have

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{H^0}] \leq \tilde{C}(\varepsilon) + 4CL \int_0^t \mathbb{E}[|y_\varepsilon(s) - v(s)|^2_{H^0}]ds,$$

where $\tilde{C}(\varepsilon) \to 0$ as $\varepsilon \to 0$ because of the exponential formula (see Theorem 8.3 in Chapter 1 in Pazy [19]). Therefore, we can apply Gronwall's inequality and get the result (i):

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{H^0}] \to 0, \text{ as } \varepsilon \to 0.$$

□

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