Electron. Commun. Probab. **18** (2013), no. 61, 1–9.
DOI: 10.1214/ECP.v18-2417
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

How big are the l^∞-valued random fields?*

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Abstract

In this paper we establish path properties and a generalized uniform law of the iterated logarithm (LIL) for strictly stationary and linearly positive quadrant dependent (LPQD) or linearly negative quadrant dependent (LNQD) random fields taking values in l^∞-space.

Keywords: linearly positive quadrant dependence, linearly negative quadrant dependence, stationary random field, law of the iterated logarithm..
AMS MSC 2010: 60F10; 60F15; 60G17; 60G60.
Submitted to ECP on November 5, 2012, final version accepted on April 30, 2013.

1 Introduction and Results

In the last years there has been growing interest in concepts of positive/negative dependence for families of random variables. Such concepts are of considerable use in deriving inequalities in probability and statistics. Recently, Csörgő et al. [1] and Choi and Csörgő [2] studied path properties and asymptotic properties for l^∞-valued Gaussian random fields, respectively. In this paper we are interested in path properties for *any* positive or negative dependent random fields with multidimensional indices taking values in l^∞-space.

Newman [3] introduced and discussed the following concepts of positive or negative dependence. The random field $\{X_i(\mathbf{t})\,;\,\mathbf{t} := (t_1, \cdots, t_N) \in [0,\infty)^N\}_{i=1}^\infty$ is said to be *linearly positive quadrant dependent* (LPQD) if, for any positive numbers λ_i and any disjoint finite subsets A, B of $\mathbb{Z}_+$(set of positive integers), the inequality

$$P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}_i) \ge x, \sum_{j\in B}\lambda_j X_j(\mathbf{t}_j) \ge y\Big\} \ge P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}_i) \ge x\Big\} P\Big\{\sum_{j\in B}\lambda_j X_j(\mathbf{t}_j) \ge y\Big\} \tag{1.1}$$

holds for all x, $y \in \mathbb{R}$(set of real numbers), where $\{\mathbf{t}_j\}_{j=1}^\infty \subset \{\mathbf{t}\}$, which is equivalent to the inequality (Lehmann [6], pp. 1137-1138)

$$P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}_i) \le x, \sum_{j\in B}\lambda_j X_j(\mathbf{t}_j) \le y\Big\} \ge P\Big\{\sum_{i\in A}\lambda_i X_i(\mathbf{t}_i) \le x\Big\} P\Big\{\sum_{j\in B}\lambda_j X_j(\mathbf{t}_j) \le y\Big\}, \tag{1.2}$$

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while the random field $\{X_i(\mathbf{t})\}_{i=1}^\infty$ is said to be *linearly negative quadrant dependent* (LNQD) if the inequalities in (1.1) and (1.2) are reversed. For the results related to such dependence, one can refer to [3, 4, 5]. In general, two random variables X and Y have been called *positively* (resp. *negatively*) *quadrant dependent* (PQD) (resp. NQD) by Lehmann [6], if $P\big(X \ge x, Y \ge y\big) \ge (\text{resp.} \le)\, P\big(X \ge x\big)P\big(Y \ge y\big)$ for all x, $y \in \mathbb{R}$.

The objective of this paper is to establish a generalized uniform law of the iterated logarithm and investigate path properties for LPQD or LNQD random fields taking values in l^∞-space, whose description now follows. For $\mathbf{s} := (s_1, \cdots, s_N)$, $\mathbf{t} := (t_1, \cdots, t_N) \in [0,\infty)^N$, denote

$$\begin{aligned}
&\mathbf{0} = (0,\cdots,0), \quad \mathbf{1} = (1,\cdots,1), \quad \mathbf{s} \pm \mathbf{t} = (s_1 \pm t_1, \cdots, s_N \pm t_N),\\
&\mathbf{s} \le \mathbf{t} \quad \text{if} \;\; s_m \le t_m \;\; \text{for each} \;\; m = 1,2,\cdots,N,\\
&a\mathbf{t} = (at_1,\cdots,at_N) \;\; \text{for } a \in (-\infty,\infty), \quad (\mathbf{s},\mathbf{t}) = (s_1,\cdots,s_N,t_1,\cdots,t_N) \in [0,\infty)^{2N}.
\end{aligned}$$

Assume that $\{X_i(\mathbf{t})\,;\, \mathbf{t} \in [0,\infty)^N\}_{i=1}^\infty$ is a sequence of centered strictly stationary and LPQD (or LNQD) random fields with $X_i(\mathbf{0}) = 0$ and stationary increments

$$\sigma_i(\|\mathbf{t}\|) := \sqrt{E\{X_i(\mathbf{s}+\mathbf{t}) - X_i(\mathbf{s})\}^2}, \quad i \ge 1,$$

where $\sigma_i(t)$ are nondecreasing continuous functions of $t > 0$, and $\|\cdot\|$ denotes the Euclidean norm such that $\|\mathbf{t}\| = \big(\sum_{m=1}^N t_m^2\big)^{1/2}$. Put

$$\sigma_*(t) = \sup_{i\ge1} \sigma_i(t)$$

and assume that $\sigma_*(\cdot)$ is a regularly varying function with exponent $\alpha > 0$ at ∞. Recall that a positive function $\sigma(t)$ of $t > 0$ is said to be *regularly varying* with exponent $\alpha > 0$ at $b \ge 0$ if $\lim_{t\to b}\{\sigma(xt)/\sigma(t)\} = x^\alpha$ for $x > 0$.

Let $\{\mathbb{X}(\mathbf{t}) := (X_1(\mathbf{t}), X_2(\mathbf{t}), \cdots)\,;\, \mathbf{t} \in [0,\infty)^N\}$ be a centered strictly stationary and LPQD (LNQD) random field taking values in l^∞-space (i.e. *l^∞-valued random field*) with l^∞-norm $\|\cdot\|_\infty$ defined by $\|\mathbb{X}(\mathbf{t})\|_\infty = \sup_{i\ge1}|X_i(\mathbf{t})|$.

For each $m = 1,2,\cdots,N$, let $a_m(x)$ and $b_m(x)$ be positive nondecreasing functions of $x > 0$ such that $a_m(x) \le b_m(x)$ and $\lim_{x\to\infty} b_m(x) = \infty$. Denote

$$\begin{aligned}
&\boldsymbol{a}_x = (a_1(x),\cdots,a_N(x)), \quad \mathbf{b}_x = (b_1(x),\cdots,b_N(x)),\\
&\gamma(x) = \sqrt{2\big\{\log\big(\|\mathbf{b}_x\|/\|\boldsymbol{a}_x\|\big) + \log\log\|\mathbf{b}_x\|\big\}},
\end{aligned}$$

where $\log z = \log(\max\{z, e\})$.

The main results are as follows. Let $\{x_k\,;\, x_k > 0\}_{k=1}^\infty$ be an increasing sequence with $x_0 > 0$ and $\lim_{k\to\infty} x_k = \infty$, and let $u_k = O(v_k)$ denote $\limsup_{k\to\infty} u_k/v_k < \infty$.

Theorem 1.1. *Let $\{\mathbb{X}(\mathbf{t})\,;\, \mathbf{t} \in [0,\infty)^N\}$ be a centered strictly stationary and LPQD (LNQD) l^∞-valued random field with l^∞-norm $\|\cdot\|_\infty$ and $E|X_1(\mathbf{t})|^{2+\delta} < \infty$ for some $\delta \in (0,1]$, which satisfies conditions*

(i) $\displaystyle\sum_{j\ge k+1} |\operatorname{Cov}(X_i(\mathbf{1}), X_i(\mathbf{b}_j))| = O(\|\mathbf{b}_k\|^{-\lambda})$ for each $i, k \ge 1$ and some $\lambda > 2$,

(ii) $\displaystyle\inf_{x\ge1} \sigma_*^2(x)/x > 0$.

For each $m = 1,2,\cdots,N$, let the functions $a_m(x)$ and $b_m(x)$ satisfy conditions

(iii) $b_m(x)/a_m(x)\,(>1)$ *is increasing,*

(iv) *there exists $c_0 > 1$ such that* $b_m(x_k) \le c_0\, b_m(x_{k-1})$ for $k \ge 1$.

Then we have

$$\limsup_{x\to\infty} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_x\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_x\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t}) - \mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\gamma(x)} = \limsup_{x\to\infty} \frac{\|\mathbb{X}(\mathbf{b}_x)\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\gamma(x)} = 1 \qquad \text{a.s.} \quad (1.3)$$

Theorem 1.1 presents a path property for l^∞-valued random field, while we can obtain the following law of the iterated logarithm (LIL) without conditions (iii)-(iv) of Theorem 1.1.

Theorem 1.2. *Let* $\{\mathbb{X}(\mathbf{t})\,;\,\mathbf{t}\in[0,\infty)^N\}$ *be as in* Theorem 1.1 *with conditions* (i)-(ii). *Then*

$$\limsup_{x\to\infty}\sup_{\|\mathbf{s}\|\le\|\mathbf{b}_x\|}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_x\|}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\sqrt{2\log\log\|\mathbf{b}_x\|}}$$
$$=\limsup_{x\to\infty}\frac{\|\mathbb{X}(\mathbf{b}_x)\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\sqrt{2\log\log\|\mathbf{b}_x\|}}=1\qquad\text{a.s.}\tag{1.4}$$

Note that the first result in (1.4) implies a *generalized uniform law of the iterated logarithm* for LPQD or LNQD l^∞-valued random fields, but the second one in (1.4) is a standard form of the ordinary LIL for any dependent (or independent) l^∞-valued random fields, which is an extension of some theorems in [1, 2, 4, 8].

Returning to our present exposition of Theorem 1.2, we present the following examples.

Example 1.3. Let $\{X_i(\mathbf{t})\,;\,\mathbf{t}\in[0,\infty)^N\}_{i=1}^\infty$ be a sequence of centered stationary and independent l^∞-valued Gaussian random fields with exponent $\alpha=1/2$ (e.g. *Wiener* random field). For each $i=1,2,\dots,N$, let $b_i(x)=\sqrt{i}\,x$. Then

$$\mathbf{b}_x:=(b_1(x),\cdots,b_N(x))=(1,\sqrt{2},\cdots,\sqrt{N})x,\quad \|\mathbf{b}_x\|=\sqrt{N(N+1)/2}\,x.$$

Hence, by Theorem 1.2, we have the *uniform law of the iterated logarithm*

$$\limsup_{x\to\infty}\sup_{\|\mathbf{s}\|\le\|\mathbf{b}_x\|}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_x\|}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sqrt{\|\mathbf{b}_x\|}\sqrt{2\log\log\|\mathbf{b}_x\|}}=1\qquad\text{a.s.}$$

From the ordinary LIL in (1.4), one can obtain Theorem 1 in [4] for LPQD random sequence $\{\xi_n\,;\,n\ge1\}$, as in Example 1.4 below. In (1.1) and Theorem 1.2, denote $X(n)=X_n(\mathbf{t}_n)$, $X(n)=S_n:=\xi_1+\cdots+\xi_n$ and $\sigma(n):=\sqrt{E(S_n)^2}$, when indexed by a single time-parameter n in Theorems 1.1-1.2.

Example 1.4. Let $\{\xi_n\,;\,n\ge1\}$ be a centered strictly stationary and LPQD (or LNQD) random sequence with $E\xi_1^2>0$, which satisfies conditions

(i) $E|\xi_1|^p<\infty$ for each $p>2$,

(ii) $\displaystyle\sum_{j\ge k+1}|\operatorname{Cov}(\xi_1,\xi_j)|=O(k^{-\lambda})$ for each $k\ge1$ and some $\lambda>2$,

(iii) $\displaystyle\sigma^2:=E\xi_1^2+2\sum_{j=2}^\infty|\operatorname{Cov}(\xi_1,\xi_j)|<\infty.$

Then we have

$$\limsup_{n\to\infty}\frac{S_n}{\sigma(n)\sqrt{2\log\log n}}=1\qquad\text{a.s.,}$$

where it is easy to prove that $\sigma(n)\approx\sigma\sqrt{n}$ for n large enough.

2 Proofs

In this section, let c denote a positive constant which may take different values whenever they appear in different lines. We need the following properties.

(P_1) Two random variables X and Y are PQD (resp. NQD) if and only if $\mathrm{Cov}\,\big(f(X), g(Y)\big) \geq (\text{resp.} \leq)\ 0$ for all real-valued nondecreasing functions f and g (such that $f(X)$ and $g(Y)$ have finite variances) (see Lehmann [6]);

(P_2) (*Hoeffding equality*): For any absolutely continuous functions f and g on the real line and for any random variables X and Y satisfying $Ef^2(X) + Eg^2(Y) < \infty$, we have

$$\begin{aligned}&\mathrm{Cov}\,\big(f(X), g(Y)\big)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(x)g'(y)\big\{P(X \geq x,\ Y \geq y) - P(X \geq x)P(Y \geq y)\big\}dxdy.\end{aligned}$$

The main ingredients of the proofs of Theorems 1.1-1.2 are Propositions 2.1-2.3 below. Note that conditions (i)-(ii) in Theorem 1.1 imply conditions (C2) and (I)-(II) in [4] and [5], respectively. Moreover, $\|\mathbb{X}(\mathbf{t})\|_\infty/\sigma_*(\|\mathbf{t}\|)$ is a standardized random variable. Thus Lemma 2 in [4] and Corollary 2.1 in [5] are easily changed to the following Berry-Esseen type theorem.

Proposition 2.1 (Berry-Esseen type theorem)**.** *Let* $\{\mathbb{X}(\mathbf{t})\,;\,\mathbf{t} \in [0,\infty)^N\}$ *be as in* Theorem 1.1 *with conditions* (i)-(ii). *Then*

$$\sup_{z\in\mathbb{R}}\Big|P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_x)\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)} \leq z\Big\} - \Phi(z)\Big| = O\big(\|\mathbf{b}_x\|^{-1/5}\big), \quad x \to \infty,$$

where $\Phi(\cdot)$ *is a standard normal distribution function and* $\|\mathbf{b}_x\| \to \infty$ *as* $x \to \infty$.

Denote $\mathbf{b}_k = \mathbf{b}_{x_k}$ for a positive increasing sequence $\{x_k\}_{k=1}^\infty$. Using Proposition 2.1 above, the following proposition is immediate from the proof of Lemma 9 in Petrov [7, p. 311].

Proposition 2.2. *Let* $\{\mathbb{X}(\mathbf{t})\}$ *be as in* Proposition 2.1. *Assume that* $g(x)$ *is a positive nondecreasing function of* $x > 0$ *and that* $\{\|\mathbf{b}_k\|\,;\,k \geq 1\}$ *is a positive nondecreasing sequence such that* $\sum_{k=1}^\infty \|\mathbf{b}_k\|^{-1/5} < \infty$. *Then the following statements are equivalent.*

(A) $$\sum_{k=1}^\infty P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_k)\|_\infty}{\sigma_*(\|\mathbf{b}_k\|)} > g(\|\mathbf{b}_k\|)\Big\} < \infty,$$

(B) $$\sum_{k=1}^\infty \frac{1}{g(\|\mathbf{b}_k\|)}\exp\Big(-\frac{1}{2}g^2(\|\mathbf{b}_k\|)\Big) < \infty.$$

The next proposition on the large deviation probability is essential to prove our theorems for *any* strictly stationary l^∞-valued random field, which is proved in a way similar to those of Lemmas 2.2 and 2.3 in [8].

Proposition 2.3. *Let* $\{\mathbb{X}(\mathbf{t})\,;\,\mathbf{t} \in [0,\infty)^N\}$ *be a centered strictly stationary* l^∞*-valued random field. Then, for any* $\varepsilon > 0$ *there exists a constant* $c_\varepsilon > 0$ *such that, for* $v > 1$,

$$\begin{aligned}&P\Big\{\sup_{\|\mathbf{s}\|\leq\|\mathbf{b}_x\|}\ \sup_{\|\mathbf{t}\|\leq\|\mathbf{b}_x\|}\frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t}) - \mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)} \geq v\Big\}\\ &\quad\leq c_\varepsilon\Big(P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_x)\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)} \geq \frac{v}{1+\varepsilon}\Big\} + \sum_{n=1}^\infty 2^{2N2^n}P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_x)\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)} \geq \frac{v}{1+\varepsilon}\sqrt{1+2N\log 3}\cdot 2^{n/2}\Big\}\Big).\end{aligned}$$

Proof of Theorem 1.1. Since $\gamma(x) \geq \sqrt{2\log\log\|\mathbf{b}_x\|}$, we will first prove the sharper result

$$\limsup_{x\to\infty} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_x\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_x\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\sqrt{2\log\log\|\mathbf{b}_x\|}} \le 1 \qquad \text{a.s.} \tag{2.1}$$

without conditions (iii)-(iv), whose result is used to prove (1.4). For $\theta > 1$ and $k \geq 1$, set $A_k = \{x;\, \theta^k \le \|\mathbf{b}_x\| \le \theta^{k+1}\}$. Note that $\sqrt{2\log\log\theta^k} \geq \theta^{-1}\sqrt{2\log\log\theta^{k+1}}$ since $(\log u)/u$ is decreasing for $u > e^e$. By the regularity of $\sigma_*(\cdot)$, we get $\sigma_*(\|\mathbf{b}_x\|)/\sigma_*(\theta^{k+1}) \geq \theta^{-2\alpha}$ and hence

$$\begin{aligned}
&\limsup_{x\to\infty} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_x\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_x\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\sqrt{2\log\log\|\mathbf{b}_x\|}} \\
&\quad\le \limsup_{k\to\infty} \sup_{x\in A_k} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}_x\|} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_x\|} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\sqrt{2\log\log\theta^k}} \\
&\quad\le \theta^{1+2\alpha} \limsup_{k\to\infty} \sup_{\|\mathbf{s}\|\le\theta^{k+1}} \sup_{\|\mathbf{t}\|\le\theta^{k+1}} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\theta^{k+1})\sqrt{2\log\log\theta^{k+1}}}.
\end{aligned} \tag{2.2}$$

For convenience, let $\|\mathbf{b}_k\| = \theta^k$, where $\mathbf{b}_k := \mathbf{b}_{x_k}$ for a positive increasing sequence $\{x_k\}_{k=1}^\infty$. Using Proposition 2.3, it follows that for any $\varepsilon > 0$ there exists a positive constant c_ε such that

$$\begin{aligned}
&P\Big\{\sup_{\|\mathbf{s}\|\le\theta^{k+1}} \sup_{\|\mathbf{t}\|\le\theta^{k+1}} \frac{\|\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\|_\infty}{\sigma_*(\theta^{k+1})\sqrt{2\log\log\theta^{k+1}}} > 1+2\varepsilon\Big\} \\
&\quad\le c_\varepsilon\bigg(P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_{k+1})\|_\infty}{\sigma_*(\theta^{k+1})} \ge \frac{(1+2\varepsilon)\sqrt{2\log\log\theta^{k+1}}}{1+\varepsilon}\Big\} \\
&\qquad + \sum_{n=1}^\infty 2^{2N2^n} P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_{k+1})\|_\infty}{\sigma_*(\theta^{k+1})} \ge \frac{(1+2\varepsilon)\sqrt{2\log\log\theta^{k+1}}}{1+\varepsilon}\sqrt{1+2N\log 3\cdot 2^{n/2}}\Big\}\bigg).
\end{aligned} \tag{2.3}$$

Now let us apply Proposition 2.2 with $\|\mathbf{b}_k\| = \theta^k$ and $g(\theta^k) = g_1(\theta^k)$ $\big(\text{or } g_2(\theta^k)\big)$, where

$$g_1(\theta^k) := \frac{(1+2\varepsilon)\sqrt{2\log\log\theta^{k+1}}}{1+\varepsilon}, \quad g_2(\theta^k) := \frac{(1+2\varepsilon)\sqrt{2\log\log\theta^{k+1}}}{1+\varepsilon}\sqrt{1+2N\log 3\cdot 2^{n/2}}$$

in (2.3). Considering the right hand side of (2.3) and equivalence of Proposition 2.2, we have

$$\begin{aligned}
&\sum_{k=1}^\infty \frac{1}{g_1(\theta^k)} e^{-g_1^2(\theta^k)/2} \le c\sum_{k=1}^\infty \big(\log\theta^{k+1}\big)^{-1-\varepsilon'} < \infty \\
&\Rightarrow \sum_{k=1}^\infty P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_{k+1})\|_\infty}{\sigma_*(\theta^{k+1})} \ge g_1(\theta^k)\Big\} < \infty,
\end{aligned}$$

where $\varepsilon' = \varepsilon/(1+\varepsilon)$, by the strict stationarity of $\mathbb{X}(\mathbf{t})$, and also

$$\begin{aligned}
\frac{1}{g_2(\theta^k)}\exp\Big(-\frac{1}{2}g_2^2(\theta^k)\Big) &\le \exp\Big(-\frac{1}{2}\Big(\frac{1+2\varepsilon}{1+\varepsilon}\Big)^2(2\,\log\log\theta^{k+1})(1+2N\log 3)2^n\Big) \\
&\le ((k+1)\log\theta)^{-(1+\varepsilon')(1+2N\log 3)2^n} \le c\,(k+1)^{-(1+\varepsilon')(1+2N\log 3)2^n}
\end{aligned}$$

which implies

$$P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_{k+1})\|_\infty}{\sigma_*(\theta^{k+1})} > g_2(\theta^k)\Big\} \le c\,(k+1)^{-(1+\varepsilon')(1+2N\log 3)2^n}$$

by Proposition 2.2. Thus

$$\sum_{k=1}^\infty\sum_{n=1}^\infty 2^{2N2^n}P\Big\{\frac{\|\mathbb{X}(\mathbf{b}_{k+1})\|_\infty}{\sigma_*(\theta^{k+1})}>g_2(\theta^k)\Big\}\le c\sum_{k=1}^\infty\sum_{n=1}^\infty 2^{2N2^n}(k+1)^{-(1+\varepsilon')(1+2N\log 3)2^n}$$
$$\le c\sum_{k=1}^\infty\sum_{n=1}^\infty 2^{-N2^n\log_2(k+1)}\cdot 2^{-n}\le c\sum_{k=1}^\infty\sum_{n=1}^\infty k^{-2}\cdot 2^{-n}<\infty.$$

In conclusion, it follows from (2.3) that

$$\sum_{k=1}^\infty P\Big\{\sup_{\|\mathbf{s}\|\le\theta^{k+1}}\ \sup_{\|\mathbf{t}\|\le\theta^{k+1}}\frac{\|\,\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\,\|_\infty}{\sigma_*(\theta^{k+1})\sqrt{2\log\log\theta^{k+1}}}>1+2\varepsilon\Big\}<\infty$$

and the Borel-Cantelli lemma yields

$$\limsup_{k\to\infty}\ \sup_{\|\mathbf{s}\|\le\theta^{k+1}}\ \sup_{\|\mathbf{t}\|\le\theta^{k+1}}\frac{\|\,\mathbb{X}(\mathbf{s}+\mathbf{t})-\mathbb{X}(\mathbf{s})\,\|_\infty}{\sigma_*(\theta^{k+1})\sqrt{2\log\log\theta^{k+1}}}\le 1+2\varepsilon\qquad\text{a.s.}$$

Combining this with (2.2) implies (2.1) since ε and θ are arbitrary.

By virtue of (2.1), the proof of (1.3) is completed if we show that

$$\limsup_{x\to\infty}\frac{\|\,\mathbb{X}(\mathbf{b}_x)\,\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\gamma(x)}\ge 1\qquad\text{a.s.}\tag{2.4}$$

Let $\{x_k\,;\,x_k>0\}_{k=1}^\infty$ be an increasing sequence such that $x_0>0$ and the $(k-1)$st point x_{k-1} is placed by the relation $b_m(x_k)-a_m(x_k)=b_m(x_{k-1}),1\le m\le N$, with x_k defined by induction, since $b_m(x)-a_m(x)$ is increasing by (iii). For convenience, put $\boldsymbol{a}_k=\boldsymbol{a}_{x_k}$ and $\mathbf{b}_k=\mathbf{b}_{x_k}$, and let $i_0\ge 1$ be an integer such that $\sigma_{i_0}(\|\mathbf{b}_k\|)=\sigma_*(\|\mathbf{b}_k\|)$, where $\|\mathbf{b}_k\|:=\theta^k$ as above. Then,

$$\limsup_{k\to\infty}\frac{\|\,\mathbb{X}(\mathbf{b}_k)\,\|_\infty}{\sigma_*(\|\mathbf{b}_k\|)\gamma(x_k)}\ge\limsup_{k\to\infty}\frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)\gamma(x_k)}\tag{2.5}$$

and the inequality (2.4) is immediate from (2.5) if we prove

$$\limsup_{k\to\infty}\frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)\gamma(x_k)}>1-4\varepsilon\qquad\text{a.s.}\tag{2.6}$$

for any small $\varepsilon>0$. For each $k\ge 1$, we see that

$$U_k:=\frac{X_{i_0}(\mathbf{b}_k)-X_{i_0}(\mathbf{b}_{k/2})}{\sigma_{i_0}(\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|)}$$

is a standardized random variable. Let $B_k=\{U_k>(1-2\varepsilon)\gamma(x_k)\}$. If $\mathcal{N}$ is a standard normal random variable, then it follows from Proposition 2.1 and the strict stationarity of $\mathbb{X}(\mathbf{t})$ that

$$\begin{aligned}P(B_k)&=\Big(1-P\Big\{U_k\le(1-2\varepsilon)\gamma(x_k)\Big\}-1+P\{\mathcal{N}\le(1-2\varepsilon)\gamma(x_k)\}\Big)\\&\quad+P\{\mathcal{N}>(1-2\varepsilon)\gamma(x_k)\}\\&\ge -c_1\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|^{-1/5}+\frac{1}{\sqrt{2\pi}(1-2\varepsilon)^2\,\gamma^2(x_k)}\ \exp\Big(-\frac{1-2\varepsilon}{2}\gamma^2(x_k)\Big)\\&\ge -c_1\frac{1}{(\log\|\,\mathbf{b}_k\,\|)^{1-\varepsilon}}+c\left(\frac{\|\,\boldsymbol{a}_k\,\|}{\|\,\mathbf{b}_k\,\|\log\|\,\mathbf{b}_k\,\|}\right)^{1-\varepsilon}\ge c\frac{1}{(\log\|\,\mathbf{b}_k\,\|)^{1-\varepsilon}}\Big(\varepsilon\frac{\|\,\boldsymbol{a}_k\,\|}{\|\,\mathbf{b}_k\,\|}\Big)\end{aligned}$$

for all large k by (iii), where c and c_1 are positive constants, and further

$$\sum_{k=k_0}^{\ell} P(B_k) \geq \varepsilon \frac{1}{(\log \| \mathbf{b}_\ell \|)^{1-\varepsilon}} \sum_{k=k_0}^{\ell} \frac{\| \boldsymbol{a}_k \|}{\| \mathbf{b}_k \|}$$

for some $k_0 \geq 1$ with $k_0 \leq k \leq \ell$. Also there exist constants $c_2, c_3 > 1$ such that

$$\log \|\mathbf{b}_\ell\| \leq c_2 \sum_{k=k_0}^{\ell} \log \frac{\|\mathbf{b}_k\|}{\|\mathbf{b}_{k-1}\|} \leq c_2 \sum_{k=k_0}^{\ell} \log \left(c_0 + \frac{\|\boldsymbol{a}_k\|}{\|\mathbf{b}_{k-1}\|} \right) \leq c_2 \sum_{k=k_0}^{\ell} \log \left(c_0 + \frac{c_3 \|\boldsymbol{a}_{k-1}\|}{\|\mathbf{b}_{k-1}\|} \right) \tag{2.7}$$

since $c_0 \| \mathbf{b}_{k-1} \| \geq \| \mathbf{b}_k \| - \| \boldsymbol{a}_k \|$ by (iv). The last inequality of (2.7) follows from the fact that

$$\frac{\|\boldsymbol{a}_k\|}{\|\boldsymbol{a}_{k-1}\|} \leq \frac{\|\mathbf{b}_k\|}{\|\mathbf{b}_{k-1}\|} \leq \frac{c_0 \|\mathbf{b}_k\|}{\|\mathbf{b}_k\| - \|\boldsymbol{a}_k\|} = \frac{c_0}{1 - (\|\boldsymbol{a}_k\| / \|\mathbf{b}_k\|)} \leq c_3$$

by (iii). Thus, by (2.7), there exists a constant $K > 1$ such that

$$\log \|\mathbf{b}_\ell\| \leq c_2 \sum_{k=k_0}^{\ell} \log \Big(c_0 + \frac{c_3 \|\boldsymbol{a}_k\|}{\|\mathbf{b}_{k-1}\|} \Big) \leq K \sum_{k=k_0}^{\ell} \frac{{c_3}^2 \|\boldsymbol{a}_k\|}{\|\mathbf{b}_k\|}.$$

Therefore, we have $\sum_{k=1}^{\ell} P(B_k) \geq \varepsilon (\log \|\mathbf{b}_\ell\|)^\varepsilon / (K {c_3}^2) \to \infty$ as $\ell \to \infty$; that is, we get $\sum_{k=1}^{\infty} P(B_k) = \infty$.

Next, let $B'_k = \{U_k > (1 - 3\varepsilon)\gamma(x_k)\}$. We will show that $P\big(B'_k,\ i.o.\big) = 1$. Choose a differential function $f(x)$ on $\mathbb{R}$ such that $|f'(x)| \leq \kappa$ for some $0 < \kappa < \infty$ and

$$0 \leq I\Big\{x > (1-2\varepsilon)\gamma(x_k)\Big\} \leq f(x) \leq I\Big\{x > (1-3\varepsilon)\gamma(x_k)\Big\} \leq 1, \tag{2.8}$$

where $I\{\cdot\}$ is an indicator function. In order to prove $P\big(B'_k,\ i.o.\big) = 1$, it is enough to show that $\sum_{k=1}^{\infty} f(U_k) = \infty$ a.s. Since $\sum_{k=1}^{\infty} P(B_k) = \infty$ in the above statement, it follows from (2.8) that $\sum_{k=1}^{\infty} Ef(U_k) \geq \sum_{k=1}^{\infty} P(B_k) = \infty$. By Markov inequality, we have

$$\begin{aligned} P\left\{ \sum_{k=1}^{\infty} f(U_k) < \frac{1}{2} \sum_{k=1}^{n} Ef(U_k) \right\} &\leq P\left\{ \left| \sum_{k=1}^{n} f(U_k) - \sum_{k=1}^{n} Ef(U_k) \right| > \frac{1}{2} \sum_{k=1}^{n} Ef(U_k) \right\} \\ &\leq 4 \operatorname{Var} \left(\sum_{k=1}^{n} f(U_k) \right) \Bigg/ \left(\sum_{k=1}^{n} Ef(U_k) \right)^2 \\ &\leq \frac{4}{\sum_{k=1}^{n} Ef(U_k)} + \frac{8 \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} |\operatorname{Cov}\big(f(U_k), f(U_j)\big)|}{\Big(\sum_{k=1}^{n} Ef(U_k)\Big)^2}. \end{aligned} \tag{2.9}$$

Noting that U_k and U_j are LPQD (resp. LNQD) from the definition of LPQD (resp.

LNQD), it follows from (P_1), (P_2), condition (i) and the regularity of $\sigma_*(\cdot)$ that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty}\big|\operatorname{Cov}\big(f(U_k),f(U_j)\big)\big| \\
&\le \kappa^2\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty}\Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(P\{U_k\ge x,U_j\ge y\}-P\{U_k\ge x\}P\{U_j\ge y\}\Big)dxdy\Big| \\
&\le c\sum_{k=1}^{\infty}\frac{\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|}{\sigma_{i_0}^2(\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|)}\sum_{j=k+1}^{\infty}\Big|\operatorname{Cov}\big(X_{i_0}(\mathbf{1}),\,X_{i_0}(\mathbf{b}_j)-X_{i_0}(\mathbf{b}_{j/2})\big)\Big| \qquad (2.10)\\
&\le c\sum_{k=1}^{\infty}(\theta^k)^{1-2\alpha}\|\,\mathbf{b}_{(k+1)/2}\,\|\sum_{j\ge k+1}\Big|\operatorname{Cov}\big(X_{i_0}(\mathbf{1}),\,X_{i_0}(\mathbf{b}_j)\big)\Big| \\
&\le c\sum_{k=1}^{\infty}\theta^{-(\lambda-2+2\alpha)k}<\infty.
\end{aligned}$$

Since $\sum_{k=1}^{\infty}Ef(U_k)=\infty$ as above, letting $n\to\infty$ in (2.9) yields $P\left\{\sum_{k=1}^{\infty}f(U_k)<\infty\right\}=0$ by (2.10). This proves $\sum_{k=1}^{\infty}f(U_k)=\infty$ a.s. and consequently $P\big(B_k',\ i.o.\big)=1$. Let

$$C_k=\left\{\frac{X_{i_0}(\mathbf{b}_{k/2})}{\sigma_{i_0}(\|\mathbf{b}_{k/2}\|)}\ge -2\gamma(x_{k/2})\right\}.$$

Since $P\big(B_k',\ i.o.\big)=1$, it follows from (2.1) that $P(B_k'\cap C_k,\ i.o.)=1$. It is easy to see that

$$\begin{aligned}
&P\Big\{\frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)}>(1-4\varepsilon)\gamma(x_k),\ i.o.\Big\}\\
&\quad\ge P\Big\{\frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)}>(1-3\varepsilon)\gamma(x_k)-2\gamma(x_{k/2}),\ i.o.\Big\}\\
&\quad\ge P\Big\{B_k'\cap C_k,\ i.o.\Big\}=1
\end{aligned}$$

for k large enough, by the stationarity of $\mathbb{X}(\mathbf{t})$. This implies (2.6) and hence (2.4) holds true.

Proof of Theorem 1.2. Since we have proved (2.1) without conditions (iii)-(iv) of Theorem 1.1, it is enough to show that

$$\limsup_{x\to\infty}\frac{\|\mathbb{X}(\mathbf{b}_x)\|_\infty}{\sigma_*(\|\mathbf{b}_x\|)\sqrt{2\log\log\|\mathbf{b}_x\|}}\ge 1\qquad\text{a.s.}\tag{2.11}$$

Set $\mathbf{b}_k=\mathbf{b}_{x_k}$ for a positive increasing sequence $\{x_k\}_{k=1}^{\infty}$, and let $i_0\ge 1$ be an integer such that $\sigma_{i_0}(\|\mathbf{b}_k\|)=\sigma_*(\|\mathbf{b}_k\|)$. Then

$$\limsup_{k\to\infty}\frac{\|\,\mathbb{X}(\mathbf{b}_k)\,\|_\infty}{\sigma_*(\|\mathbf{b}_k\|)\sqrt{2\log\log\|\mathbf{b}_k\|}}\ge\limsup_{k\to\infty}\frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)\sqrt{2\log\log\|\mathbf{b}_k\|}}\tag{2.12}$$

and (2.11) is immediate from (2.12) if we prove

$$\limsup_{k\to\infty}\frac{X_{i_0}(\mathbf{b}_k)}{\sigma_{i_0}(\|\mathbf{b}_k\|)\sqrt{2\log\log\|\mathbf{b}_k\|}}>1-4\varepsilon\qquad\text{a.s.}$$

for any small $\varepsilon>0$. For $\theta>1$, set $\|\mathbf{b}_k\|=\theta^k$ and $B_k^*=\big\{U_k>(1-2\varepsilon)\sqrt{2\log\log\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|}\big\}$ as in the proof of (2.6). Then $\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|\approx\theta^k$ for sufficiently large k. If we apply Proposition 2.2 with $g(\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|)=(1-2\varepsilon)\sqrt{2\log\log\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|}$, then

$$\sum_{k=1}^{\infty}\frac{1}{g\big(\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|\big)}\exp\Big(-\frac{1}{2}g^2\big(\|\,\mathbf{b}_k-\mathbf{b}_{k/2}\,\|\big)\Big)\ge c\sum_{k=1}^{\infty}k^{-1+\varepsilon}=\infty$$

and hence $\sum_{k=1}^{\infty} P(B_k^*) = \infty$ by the strict stationarity of $X_i(\mathbf{t})$ for $i \geq 1$. The remainder of the proof is the same as the corresponding proof in (2.8)-(2.10). The details are omitted. This completes the proof of Theorem 1.2.

Acknowledgments. The authors wish to thank the referees for careful reading of their manuscript, and for helpful suggestions that have led to improving the presentation of our results.

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