Random pure quantum states via unitary Brownian motion

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Abstract

We introduce a new family of probability distributions on the set of pure states of a
finite dimensional quantum system. Without any a priori assumptions, the most nat-
ural measure on the set of pure state is the uniform (or Haar) measure. Our family of
measures is indexed by a time parameter t and interpolates between a deterministic
measure ($t = 0$) and the uniform measure ($t = \infty$). The measures are constructed
using a Brownian motion on the unitary group $\mathcal{U}_N$. Remarkably, these measures have
a $\mathcal{U}_{N-1}$ invariance, whereas the usual uniform measure has a $\mathcal{U}_N$ invariance. We
compute several averages with respect to these measures using as a tool the Laplace
transform of the coordinates.

1 Introduction

Defining models of randomness for quantum objects has become a central problem
in quantum information theory which has found many interesting and novel applica-
tions. Probability measures on the set of quantum states have been investigated thor-
oughly in recent years [17, 14, 2] both in the physical and the mathematical literature.
Random quantum channels have also been a subject of interest [10, 5, 6]. In particular,
ensembles of quantum channels are the central idea behind the recent breakthroughs
in the additivity conjecture [9]. Random quantum states can arise in two different ways.
First, they describe states of open systems which are subjected to random interaction
with an (unknown) environment. This aspect has been the starting point of the so-called
induced measures which describe finite dimensional systems in interaction with a usu-
ally larger, but finite dimensional environment. Statistical ensembles of quantum states
can also be used to study physical properties of generic states, such as entanglement,
purity or other physically relevant quantities.

Defining a model of randomness for quantum states amounts to specifying a proba-
bility measure on the set of density matrices on the corresponding Hilbert space. When
one considers only *pure* states (rank one density matrices), it turns out that there ex-
ists a unique natural candidate for such a probability measure: the Lebesgue measure
on the unit sphere of the underlying Hilbert space, π_∞. This measure on the set unit

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vectors, called the *Fubini-Study* or the *uniform* distribution, is canonical in the sense that it is invariant under changes of bases: an element ψ of this ensemble has the same distribution as one of its rotations $U\psi$, for any given unitary matrix $U \in \mathcal{U}_N(\mathbb{C})$. This invariance property, which characterizes the uniform distribution, justifies its use when no information about the internal structure of the system is known.

Very recently, new ensembles of pure states have been constructed, in order to take into account any available *a priori* information on the system. In [7], the authors introduce an ensemble of states for quantum multipartite systems. Given a graph which encodes the entanglement between the different parties, a probability measure on the pure states of the total system is constructed. The ensemble of *graph states* is different from the uniform ensemble because it contains the structure information about the initial entanglement present in the system.

In the present work, we are going to generalize the uniform ensemble in another direction, by removing the rotation invariance condition. Pure state ensembles on a single-partite system with no *full* invariance property exhibit *preferred* states which have a larger probability than other states. The model introduced in the present work has a symmetry group of smaller dimension $\mathcal{U}_{N-1}(\mathbb{C})$ and this feature makes is suitable for modelling systems on which some partial information is available. Given a fixed state ψ of a quantum system, we shall introduce a one parameter family of probability measures π_t^ψ indexed by a real parameter $t \geq 0$. The parameter t can be interpreted as a *time parameter* is such a way that for $t = 0$, the measure is deterministic, being supported on the state ψ, and in the limit $t \to \infty$, the measure π_t^ψ approaches the uniform measure π_∞. For each value of the parameter t, the measure π_t^ψ is invariant under the subgroup of rotations which leave invariant the vector ψ, making it the *preferred* state of the measure. Our construction is based on the *unitary Brownian motion*, a stochastic process valued in the set of unitary matrices. At fixed time t, this process itself is an interpolation between the identity matrix (at $t = 0$) and the unique invariant Haar measure on the compact group of unitary matrices (when $t \to \infty$). The construction is motivated by the similar procedure that was used in the definition of the Fubini-Study measure.

The paper is organized as follows. In section 2 we review the definition and some basic properties of the unitary Brownian motion. Section 3 contains the definition of the new family of ensembles of pure states, that are analyzed in section 4 using the Laplace transform. Finally, we compute in section 5 averages of some quantities of interest in quantum information theory.

Let us now introduce some notation. In quantum information theory, any norm 1 vector (or pure state) x gives rise to a probability vector. More precisely, if e_i is the canonical basis of $\mathcal{H} \simeq \mathbb{C}^N$, then we can decompose ψ in the following form $\psi = \sum \psi_i e_i$. To such a state we naturally associate the probability vector $p(\psi) = (|\psi_1|^2, \dots, |\psi_N|^2)$. We denote $\Delta_N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x \geq 0, \sum_{i=1}^N x_i = 1\}$, the set of probability vectors. Physically, if e_i determines the level of energies of an atom and if x represents the wave function describing this atom, the quantity $|\psi_i|^2$ represents the probability to be in the energy level e_i, that is

$$\mathbb{P}[\text{to be in the state } e_i] = |\psi_i|^2.$$

Another physical motivation related to probability theory concerns the measurement of observables. It is known in quantum mechanics that a physical quantity of a quantum system $\mathcal{H} \simeq \mathbb{C}^N$ is represented by an observable, which is an auto-adjoint operator on $\mathcal{H}$. Let A be an observable and $A = \sum_{i=1}^p \lambda_i P_i$ be its spectral decomposition. If ψ is a reference vector state of $\mathcal{H}$, it follows from the axioms of quantum mechanics that a

measurement of the observable A gives a random result λ_i:

$$\mathbb{P}[\text{to observe } \lambda_i] = \|P_i\psi\|^2.$$

In particular if the projectors P_i are the one dimensional projectors on $\mathbb{C}e_i$ we recover the previous probability.

2 Unitary Brownian Motion

This section is devoted to the presentation of the unitary Brownian motion: definition, properties, stochastic calculus, invariant measure. In particular, we present all the ingredients that we are going to use for generating random quantum states. The unitary Brownian motion refers to the natural definition of a Brownian motion on the unitary group of complex matrices.

2.1 Definition

We now define the unitary Brownian motion (UBM). For $N \in \mathbb{N}$, we denote $\mathcal{U}_N(C)$ the unitary group on $\mathcal{M}_N(\mathbb{C})$, that is, $\mathcal{U}_N(C) = \{U \in \mathrm{GL}_n(\mathbb{C})/UU^* = I\}$ and let denote by $\mathcal{M}_N^{sa}(\mathbb{C})$ the set of Hermitian matrices on $\mathcal{M}_N(\mathbb{C})$, that is, $\mathcal{M}_N^{sa}(\mathbb{C}) = \{H \in \mathcal{M}_N(\mathbb{C})/H^* = H\}$. The set $\mathcal{M}_N^{sa}(\mathbb{C})$ is a real linear subspace of $\mathcal{M}_N(\mathbb{C})$, that we endow with the scalar product

$$\langle A, B \rangle = N \operatorname{Tr}[A^*B] = N \operatorname{Tr}[AB].$$

This way, we can consider the Brownian motion on $\mathcal{M}_N^{sa}(\mathbb{C})$, the unique Gaussian process (H_t) which satisfies

$$\forall s, t, \forall A, B \in \mathcal{M}_N^{sa}(\mathbb{C}), \ \mathbb{E}[\langle A, H_s \rangle \langle B, H_t \rangle] = (s \wedge t)\langle A, B \rangle. \tag{2.1}$$

In an equivalent way, the process (H_t) has the same distribution of the random Hermitian matrix whose upper-diagonal coefficients are $\frac{1}{\sqrt{2N}}(B_t^{kl} + iC_t^{kl})$ and whose diagonal coefficients are $\frac{1}{\sqrt{N}}D_t^k$, where $(B_t^{kl}, C_t^{kl}, D_t^k)$ are independent standard real Brownian motions.

We have now all the ingredients to define the unitary Brownian motion. This is the process $(U_t)_{t \geq 0}$, solution of the stochastic differential equation[1]

$$dU_t = i(dH_t)\, U_t - \frac{1}{2}U_t dt, \quad U_0 = I. \tag{2.2}$$

In particular, we have $dU_t^* = -iU_t^* dH_t - \frac{1}{2}U_t^* \mathrm{d}t$. This way it is easy to check that (U_t) is a unitary process.

2.2 Laplace Beltrami operator, Markov generator.

For the sake of completeness, let us describe the Markov generator of this process. In particular, this allows us to motivate the definition of the Brownian motion from a geometric point of view. To this end, we denote by $\mathfrak{u}_N(\mathbb{C})$ the Lie algebra of $\mathcal{U}_N(C)$ and by $(X_1, \ldots, X_{N^2})$ an orthonormal basis of $\mathfrak{u}(N)$. For all smooth functions $F : \mathcal{U}_N(C) \to \mathbb{R}$, we define

$$(\mathcal{L}_{X_i}F)(U) = \frac{d}{dt}_{|t=0}\left(F(Ue^{tX_i})\right), \quad \forall i. \tag{2.3}$$

The operator $\frac{1}{2}\Delta$, where $\Delta = \sum_{i=1}^{N^2} \mathcal{L}_{X_i}^2$, is then the Markov generator of the unitary Brownian motion. The operator Δ is actually the Laplace-Beltrami operator on the

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[1]This equation is written in Ito form.

Riemmanian manifold $\mathcal{U}_N(C)$ endowed with the Riemmanian metric induced by the scalar product on matrices $\langle A, B \rangle = N \operatorname{Tr}[A^* B]$.

Proposition 2.1. *[13, Proposition 2.1] Let $F : \mathcal{U}_N(\mathbb{C}) \to \mathbb{C}$ be a function of class C^2. Then for all $t \geq 0$, we have*

$$F(U_t) = F(I) + \sum_{i=1}^{N^2} \int_0^t (\mathcal{L}_{X_i} F)(U_s) d\langle X_k, iH_s \rangle + \int_0^t \frac{1}{2} \Delta F(U_s) ds \tag{2.4}$$

and the processes $(\langle X_k, iH_t \rangle), k = 1, \ldots, N^2$ are independent standard real Brownian motions.

This proposition is a classical result in stochastic analysis on manifolds.

2.3 Invariant measure

As announced in the introduction, the UBM will help us to define a new family of random states which interpolates between deterministic and uniformly distributed random states. This relies on the large time behavior and the invariant measure of the UBM.

Theorem 2.2. *For all initial unitary conditions U_0 the solution of the stochastic differential equation*

$$dU_t = i(dH_t) U_t - \frac{1}{2} U_t dt, \tag{2.5}$$

converges in distribution to the Haar measure Haar_N on $\mathcal{U}_N(\mathbb{C})$. In other words, the Haar measure is the unique invariant measure of the Markov process, solution of (2.5).

The theorem above justifies the property of interpolation between the identity operator ($U_0 = I$) and the Haar measure for large time (t goes to infinity) for the unitary operator U_t. This property is essential for our definition of new ensembles of random pure states. This fact will be made precise in Section 3. Theorem 2.2 also shows that in general the distribution of the UBM (U_t) is not invariant by unitary multiplication (except under the invariant measure) but the distribution of (U_t) is nevertheless invariant by unitary conjugation and by inversion.

Proposition 2.3. *Let (U_t) be the UBM defined by the SDE (2.5) and let V be any unitary matrix in $\mathcal{U}_N(\mathbb{C})$. The processes (VU_tV^*) and (U_t^{-1}) have the same distribution than (U_t).*

Proof. The property concerning the stochastic process (VU_tV^*) follows from the fact that it satisfies the same stochastic differential equation (2.5) with the same initial condition. Let (W_t) defined by $W_t = VU_tV^*$ for all t, we have

$$dW_t = V(i \cdot dH_t U_t)V^* - \frac{1}{2}VU_tV^* dt = i \cdot d(VH_tV^*)W_t - \frac{1}{2}W_t dt.$$

Since (VH_tV^*) is a Brownian motion on the Hermitian matrices, we see that (W_t) and (U_t) satisfies the same SDE. Hence, as they start with the same initial condition, the two processes must have the same distribution.

The statement for the inverse is an easy consequence of the fact that $U_t^{-1} = U_t^*$ and the process (U_t^*) has the same Markov generator as (U_t). This can be checked using the linearity of the adjoint operation and Proposition 2.1.

$\square$

2.4 Useful formulas

We continue by investigating some properties of the UBM which are going to be useful for studying the random pure states generated by the UBM. In particular, we will be interested in the properties of the coefficients of the matrix (U_t) which we denote by U_t^{jk}, for $1 \leq j, k \leq n$. The stochastic differential equations satisfied by these elements are of the following form

$$dU_t^{jk} = i \sum_{s=1}^{N} (dH_t^{js}) U_t^{sk} - \frac{1}{2} U_t^{jk} dt, \quad d\overline{U_t^{jk}} = -i \sum_{s=1}^{N} \overline{U_t^{sk}} dH_t^{sj} - \frac{1}{2} \overline{U_t^{jk}} dt. \qquad (2.6)$$

Using[2] $d\langle\!\langle H_t^{ij}, H_t^{kl} \rangle\!\rangle = \frac{dt}{N} \delta_{il} \delta jk$, we obtain the following expression

$$d|U_t^{j1}|^2 = i \sum_{s=1}^{N} (dH_t^{js}) U_t^{s1} \overline{U_t^{j1}} - i \sum_{s=1}^{N} U_t^{j1} \overline{U_t^{s1}} dH_t^{sj} + \left(-|U_t^{j1}|^2 + \frac{1}{N} \right) dt. \qquad (2.7)$$

From this, we obtain the brackets of the matrix coordinates which are given by

$$d\langle\!\langle |U_t^{j1}|^2, |U_t^{k1}|^2 \rangle\!\rangle = \frac{2}{N} \left(|U_t^{j1}|^2 \delta_{jk} - |U_t^{j1}|^2 |U_t^{k1}|^2 \right) dt. \qquad (2.8)$$

3 Random Pure States Generated by Unitary Brownian Motion

Before developing our theory for random pure states generated by UBM, we review the definition and the basic properties of the *uniform* (or Fubini-Study) probability measure on the set of pure states (or unit vectors). The lack of any *a priori* information on the state ψ of a quantum system described by a Hilbert space $\mathcal{H} \simeq \mathbb{C}^N$ imposes the choice of a measure which should be invariant by changes of bases. In our setting of finite dimensional complex Hilbert spaces, changes of bases are implemented by unitary operators $U \in \mathcal{U}_N(\mathbb{C})$. As a consequence, we ask that the uniform probability measure should be unitarily invariant. A probability measure π on the unit ball of $\mathcal{H}$ is called *unitarily invariant* if for all Borel subsets A and for all unitary operators $U \in \mathcal{U}_N(\mathbb{C})$, $\pi(UA) = \pi(A)$. This condition determines uniquely the measure π: it is the normalized surface area of the unit ball of $\mathbb{C}^N$, which we shall denote by π_∞, for consistency reasons which shall be clear later. Moreover, we introduce the image measure $\sigma_\infty = \mathrm{sq}_\# \pi_\infty$, where $\mathrm{sq} : \mathbb{C}^N \to \Delta_N$, $\mathrm{sq}[(\psi_i)] = (|\psi_i|^2)$. In other words, if the unit vector ψ has distribution π_∞, then the probability vector $(|\psi_i|^2)_{i=1}^N$ has distribution σ_∞.

Other that the abstract definition of the invariant measure π_∞, one can characterize this probability in the following way: let $U \in \mathcal{U}_N(\mathbb{C})$ be a Haar-distributed random unitary matrix. Then, the first column (or any column, or any line) of U has distribution π_∞. This characterization will be the starting point for the definition of new probability measures on the unit sphere of $\mathbb{C}^N$.

Start with a fixed vector $\psi \in \mathbb{C}^N$ of norm 1, and define the stochastic process (ψ_t), where

$$\psi_t = U_t \psi, \quad \forall t \geq 0, \qquad (3.1)$$

and (U_t) is a UBM starting at $U_0 = I_N$. This gives rise to a stochastic process valued in the unit sphere, with $\psi_0 = \psi$. In the sequel, we study the properties of this process, whose distribution at time t we denote by π_t^ψ. As before, the distribution of the probability vector $|\psi_t^j|^2$ is denoted by σ_t^ψ, making explicit the dependence in the initial condition $\psi_0 = \psi$.

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[2]Here we have adopted the notations $\langle\!\langle \, , \, \rangle\!\rangle$ for the stochastic bracket not to be confused with the scalar product $\langle \, , \, \rangle$.

Definition 3.1. *The ensemble of pure states (unit vectors) of $\mathbb{C}^N$ having distribution π_t^ψ is called the* unitary Brownian motion induced ensemble *at time t. The distribution of the square moduli of the coordinates of a random vector ψ in this ensemble will be denoted by σ_t^ψ.*

Let us first discuss the connection between the distribution π_t^ψ and the Haar measure π_∞. First, note that the distribution of ψ_t depends on the initial value ψ, in contrast with Haar distributed random states, whose distribution is invariant. In particular, we do not have for π_t^ψ invariance by all unitary transformations, that is, if V denotes a unitary operator, in general the processes $(V\psi_t)$ and (ψ_t) have different distributions. More precisely we have the following result.

Proposition 3.2. *Consider a unitary operator V and a (possibly random) unit vector ψ. Let (ψ_t) be the process generated by a unitary Brownian motion independent of ψ, with initial condition $\psi_0 = \psi$. Then, the processes $(V\psi_t)$ and $(U_t V\psi)$ have the same distribution. In particular, the processes $(V\psi_t)$ and (ψ_t) have the same distribution if and only if $V\psi \sim \psi$.*

Proof. This follows from the fact that $V\psi_t = U_t \psi = (V U_t V^*)V\psi = \tilde{U}_t V\psi$. Since (U_t) and $(\tilde{U}_t) = (V U_t V^*)$ have the same distribution and are independent of ψ and $V\psi$, the first part is then straightforward. The second part follows form the first part. $\quad\square$

Corollary 3.3. *Let ψ be a random uniform unit vector, i.e. $\psi \sim \pi_\infty$. Consider an independent UBM (U_t) which induces a process (ψ_t) with initial condition ψ. Then, for all t, the vector ψ_t has distribution π_∞, i.e. $\pi_t^\psi = \pi_\infty$.*

Proposition 3.2 is more instructive when we look at deterministic initial conditions. In particular, the processes $(V\psi_t)$ and (ψ_t) have the same distributions if and only if $V\psi = \psi$, restricting the class of unitary transformations which leave invariant the process. In other words, the distribution of (ψ_t) is invariant under all unitary transformations which fix the initial condition. In conclusion, the process (ψ_t) has a $\mathcal{U}_{N-1}(\mathbb{C})$ invariance group, whereas a uniform unit vector $\psi \sim \pi_\infty$ has a $\mathcal{U}_N(\mathbb{C})$ invariance group.

4 PDE for the Laplace transform of σ_t^ψ

We are now in the position to develop our model in more detail, using as a main tool a partial differential equation satisfied by the Laplace transform of the amplitude vector. Let $\psi \in \mathbb{C}^N$ a fixed unit vector and consider the process $\psi_t = \tilde{U}_t \psi$ generated by a UBM $(\tilde{U}_t)$ with $\tilde{U}_0 = I$. Consider also a fixed unitary matrix V such that $Ve_1 = \psi$, where $e_1 = (1, 0, \ldots, 0)$ is the first element of the canonical basis of $\mathbb{C}^N$. Then one can write

$$\psi_t = \tilde{U}_t V e_1 = U_t e_1, \tag{4.1}$$

where (U_t) is another UBM starting at $U_0 = V$. Note that the choice of the matrix V satisfying $Ve_1 = \psi$ is not important, because of Proposition 3.2. In this way, the initial condition of the problem has been transferred into the UBM (U_t) and we have

$$\psi_t = U_t e_1 = (U_t^{11}, U_t^{21}, \ldots, U_t^{N1}),$$

which corresponds to the first column of the unitary Brownian motion U_t. Such a unit norm vector gives rise to a probability vector $(|\psi_t^j|^2)_j = (|U_t^{j1}|^2)_j$ having distribution σ_t^ψ. This random variable is compactly supported, hence its Laplace transform determines its distribution. Let us define the Laplace transform by (notice the positive sign in the exponential)

$$\varphi(\lambda_1, \ldots, \lambda_N; t) = \mathbb{E}e^{\langle \lambda, |U_t^{\cdot 1}|^2 \rangle} = \mathbb{E}\left[\exp \sum_{j=1}^N \lambda_j |U_t^{j1}|^2\right], \tag{4.2}$$

for all $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ and all $t \geq 0$. Since the random variable is bounded, the function $\lambda \mapsto \varphi(\lambda; t)$ is complex analytic for each t.

The partial derivatives of the function φ read:

$$\partial_j \varphi(\lambda; t) = \partial_{\lambda_j} \varphi(\lambda; t) = \mathbb{E}\left[|U_t^{j1}|^2 \, e^{\langle \lambda, |U_t^{\cdot 1}|^2 \rangle} \right]; \tag{4.3}$$

$$\partial_{jk} \varphi(\lambda; t) = \partial_{\lambda_j \lambda_k} \varphi(\lambda; t) = \mathbb{E}\left[|U_t^{j1}|^2 |U_t^{k1}|^2 \, e^{\langle \lambda, |U_t^{\cdot 1}|^2 \rangle} \right]. \tag{4.4}$$

The following theorem is the main result of this paper, establishing a partial differential equation for the Laplace transform φ. In principle, it allows to recover φ and then, by Laplace inversion, the probability vector $(|U_t^{j1}|^2)_{j=1}^N$.

Theorem 4.1. *The Laplace transform φ of the random vector $(|U_t^{j1}|^2)$, $j = 1, \ldots, N$ satisfies the following partial differential equation*

$$\partial_t \varphi = \frac{\sum_{j=1}^N \lambda_j}{N} \varphi + \left\langle \frac{\lambda^2}{N} - \lambda, \nabla_\lambda \varphi \right\rangle - \frac{1}{N} \langle \lambda, H(\varphi) \lambda \rangle, \tag{4.5}$$

where λ^2 represents the vector $(\lambda^2)_j = \lambda_j^2$ and ∇_λ and H represent the gradient and the Hessian operators, i.e. $(\nabla_\lambda \varphi)_j = \partial_j \varphi$, $[H(\varphi)]_{jk} = \partial_j \partial_k \varphi$.

Proof. Using the multivariate Ito formula for the function $(x_1, \ldots, x_N) \mapsto e^{\langle \lambda, x \rangle}$ applied to the multidimensional process $(|U_t^{\cdot 1}|^2)$, we obtain:

$$\mathrm{d} \exp \langle \lambda, |U_t^{\cdot 1}|^2 \rangle = \sum_{j=1}^N \lambda_j \exp \langle \lambda, |U_t^{\cdot 1}|^2 \rangle \mathrm{d} |U_t^{j1}|^2 + \frac{1}{2} \sum_{j,k=1}^N \lambda_j \lambda_k \exp \langle \lambda, |U_t^{\cdot 1}|^2 \rangle \mathrm{d} \langle\!\langle |U_t^{j1}|^2, |U_t^{k1}|^2 \rangle\!\rangle. \tag{4.6}$$

Taking the expectation and using (2.7), (2.8), we obtain

$$\mathrm{d} \mathbb{E} \exp \langle \lambda, |U_t^{\cdot 1}|^2 \rangle = \sum_{j=1}^N \lambda_j \mathbb{E}\left[\exp \langle \lambda, |U_t^{\cdot 1}|^2 \rangle \left(-|U_t^{j1}|^2 + \frac{1}{N} \right) \right] dt$$

$$+ \frac{1}{2} \sum_{j,k=1}^N \lambda_j \lambda_k \mathbb{E}\left[\exp \langle \lambda, |U_t^{\cdot 1}|^2 \rangle \frac{2}{N} \left(|U_t^{j1}|^2 \delta_{jk} - |U_t^{j1}|^2 |U_t^{k1}|^2 \right) \right] dt. \tag{4.7}$$

Using formulas (4.3), (4.4), we obtain the announced partial differential equation (4.5). $\square$

From the above PDE one can obtain, in principle, all the information about the distribution σ_t^ψ of the random vector $(|U_t^{j1}|^2)$. In the remainder of this section, we shall focus on the marginals $|U_t^{j1}|^2$ (j fixed); covariances $|U_t^{j1}|^2 |U_t^{k1}|^2$ and other statistical quantities will be investigated in the next section.

In the case of marginals, we compute in the following proposition the Laplace transform of the square modulus of one coordinate, in terms of the Kummer confluent hypergeometric function $_1F_1$, whose definition we recall :

$$_1F_1(a; b; z) = \sum_{k=0}^\infty \frac{(a)_k z^k}{(b)_k k!}, \tag{4.8}$$

where $(x)_n$ is the Pochhammer symbol, $(x)_n = x(x+1) \cdots (x+n-1)$.

Proposition 4.2. *Let* $\varphi_j(\lambda; t) = \varphi(0, \ldots, 0, \lambda, 0, \ldots, 0; t) = \mathbb{E} \exp(\lambda |U_t^{j1}|^2)$ *be the Laplace transform of the j-th coordinate of the first column of U_t. Then*

$$\partial_t \varphi_j = \frac{\lambda}{N} \varphi_j + \left(\frac{\lambda^2}{N} - \lambda \right) \partial_\lambda \varphi_j - \frac{\lambda^2}{N} \partial_{\lambda\lambda} \varphi_j, \tag{4.9}$$

with the notation

$$\partial_t \varphi_j = \frac{\partial \varphi_j}{\partial t}, \quad \partial_\lambda \varphi_j = \frac{\partial \varphi_j}{\partial \lambda}, \quad \partial_{\lambda\lambda} \varphi_j = \frac{\partial^2 \varphi_j}{\partial \lambda^2}. \tag{4.10}$$

Given an initial condition $|U_0^{j1}|^2 = c \in [0,1]$, there exists a sequence $(a_n)_{n \geq 0}$ of real numbers (depending on c) such that

$$\varphi_j(\lambda; t) = \sum_{n=0}^{\infty} a_n e^{-\Lambda_n t} \lambda^n {}_1F_1(n+1; N+2n; \lambda), \tag{4.11}$$

where

$$\Lambda_n = n + \frac{n(n-1)}{N}. \tag{4.12}$$

Proof. Equation (4.9) follows from equation (4.5) of Theorem 4.1 by letting $\lambda_k = 0$ for all $k \neq j$. In the rest of the proof, we shall drop the index j, since the initial condition will be encoded into $\varphi(\lambda; 0) = \exp(\lambda |U_t^{j1}|^2)$.

Using separation of variables, we look for solutions of the form $\varphi(\lambda; t) = f(\lambda) g(t)$. Neither of f or g can be zero, hence we obtain

$$\frac{g'}{g} = -\frac{\lambda^2}{N} \frac{f''}{f} + \left(\frac{\lambda^2}{N} - \lambda \right) \frac{f'}{f} + \frac{\lambda}{N}. \tag{4.13}$$

Note that the left hand side of the above equation depends only on t and the right-hand since depends only on λ. This is impossible unless both are equal to a constant C, in which case $g(t) = e^{Ct}$ (we can move the constant factor to f) and f satisfies the ordinary differential equation

$$-\frac{\lambda^2}{N} f'' + \left(\frac{\lambda^2}{N} - \lambda \right) f' + \left(\frac{\lambda}{N} - C \right) f = 0. \tag{4.14}$$

Writing f as a power series $f(\lambda) = \sum_{n \geq 0} a_n \lambda^n$, we get:

$$C a_0 = 0, \qquad a_1(1 + C) = \frac{1}{N} a_0, \qquad a_k \left(\frac{k(k-1)}{N} + k + C \right) = \frac{k}{N} a_{k-1} \qquad \forall k \geq 2. \tag{4.15}$$

These equations can be summarized as (we put $a_{-1} = 0$)

$$a_k(C + \Lambda_k) = \frac{k}{N} a_{k-1} \qquad \forall k \geq 0, \tag{4.16}$$

If $C \notin \{-\Lambda_k\}_{k \geq 0}$, it follows that $f = 0$, which is impossible. Hence, $C = -\Lambda_n$ for some $n \geq 0$. We can compute all the coefficients of the series expansion of f from the recurrence relations above:

$$a_m = 0 \qquad 0 \leq m < n \tag{4.17}$$

$$a_n \text{ is free} \tag{4.18}$$

$$a_{n+m} = a_n \frac{(n+1)_m}{m!(N+2n)_m} = a_n \binom{n+m}{m} \frac{1}{(N+2n)_m} \qquad \forall m \geq 1. \tag{4.19}$$

We obtain the final expression for the Laplace transform:

$$\varphi(\lambda; t) = \mathbb{E} \exp(\lambda |U_t^{j1}|^2) = \sum_{n=0}^{\infty} a_n e^{-\Lambda_n t} \sum_{m=0}^{\infty} \binom{n+m}{m} \frac{\lambda^{n+m}}{(N+2n)_m}. \qquad (4.20)$$

Note that the above series converges The second sum in the above formula admits a more compact expression using the Kummer confluent hypergeometric function $_1F_1$:

$$\sum_{m=0}^{\infty} \binom{n+m}{m} \frac{\lambda^{n+m}}{(N+2n)_m} = \lambda^n {}_1F_1(n+1; N+2n; \lambda), \qquad (4.21)$$

and thus $\varphi(\lambda; t) = \sum_{n=0}^{\infty} a_n e^{-\Lambda_n t} \lambda^n {}_1F_1(n+1; N+2n; \lambda)$. $\qquad \square$

Remark 4.3. *Note that in the limit $t \to \infty$, only the $n = 0$ term survives and we obtain*

$$\lim_{t \to \infty} \varphi(\lambda; t) = \sum_{m=0}^{\infty} \frac{\lambda^m}{(N)_m} \qquad (4.22)$$

which is the result for the Haar measure. This is consistent with [11], Lemma 4.2.4, where the m-th moment in the Haar case was shown to be $\binom{N+m-1}{N-1}$.

It is interesting to note that equation (4.9) does not depend on the actual value of j. However, it does depend on the initial condition c. Next, we compute explicitly φ for particular values of the initial condition $c = |U_0^{j1}|^2 \in [0, 1]$. The values of the coefficients a_n appearing in the proposition can be computed in principle from the following initial conditions:

$$\varphi(\lambda; 0) = e^{\lambda c} \quad \varphi(0; t) = 1.$$

As a first observation, note that the latter relation fixes the value of the constant coefficient in the series, $a_0 = 1$. The first condition translates to ($p = n + m$):

$$\sum_{n=0}^{p} a_n \binom{p}{n} \frac{1}{(N+2n)_{p-n}} = \frac{c^p}{p!}, \quad \forall p \geq 0. \qquad (4.23)$$

The above infinite triangular system of linear equations can be solved in principle and explicit formulas for the coefficients a_n can be found.

Analytical formulas can be obtained (and easily proved by induction) in two particular cases. For $c = 0$, one can show that the unique solution to the equations above are given by

$$a_n = \frac{(-1)^n}{(N+n-1)_n}, \quad \forall n \geq 0. \qquad (4.24)$$

Similarly, for $c = 1$, one has

$$a_n = \frac{(N-1)_n}{n!(N+n-1)_n}, \quad \forall n \geq 0. \qquad (4.25)$$

5 Properties of the measure σ_t^{ψ}

This section contains a list of results which address important statistical properties (moments, covariances) of probability vectors distributed along the measure σ_t^{ψ}.

From Proposition 4.2, it is easy to obtain the expression of the moments of the j-th coordinate $|\psi_t^j|^2$. To this end, we define a family of maps $y_p : [0, \infty) \to [0, 1]$, $y_p(t) = \mathbb{E}|U_t^{j1}|^{2p}$, which depend implicitly on the size parameter N. The dependence on the

index j is encoded in the initial condition $y_p(0) = |\psi^j|^{2p} = |U_0^{j1}|^{2p}$. Interchanging the derivation operator ∂^p and the expectation $\mathbb{E}$, we obtain

$$y_p(t) = \frac{\partial^p \varphi}{\partial \lambda^p}(0; t). \tag{5.1}$$

The applications y_p satisfies a particular system of ordinary differential equations.

Proposition 5.1. *With the convention that $y_0 \equiv 1$, the applications y_p satisfy the following ordinary differential equations on $\mathbb{R}_+$:*

$$y_p' = -\Lambda_p y_p + \frac{p^2}{N} y_{p-1} \tag{5.2}$$

The solution of the system (5.2) *can be expressed in terms of a_n and Λ_n in the following way*

$$y_p(t) = \sum_{n=0}^{p} \binom{p}{p-n} \frac{a_n}{(N+2n)_{p-n}} e^{-\Lambda_n t}. \tag{5.3}$$

Proof. Using Proposition 4.2, we can rewrite the formula (4.20) for $\varphi(\lambda; t)$ in the form

$$\varphi(\lambda; t) = \mathbb{E} e^{\lambda |U_t^{j1}|^2} = \sum_{p=0}^{\infty} \sum_{n=0}^{p} \binom{p}{p-n} \frac{a_n e^{-\Lambda_n t}}{(N+2n)_{p-n}} \lambda^p \tag{5.4}$$

and then, taking the p-th derivative of the expression above, we obtain (the coefficients a_n depend on the initial condition)

$$y_p(t) = p! \sum_{n=0}^{p} \binom{p}{p-n} \frac{a_n}{(N+2n)_{p-n}} e^{-\Lambda_n t}. \tag{5.5}$$

$\square$

From the explicit computations in the previous section, we can specify to the coefficients a_n for the initial condition $\psi = (1, 0 \ldots, 0)$. Indeed, the moments of $|U_t^{11}|^2$, are

$$\mathbb{E}\left[|U_t^{11}|^{2p}\right] = p! \sum_{n=0}^{p} \binom{p}{p-n} \frac{(N-1)_n}{(N+n-1)_n (N+2n)_{p-n}} e^{-\Lambda_n t} \tag{5.6}$$

and the moments for $|U_t^{j1}|^2, j > 1$, are given

$$\mathbb{E}\left[|U_t^{j1}|^{2p}\right] = p! \sum_{n=0}^{p} \binom{p}{p-n} \frac{(-1)^n}{(N+n-1)_n (N+2n)_{p-n}} e^{-\Lambda_n t}. \tag{5.7}$$

For general initial conditions, one has to compute recursively all the moments y_p. In the next proposition we give the general form of y_1 and y_2 for general initial conditions.

Proposition 5.2. *The moments y_1 and y_2 are given by*

$$y_1(t) = \left(y_1(0) - \frac{1}{N}\right) e^{-t} + \frac{1}{N}, \tag{5.8}$$

$$y_2(t) = \left[y_2(0) - \frac{1}{N+2}\left(y_1(0) - \frac{1}{N}\right) - \frac{2}{N(N+1)}\right] e^{-(2+2/N)t}$$

$$+ \frac{1}{N+2}\left[y_1(0) - \frac{1}{N}\right] e^{-t} + \frac{2}{N(N+1)}, \tag{5.9}$$

where $y_1(0) = |U_0^{j1}|^2 = |\psi^j|^2$ and $y_2(0) = |U_0^{j1}|^4 = |\psi^j|^4$.

We now move on to study the covariance of elements $|U_t^{j1}|^2$. We aim to compute the quantities $\mathbb{E}\left[|U_t^{j1}|^2|U_t^{k1}|^2\right]$, for all $j \neq k$. Note that for $j = k$, one can use the more general moment formula in Proposition 5.1. Let us define define a family of covariance maps $f_{jk} : [0, \infty) \to [0, 1]$, with $f_{jk}(t) = \mathbb{E}\left[|U_t^{j1}|^2|U_t^{k1}|^2\right]$. Using the Laplace transform, we have

$$f_{jk}(t) = \frac{\partial^2 \varphi}{\partial \lambda_j \lambda_k}(0; t). \tag{5.10}$$

We put also $f_j(t) = E\left[|U_t^{j1}|^2\right], \forall t \geq 0$. From Proposition 5.1, we have that

$$f_j(t) = \partial_{\lambda_j}\varphi(0; t) = \mathbb{E}\left[|U_t^{j1}|^2\right] = \left(f_j(0) - \frac{1}{N}\right)e^{-t} + \frac{1}{N}.$$

Proposition 5.3. *For all $j \neq k$, the covariance applications satisfy the following first order ODE*

$$f'_{jk} = \frac{1}{N}\left(f_j + f_k\right) - \left(2 + \frac{2}{N}\right)f_{jk}, \tag{5.11}$$

which admits the solution

$$\begin{aligned}
f_{jk}(t) &= \left[f_{jk}(0) - \frac{1}{N+2}\left(f_j(0) + f_k(0) - \frac{2}{N}\right) - \frac{1}{N(N+1)}\right]e^{-t(2+2/N)} \\
&\quad + \frac{1}{N+2}\left[f_j(0) + f_k(0) - \frac{2}{N}\right]e^{-t} + \frac{1}{N(N+1)},
\end{aligned} \tag{5.12}$$

with $f_j(0) = |\psi^j|^2$, $f_k(0) = |\psi^k|^2$ and $f_{jk}(0) = |\psi^j|^2|\psi^k|^2$.

Proof. To compute the two times derivative it is sufficient to consider the functions

$$\varphi(\lambda_j, \lambda_k; t) = \varphi(0, \ldots, 0, \lambda_j, 0, \ldots, 0, \lambda_k, 0, \ldots, 0; t),$$

with λ_j (resp. λ_k) in the j-th (resp. k-th) position. Indeed, we have $f_{jk}(t) = \partial_{\lambda_j \lambda_k}\varphi(0, 0; t)$. Using the Laplace transform of Theorem 4.1, we obtain

$$\begin{aligned}
\partial_t \varphi &= \frac{\lambda_j}{N}\varphi + \frac{\lambda_k}{N}\varphi + \left(\frac{\lambda_j^2}{N} - \lambda_j\right)\partial_{\lambda_j}\varphi + \left(\frac{\lambda_k^2}{N} - \lambda_k\right)\partial_{\lambda_k}\varphi \\
&\quad - \frac{1}{N}\left(\lambda_j^2\partial_{\lambda_j \lambda_j}\varphi + \lambda_k^2\partial_{\lambda_k \lambda_k}\varphi + 2\lambda_j\lambda_k\partial_{\lambda_j \lambda_k}\varphi\right).
\end{aligned} \tag{5.13}$$

Applying $\partial_{\lambda_j \lambda_k}$ on both side and taking next $\lambda_j = \lambda_k = 0$, we get the expression

$$f'_{jk}(t) = \frac{1}{N}\left(\partial_{\lambda_k}\varphi(0, 0; t) + \partial_{\lambda_j}\varphi(0, 0; t)\right) - \left(2 + \frac{2}{N}\right)f_{jk}(t), \tag{5.14}$$

which corresponds exactly to the expression (5.11). □

Naturally this expression depends on the initial condition. For example, in the case of $\psi = e_1$, we get

$$\begin{aligned}
f_{1j}(t) &= \left(-\frac{1}{N+2}\left(1 - \frac{2}{N}\right) - \frac{1}{N(N+1)}\right)e^{-t(2+2/N)} \\
&\quad + \frac{1}{N+2}\left(1 - \frac{2}{N}\right)e^{-t} + \frac{1}{N(N+1)}, \quad \text{for } j \neq 1 \\
f_{jk}(t) &= \left(\frac{2}{(N+2)N} - \frac{1}{N(N+1)}\right)e^{-t(2+2/N)} \\
&\quad - \frac{2}{(N+2)N}e^{-t} + \frac{1}{N(N+1)}, \quad \text{for } j \neq k \text{ and } j, k \neq 1. \tag{5.15}
\end{aligned}$$

Finally, we compute the average value of an observable with respect to a pure state with distribution π_t^ψ. As stated in the introduction, the main motivation for this work was to define a new ensemble of random pure states. If a quantum system is in a state described by the vector ψ_t having distribution π_t^ψ and an observable $A \in \mathcal{M}_N(\mathbb{C})$ is measured, quantum theory predicts that the average value observed is $\langle \psi_t, A\psi_t \rangle$. We shall compute the average value $\mathbb{E}\langle \psi_t, A\psi_t \rangle = \mathbb{E}\langle \psi, U_t^* A U_t \psi \rangle = \mathbb{E}\langle \psi, U_t A U_t^* \psi \rangle$.

Lemma 5.4. *The average value of the measure of a fixed observable $A \in \mathcal{M}_N(\mathbb{C})$ on a quantum system described by the ensemble π_t^ψ is*

$$\mathbb{E}\langle \psi_t, A\psi_t \rangle = \left(\langle \psi, A\psi \rangle - \frac{\mathrm{Tr}(A)}{N} \right) e^{-t} + \frac{\mathrm{Tr}(A)}{N} \tag{5.16}$$

Proof. The result follows from the computation of $dU_t A U_t^*$ with the Ito calculus. Using the Ito formula, we have

$$dU_t A U_t^* = U_t(d\, A U_t^*) + dU_t\, (A U_t^*) + \frac{\mathrm{Tr}(A)}{N} I \mathrm{d}t. \tag{5.17}$$

Computing $d\langle \psi, U_t A U_t^* \psi \rangle$ and taking the expectation, we get

$$d\mathbb{E}\langle \psi, U_t A U_t^* \psi \rangle = \left(-\mathbb{E}\langle \psi, U_t A U_t^* \psi \rangle + \frac{\mathrm{Tr}(A)}{N} \right) \mathrm{d}t, \tag{5.18}$$

which implies equation (5.16). $\square$

In particular, when t goes to infinity, we recover the usual result for the Haar measure π_∞. It is easy to compute average value of an observable $A \in \mathcal{M}_N(\mathbb{C})$ for the Fubini-Study ensemble:

$$\langle A \rangle = \int \langle \psi | A | \psi \rangle \mathrm{d}\pi_\infty(\psi) = \int \langle e_1 | U^* A U | e_1 \rangle \mathrm{d}\mathrm{Haar}(U) = \langle e_1 | \frac{\mathrm{Tr}\, A}{N} I_N | e_1 \rangle = \frac{\mathrm{Tr}\, A}{N}. \tag{5.19}$$

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