On the one-sided exit problem for stable processes in random scenery

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Abstract

We consider the one-sided exit problem for stable Lévy process in random scenery, that is the asymptotic behaviour for T large of the probability

$$\mathbb{P}\Big[\sup_{t \in [0,T]} \Delta_t \leq 1 \Big]$$

where

$$\Delta_t = \int_{\mathbb{R}} L_t(x) \, dW(x).$$

Here $W = \{W(x); x \in \mathbb{R}\}$ is a two-sided standard real Brownian motion and $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$ the local time of a stable Lévy process with index $\alpha \in (1,2]$, independent from the process W. Our result confirms some physicists prediction by Redner and Majumdar.

Keywords: Stable process, Random scenery, First passage time, One-sided barrier problem, One-sided exit problem, Survival exponent.
This research was supported by the french ANR project MEMEMO2 2010 BLAN 0125.
AMS MSC 2010: 60F05; 60F17; 60G15; 60G18; 60K37.
Submitted to ECP on November 16, 2012, final version accepted on March 11, 2013.

1 Introduction

Random processes in random scenery are simple models of processes in disordered media with long-range correlations. These processes have been used in a wide variety of models in physics to study anomalous dispersion in layered random flows [13, 5], diffusion with random sources, or spin depolarization in random fields (we refer the reader to Le Doussal's review paper [11] for a discussion of these models). Let us also mention the fact that these processes are functional limits of random walks in random scenery [9, 6, 7, 4, 8]. The persistence properties of these models were studied by Redner [15, 16] and Majumdar [12]. The interested reader could refer to the recent survey paper [1] for a complete description of already known persistence probabilities and exponents. Supported by physical arguments, numerical simulations and comparison with

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the Fractional Brownian Motion, Redner and Majumdar conjectured the persistence exponents. In this paper we rigorously prove their conjecture up to logarithmic factors. Before stating our main result, we present the process we are interested in.

Let $W = \{W(x); x \in \mathbb{R}\}$ be a standard two-sided real Brownian motion and $Y = \{Y_t, t \geq 0\}$ be a strictly stable Lévy process with index $\alpha \in (1, 2]$ such that $Y_0 = 0$. More precisely, for some positive scale-parameter c, the characteristic function of the random variable Y_1 is given by

$$\forall \theta \in \mathbb{R}, \quad \mathbb{E}[e^{i\theta Y_1}] = \exp\left\{ -c|\theta|^\alpha \left(1 + i\gamma\,\mathrm{sgn}(\theta)\tan(\pi\alpha/2)\right)\right\}. \tag{1.1}$$

where $\gamma \in [-1, 1]$. We will denote by $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$ a continuous version with compact support of the local time of the process $\{Y_t; t \geq 0\}$. The processes W and Y are defined on the same probability space and are assumed to be independent. We consider the **random process in random scenery** $\{\Delta_t; t \geq 0\}$ defined as

$$\Delta_t = \int_\mathbb{R} L_t(x)\, dW(x).$$

The process Δ is known to be a continuous δ-self-similar process with stationary increments, with

$$\delta := 1 - \frac{1}{2\alpha}.$$

This process can be seen as a mixture of Gaussian processes, but it is neither Gaussian nor Markovian. In this article, we study the asymptotic behaviour of

$$\mathcal{F}(T) := \mathbb{P}\left[\sup_{t \in [0,T]} \Delta_t \leq 1\right]$$

as $T \to +\infty$. Our main result is the following one.

Theorem 1. *For any $\alpha \in (1, 2]$, there exists a constant $c = c(\alpha) > 0$, such that for T large enough,*

$$T^{-1/(2\alpha)}(\log T)^{-c} \leq \mathbb{P}\left[\sup_{t \in [0,T]} \Delta_t \leq 1\right] \leq T^{-1/(2\alpha)}(\log T)^{+c}.$$

2 Lower Bound

For a certain class of stochastic processes $\{X_t; t \geq 0\}$ (to be specified below), Molchan [14] proved that the asymptotic behavior of

$$\mathbb{P}\left[\sup_{t \in [0,T]} X_t \leq 1\right]$$

is related to the quantity

$$I(T) := \mathbb{E}\left[\left(\int_0^T e^{X_t}\, dt\right)^{-1}\right].$$

We refer to [2] where the relationship between both quantities is clearly explained as well as the heuristics.

Theorem 2 (Statement 1, [14]). *Let $\{X_t; t \geq 0\}$ be a continuous process, self-similar with index $H \in (0, 1)$, with stationary increments s.t. for every $\theta > 0$,*

$$\mathbb{E}\left[\exp\left(\theta \max_{t \in [0,1]} |X_t|\right)\right] < +\infty.$$

Then, as $T \to +\infty$,

$$\mathbb{E}\left[\left(\int_0^T e^{X_t}\,dt\right)^{-1}\right] = HT^{-(1-H)}\left(\mathbb{E}\left[\max_{t\in[0,1]} X_t\right] + o(1)\right).$$

By applying this result to our random process Δ we get

Proposition 2.1. *For any $\alpha \in (1,2]$, as $T \to +\infty$,*

$$\mathbb{E}\left[\left(\int_0^T e^{\Delta_t}\,dt\right)^{-1}\right] = \left(1 - \frac{1}{2\alpha}\right)T^{-1/(2\alpha)}\left(\mathbb{E}\left[\max_{t\in[0,1]} \Delta_t\right] + o(1)\right).$$

Proof. The process $\{\Delta_t; t \geq 0\}$ being continuous, self-similar with index $\delta := 1 - \frac{1}{2\alpha}$, with stationary increments, it is enough to prove that for every $\theta > 0$,

$$\mathbb{E}\left[\exp\left(\theta \max_{t\in[0,1]} |\Delta_t|\right)\right] < +\infty. \tag{2.1}$$

Let $\theta > 0$. We have

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(\theta \max_{t\in[0,1]} |\Delta_t|\right)\right] &= \int_0^\infty \mathbb{P}\left(\exp\left(\theta \max_{t\in[0,1]} |\Delta_t|\right) \geq \lambda\right) d\lambda \\
&\leq 2 + \int_2^\infty \mathbb{P}\left(\max_{t\in[0,1]} |\Delta_t| \geq \frac{\log(\lambda)}{\theta}\right) d\lambda.
\end{aligned}
$$

Since the process $\{\Delta_t; t \geq 0\}$ is symmetric,

$$
\begin{aligned}
\mathbb{P}\left(\max_{t\in[0,1]} |\Delta_t| \geq \frac{\log(\lambda)}{\theta}\right) &\leq 2\,\mathbb{P}\left(\max_{t\in[0,1]} \Delta_t \geq \frac{\log(\lambda)}{\theta}\right) \\
&\leq 4\,\mathbb{P}(\Delta_1 \geq (\log\lambda)/\theta),
\end{aligned}
$$

using the maximal inequality for the process Δ (see Theorem 2.1 in [10]). Moreover, from Theorem 5.1 in [10], there exist positive constants C and γ (depending on α) s.t. for every $x > 0$,

$$\mathbb{P}(\Delta_1 \geq x) \leq C\exp(-\gamma x^{2\alpha/(1+\alpha)}).$$

Since the function $\lambda \to \exp(-\gamma((\log\lambda)/\theta)^{2\alpha/(1+\alpha)})$ is integrable at infinity for any $\alpha \in (1,2]$ and any $\theta > 0$, assertion (2.1) follows. $\qquad\square$

Aurzada's proof of the lower bound in the H-index Fractional Brownian Motion $\{B_H(t); t \geq 0\}$ case (see [2]) rests on both following arguments: the self-similarity of the FBM and the inequality (valid for a large enough)

$$(\mathbb{E}|B_H(t) - B_H(s)|^a)^{1/a} = C(a)|t-s|^H,\ t,s \geq 0 \tag{2.2}$$

with $C(a) \leq ca^\nu$, for some c and $\nu > 0$. Our random process Δ being self-similar, it is enough to prove assertion (2.2) to derive the lower bound. The increments of the process Δ being stationary, by self-similarity, we have for every $t, s \geq 0$,

$$
\begin{aligned}
\mathbb{E}[|\Delta_t - \Delta_s|^a] &= |t-s|^{\delta a}\,\mathbb{E}[|\Delta_1|^a] \\
&\leq |t-s|^{\delta a}\,\mathbb{E}[|\Delta_1|^{2([a]+1)}]^{a/(2([a]+1))}
\end{aligned}
$$

Conditionally to the process Y, the random variable Δ_1 is centered Gaussian with variance

$$V_1 := \int_{\mathbb{R}} L_1^2(x)\,dx.$$

From the independence of both processes Y and W and from the formula of the even moments of the centered reduced Gaussian law, we can derive the even moments of the random variable Δ_1, namely, for any $m \in \mathbb{N}$,

$$\mathbb{E}[\Delta_1^{2m}] = \mathbb{E}[V_1^m] \frac{(2m)!}{2^m m!}.$$

First of all, from Stirling's formula, for m large enough, we have

$$\frac{(2m)!}{2^m m!} \leq C.\left(\frac{2}{e}\right)^m m^m.$$

Moreover,

$$\mathbb{E}[V_1^m] = \int_0^\infty \mathbb{P}[V_1 \geq \lambda^{1/m}] \, d\lambda.$$

From Corollary 5.6 in [10], there exist positive constants C and ξ s.t. for every $\lambda > 0$,

$$\mathbb{P}[V_1 \geq \lambda] \leq Ce^{-\xi \lambda^\alpha}.$$

So, we have (the constant C may change from line to line but does not depend on $m \geq 1$)

$$
\begin{aligned}
\mathbb{E}[V_1^m] &\leq 2 + Cm \int_{2^{1/m}}^\infty e^{-\xi \lambda^\alpha} \lambda^{m-1} \, d\lambda \\
&\leq Cm \int_0^\infty e^{-\xi \lambda^2} \lambda^{\frac{2m}{\alpha}} \, d\lambda \\
&\leq Cc^{m/\alpha} m^{m/\alpha},
\end{aligned}
$$

for some constants $C, c > 0$. It is now easy to derive (2.2) namely

$$(\mathbb{E}|\Delta_t - \Delta_s|^a)^{1/a} = C(a)|t-s|^\delta, \ t,s \geq 0 \tag{2.3}$$

where $C(a) \leq ca^\nu$ with $\nu := \frac{1}{2}(1 + \frac{1}{\alpha})$.

3 Upper bound

As in [14] and [2], the main idea of the proof is to bound $I(T)$ from below by restricting the expectation to a well-chosen set of paths.

Observe that, conditionally to $Y = \{Y_t; t \geq 0\}$, the process $\{\Delta_t; t \geq 0\}$ is a centered Gaussian process such that for every $0 \leq t < s$,

$$\mathbb{E}[\Delta_t \Delta_s | Y] = \int_{\mathbb{R}} L_t(x) L_s(x) \, dx \geq 0,$$

$$\mathbb{E}[\Delta_t (\Delta_s - \Delta_t) | Y] = \int_{\mathbb{R}} L_t(x)(L_s(x) - L_t(x)) \, dx \geq 0,$$

since $t \to L_t(x)$ is a.s. increasing for all $x \in \mathbb{R}$. It follows then from Slepian's lemma, that for every $0 \leq u < v < w$ and every real numbers a, b,

$$\mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a, \ \sup_{t \in [v,w]} \Delta_t \leq b \Big| Y\right] \geq \mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a \Big| Y\right] \mathbb{P}\left[\sup_{t \in [v,w]} \Delta_t \leq b \Big| Y\right] \tag{3.1}$$

$$\mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a, \ \sup_{t \in [v,w]} (\Delta_t - \Delta_v) \leq b \Big| Y\right] \geq \mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a \Big| Y\right] \mathbb{P}\left[\sup_{t \in [v,w]} (\Delta_t - \Delta_v) \leq b \Big| Y\right]. \tag{3.2}$$

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Let $p > 0$ and $\beta > 0$. We define

$$a_T := (\log T)^p \quad \text{and} \quad \beta_T := \frac{a_T^{1 - \frac{1}{2\alpha} - \frac{1}{4\alpha^2}}}{(\log a_T)^\beta}.$$

For any $t > 0$, we write $|L_t|_2$ the random variable $\left(\int_\mathbb{R} L_t^2(x)\, dx \right)^{1/2}$. Let us consider the event

$$\mathcal{A}_T = \{ |L_{a_T}|_2 \geq \beta_T \}.$$

Lemma 3. *For all $p > 0$ and all $\beta > 0$,*

$$\mathbb{P}[\mathcal{A}_T^c] = O\left((\log a_T)^{-2\alpha\beta} a_T^{-\frac{1}{2\alpha}} \right) \quad \text{as } T \to +\infty.$$

Proof. First we notice that $|L_{a_T}|_2$ has the same distribution as $a_T^{1 - \frac{1}{2\alpha}} |L_1|_2$ and that, by the Cauchy-Schwartz inequality, we have

$$1 = \int_\mathbb{R} L_1(x)\, dx \leq |L_1|_2 \sqrt{S}, \quad \text{with } S := \sup_{s \in [0,1]} Y_s - \inf_{s \in [0,1]} Y_s.$$

Hence we have

$$
\begin{aligned}
\mathbb{P}[\mathcal{A}_T^c] &= \mathbb{P}[|L_{a_T}|_2 < \beta_T] \\
&= \mathbb{P}[|L_1|_2 < a_T^{-\frac{1}{4\alpha^2}} (\log a_T)^{-\beta}] \\
&\leq \mathbb{P}[S > a_T^{\frac{1}{2\alpha^2}} (\log a_T)^{2\beta}] \\
&\leq \mathbb{P}[\sup_{s \in [0,1]} Y_s > a_T^{\frac{1}{2\alpha^2}} (\log a_T)^{2\beta}/2] + \mathbb{P}[\sup_{s \in [0,1]} (-Y_s) > a_T^{\frac{1}{2\alpha^2}} (\log a_T)^{2\beta}/2] \quad (3.3)
\end{aligned}
$$

and so, for T large enough, due to Theorem 4.a in [3], we have

$$\mathbb{P}[\mathcal{A}_T^c] = O(a_T^{-\frac{1}{2\alpha}} (\log a_T)^{-2\alpha\beta}).$$

(Remark that in the case $\gamma = 1$ where γ is defined in (1.1), it follows from (8) in [3] that the first probability in (7) is zero. Theorem 4.a [3] can then be applied to the Lévy process $\{-Y_t; t \geq 0\}$ which is strictly stable with index α and $\gamma = -1$). The lemma follows. $\qquad\square$

Let us define the function

$$\phi(t) := \begin{cases} 1 & \text{for } 0 \leq t < a_T, \\ 1 - \beta_T & \text{for } a_T \leq t \leq T. \end{cases}$$

Clearly, we have

$$\mathbb{E}\left[\left(\int_0^T e^{\Delta_t}\, dt \right)^{-1} \Big| Y \right] \geq \left(\int_0^T e^{\phi(t)}\, dt \right)^{-1} \mathbb{P}\left[\forall t \in [0, T], \Delta_t \leq \phi(t) \Big| Y \right].$$

When $p(1 - \frac{1}{2\alpha} - \frac{1}{4\alpha^2}) > 1$, it is easy to show that

$$\int_0^T e^{\phi(t)}\, dt = O(a_T).$$

By Slepian's lemma (see (3.1)), we have

$$\mathbb{P}\left[\forall t \in [0, T], \Delta_t \leq \phi(t) \Big| Y \right] \geq \mathbb{P}\left[\forall t \in [0, a_T], \Delta_t \leq 1 \Big| Y \right] \mathbb{P}\left[\forall t \in [a_T, T], \Delta_t \leq 1 - \beta_T \Big| Y \right].$$

Remark that

$$\mathbb{P}\Big[\forall t \in [a_T, T], \Delta_t \leq 1 - \beta_T \Big| Y \Big] \geq \mathbb{P}\Big[\Delta_{a_T} \leq -\beta_T ; \forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \Big| Y \Big].$$

Conditionally to Y, the increments of the process $\{\Delta_t ; t \geq 0\}$ being Gaussian and positively correlated, by Slepian's lemma (see (3.2)), we get

$$\mathbb{P}\Big[\forall t \in [a_T, T], \Delta_t \leq 1 - \beta_T \Big| Y \Big] \geq \mathbb{P}\Big[\Delta_{a_T} \leq -\beta_T \Big| Y \Big] \mathbb{P}\Big[\forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \Big| Y \Big].$$

Conditionally to Y, the random variable Δ_{a_T} is centered Gaussian with variance $|L_{a_T}|_2^2$ and so $\mathbb{P}[\Delta_{a_T} \leq -\beta_T | Y] = F(-\frac{\beta_T}{|L_{a_T}|_2})$ where F is the distribution function of the Normal distribution $\mathcal{N}(0, 1)$. On the event $\mathcal{A}_T$, we then have

$$\mathbb{P}[\Delta_{a_T} \leq -\beta_T | Y] \geq F(-1).$$

Moreover,

$$\Delta_t - \Delta_{a_T} = \int \tilde{L}_{t-a_T}(x - Y_{a_T}) \, dW(x)$$

where $\tilde{L}$ is the local time of the process $\Big\{ \tilde{Y}_t ; t \geq 0 \Big\}$ defined as $\tilde{Y}_t := Y_{a_T+t} - Y_{a_T}$ for $t \geq 0$. Conditionally to Y, the processes $\{\Delta_t - \Delta_{a_T} ; t \geq a_T\}$ and $\Big\{ \int \tilde{L}_{t-a_T}(x) \, dW(x) ; t \geq a_T \Big\}$ have the same distribution and $\mathbb{P}\Big[\forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \Big| Y \Big]$ is therefore $\sigma\Big(\tilde{Y}_t, t \geq 0\Big)$-measurable. Finally, on the event $\mathcal{A}_T$, we get

$$\frac{\int_0^T e^{\phi(t)} dt}{F(-1)} \mathbb{E}\left[\left(\int_0^T e^{\Delta_t} \, dt \right)^{-1} \Big| Y \right] \geq \mathbb{P}\Big[\forall t \in [0, a_T], \Delta_t \leq 1 \Big| Y \Big] \mathbb{P}\Big[\forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \Big| Y \Big].$$

The probabilities in the right hand side are respectively measurable with respect to the σ-fields $\sigma(Y_s, s \leq a_T)$ and $\sigma(Y_{a_T+s} - Y_{a_T}, s \geq 0)$, which are independent. We get therefore

$$\frac{\int_0^T e^{\phi(t)} dt}{F(-1)} \mathbb{E}\left[\left(\int_0^T e^{\Delta_t} \, dt \right)^{-1} \right]$$

$$\geq \quad \frac{\int_0^T e^{\phi(t)} dt}{F(-1)} \mathbb{E}\left[\left(\int_0^T e^{\Delta_t} \, dt \right)^{-1} \mathbf{1}_{\mathcal{A}_T} \right]$$

$$\geq \quad \mathbb{E}\Big[\mathbb{P}\Big[\forall t \in [0, a_T], \Delta_t \leq 1 \Big| Y \Big] \mathbf{1}_{\mathcal{A}_T} \Big] \mathbb{P}\Big[\forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \Big]$$

$$\geq \quad \Big(\mathbb{P}\Big[\forall t \in [0, a_T], \Delta_t \leq 1 \Big] - \mathbb{P}\Big[\mathcal{A}_T^c \Big] \Big) \mathbb{P}\Big[\forall t \in [0, T - a_T], \Delta_t \leq 1 \Big].$$

Let c be the exponent appearing in the lower bound. We choose $\beta > c/(2\alpha)$ and p such that $p(1 - \frac{1}{2\alpha} - \frac{1}{4\alpha^2}) > 1$. Due to Lemma 3 and to the lower bound of $\mathcal{F}(T)$, the first term in the right hand side is larger than $C(\log a_T)^{-c} a_T^{-1/(2\alpha)}$ for T large enough. The second term is clearly larger than

$$\mathbb{P}\Big[\forall t \in [0, T], \Delta_t \leq 1 \Big].$$

Therefore, we get

$$\mathbb{P}\Big[\forall t \in [0, T], \Delta_t \leq 1 \Big] \leq C(\log a_T)^c (a_T)^{1 + \frac{1}{2\alpha}} \mathbb{E}\left[\left(\int_0^T e^{\Delta_t} \, dt \right)^{-1} \right]$$

and the upper bound follows using the equivalent for $I(T)$ in Proposition 2.1.

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