Electron. Commun. Probab. **18** (2012), no. 59, 1–16.
DOI: 10.1214/ECP.v18-2445
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Theory of Barnes Beta distributions

Dmitry Ostrovsky*

Abstract

A new family of probability distributions $\beta_{M,N}$, $M = 0 \cdots N$, $N \in \mathbb{N}$ on the unit interval $(0, 1]$ is defined by the Mellin transform. The Mellin transform of $\beta_{M,N}$ is characterized in terms of products of ratios of Barnes multiple gamma functions, shown to satisfy a functional equation, and a Shintani-type infinite product factorization. The distribution $\log \beta_{M,N}$ is infinitely divisible. If $M < N$, $-\log \beta_{M,N}$ is compound Poisson, if $M = N$, $\log \beta_{M,N}$ is absolutely continuous. The integral moments of $\beta_{M,N}$ are expressed as Selberg-type products of multiple gamma functions. The asymptotic behavior of the Mellin transform is derived and used to prove an inequality involving multiple gamma functions and establish positivity of a class of alternating power series. For application, the Selberg integral is interpreted probabilistically as a transformation of $\beta_{1,1}$ into a product of $\beta^{-1}_{2,2}s$.

Keywords: Multiple gamma function ; Infinite divisibility ; Selberg Integral ; Mellin transform.
AMS MSC 2010: 11M32; 30D05; 33B15; 41A60; 60E07; 60E10.
Submitted to ECP on November 24, 2012, final version accepted on July 4, 2013.

This paper addresses the question of what type of probability distributions[1] one can obtain by considering products of ratios of Barnes multiple gamma functions [2]. Classically, it is well-known that the Mellin transform[2] of the beta distribution β with parameters $b_0, b_1 > 0$ and density $\Gamma(b_0 + b_1)/\Gamma(b_0)\Gamma(b_1)\, x^{b_0-1}(1-x)^{b_1-1}$ is given by

$$\mathbf{E}[\beta^q] = \frac{\Gamma(q + b_0)}{\Gamma(b_0)} \frac{\Gamma(b_0 + b_1)}{\Gamma(q + b_0 + b_1)},\ \Re(q) > -b_0.$$

The main contribution of this paper is to construct and study the main properties of a novel family of probability distributions on the unit interval $(0, 1]$ that naturally generalize the beta distribution to arbitrary multiple gamma functions. In particular, we show that all of these distributions have infinitely divisible logarithm, satisfy a functional equation and several symmetries, and admit a remarkable infinite-product factorization. We call them Barnes beta distributions.

Our paper contributes to several areas of current interest in probability theory. First, we contribute to the probabilistic study of Barnes multiple gamma functions and, more generally, the study of infinite divisibility in the context of special functions of analytic number theory complementing [3], [10], [14], [15]. We show that a new class of Lévy-Khinchine representations is naturally associated with multiple gamma functions. Moreover, the meromorphic functions that we introduce as the Mellin transform of the

*195 Idlewood Dr., Stamford, CT 06905, USA. E-mail: dm_ostrov@aya.yale.edu

[1]The terms “probability distribution” and “random variable” are used interchangeably in this paper.

[2]It is more natural to define the Mellin transform as $\int_0^\infty x^q\, f(x)\, dx$ as opposed to the usual $\int_0^\infty x^{q-1}\, f(x)\, dx$ for our purposes.

Barnes beta distributions appear to have a number analytic significance as their pole structure depends on the rationality of the parameters of the distribution.

Second, there is a long-standing interest in the literature in the study of Dufresne distributions, whose defining property is that their Mellin transform is given in the form of a product of ratios of Euler's gamma functions, confer [4], [5], and references therein. In addition, there have recently appeared a series of papers [9], [12], [13] that computed the Mellin transform of a certain functional of the stable process in the form of a product of ratios of Alexeiewsky-Barnes $G-$functions (or, equivalently, the double gamma function). In our own work on the Selberg integral [17] we introduced a different probability distribution having the same property that its Mellin transform is given in the form of a finite product of ratios of $G-$factors. The contribution of this paper is to show that there is a whole family of Barnes beta distributions on the unit interval $(0,1]$ that extends this property to arbitrary multiple gamma functions.

Third, we contribute to the probabilistic theory of the Selberg integral complementing [22]. We show that the Selberg integral extends as a function of its dimension to the Mellin transform of a probability distribution, which factorizes in terms of $\beta_{2,2}^{-1}s$. This leads us to a new interpretation of the Selberg integral.

As an application of our results, we introduce a novel class of power series, compute their Mellin transform, and prove their positivity by relating them to the Laplace transform of the Barnes beta distribution.

The main technical tool that we rely on in this paper is the remarkable approach to multiple gamma functions due to Ruijsenaars [18]. Ruijsenaars developed a novel Malmstén-type formula for a class of functions that includes the multiple log-gamma function as a special case. We prove the key infinitely divisibility property in complete generality, that is for the whole Ruijsenaars class, before specializing to the gamma functions. Most of our proofs are elementary as the strength of his approach allows us to reduce our arguments to simple properties of multiple Bernoulli polynomials.

The plan of the paper is as follows. In Section 1 we remind the reader of the basic properties of the Barnes gamma functions following [2] and [18]. In Section 2 we state our results. Section 3 gives examples of Barnes beta distributions. Section 4 explains the connection between $\beta_{1,1}$, $\beta_{2,2}$, and the Selberg integral. In Section 5 we present the proofs. Section 6 concludes with a summary.

1 Review of Multiple Gamma Functions

Let $f(t)$ be of the Ruijsenaars class, *i.e.* analytic for $\Re(t)>0$ and at $t=0$ and of at worst polynomial growth as $t\to\infty$, confer [18], Section 2. The main example that corresponds to the case of Barnes multiple gamma functions is

$$f(t)=t^M\prod_{j=1}^M(1-e^{-a_jt})^{-1} \tag{1.1}$$

for some integer $M\geq 0$ and parameters $a_j>0$, $j=1\cdots M$. For concreteness, the reader can assume with little loss of generality that $f(t)$ is defined by (1.1). Slightly modifying the definition in [18], we define generalized Bernoulli polynomials by

$$B_m^{(f)}(x)\triangleq\frac{d^m}{dt^m}|_{t=0}\big[f(t)e^{-xt}\big]. \tag{1.2}$$

The generalized zeta function is defined by

$$\zeta_M(s,\,w)\triangleq\frac{1}{\Gamma(s)}\int\limits_0^\infty t^{s-1}e^{-wt}\,f(t)\,\frac{dt}{t^M},\;\Re(s)>M,\,\Re(w)>0. \tag{1.3}$$

It is shown in [18] that $\zeta_M(s,\,w)$ has an analytic continuation to a function that is meromorphic in $s\in\mathbb{C}$ with simple poles at $s=1,2,\cdots M$. The generalized log-gamma function is then defined by

$$L_M(w)\triangleq\partial_s\zeta_M(s,\,w)|_{s=0},\;\Re(w)>0. \tag{1.4}$$

It can be analytically continued to a function that is holomorphic over $\mathbb{C}-(-\infty,0]$. The key results of [18] that we need are summarized in the following theorem.

Theorem 1.1 (Ruijsenaars)**.** $L_M(w)$ *satisfies the Malmstén-type formula for* $\Re(w)>0$,

$$L_M(w)=\int\limits_0^\infty\frac{dt}{t^{M+1}}\Big(e^{-wt}\,f(t)-\sum_{k=0}^{M-1}\frac{t^k}{k!}\,B_k^{(f)}(w)-\frac{t^M\,e^{-t}}{M!}\,B_M^{(f)}(w)\Big). \tag{1.5}$$

$L_M(w)$ *satisfies the asymptotic expansion,*

$$L_M(w)=-\frac{1}{M!}B_M^{(f)}(w)\,\log(w)+\sum_{k=0}^{M}\frac{B_k^{(f)}(0)(-w)^{M-k}}{k!(M-k)!}\sum_{l=1}^{M-k}\frac{1}{l}+R_M(w), \tag{1.6}$$

$$R_M(w)=O(w^{-1}),\,|w|\to\infty,\,|\arg(w)|<\pi. \tag{1.7}$$

In the special case of the function $f(t)$ being defined by (1.1), the generalized zeta and gamma functions have important additional properties. It is not difficult to show that (1.3) becomes

$$\zeta_M\big(s,\,w\,|\,a\big)=\sum_{k_1,\cdots,k_M=0}^{\infty}\big(w+k_1a_1+\cdots+k_Ma_M\big)^{-s},\;\Re(s)>M,\,\Re(w)>0, \tag{1.8}$$

for $a=(a_1,\cdots,a_M)$, which is the formula given originally by Barnes [2] for the multiple zeta function. Let $L_M(w\,|\,a)$ be defined by (1.4) with $f(t)$ as in (1.1). Now, following [18], define[3] the Barnes multiple gamma function by

$$\Gamma_M(w\,|\,a)\triangleq\exp\big(L_M(w\,|\,a)\big). \tag{1.9}$$

It follows from (1.8) and (1.9) that $\Gamma_M(w\,|\,a)$ satisfies the fundamental functional equation

$$\Gamma_M(w\,|\,a)=\Gamma_{M-1}(w\,|\,\hat{a}_i)\,\Gamma_M\big(w+a_i\,|\,a\big),\,i=1\cdots M,\;\;M=1,2,3\cdots, \tag{1.10}$$

$\hat{a}_i=(a_1,\cdots,a_{i-1},\,a_{i+1},\cdots,a_M)$, and $\Gamma_0(w)=1/w$, which is also due to [2]. By iterating (1.10) one sees that $\Gamma_M(w\,|\,a)$ is meromorphic over $\mathbb{C}$ having no zeroes and poles at

$$w=-(k_1a_1+\cdots+k_Ma_M),\;k_1\cdots k_M\in\mathbb{N}, \tag{1.11}$$

with multiplicity equal the number of $M-$tuples $(k_1,\cdots,k_M)$ that satisfy (1.11).

We conclude our review of the Barnes functions by relating the general results to the classical case of Euler's gamma and Hurwitz's zeta functions. Following [18], we have the identities

$$\Gamma_1(w\,|\,a)=\frac{a^{w/a-1/2}}{\sqrt{2\pi}}\,\Gamma(w/a), \tag{1.12}$$

$$\zeta_1(s,w\,|\,a)=a^{-s}\zeta(s,\,w/a), \tag{1.13}$$

and the asymptotic expansion in Theorem 1.1 becomes Stirling's series.

[3] Barnes [2] used a slightly different normalization, which does not affect our results as we are primarily interested in ratios of Barnes gamma functions.

2 Barnes Beta Distribution

In this section we will define and describe the main properties of what we call Barnes beta distributions. We begin by introducing a combinatorial operator $\mathcal{S}_N$ that plays a central role in the formulation of our results.

Let $\{b_k\}$, $k \in \mathbb{N}$ be a sequence of positive real numbers and $N, M \in \mathbb{N}$. Let the symbol $\sum\limits_{k_1<\cdots<k_p=1}^{N}$ denote the sum over all indices $k_i = 1\cdots N$, $i = 1\cdots p$, satisfying $k_1 < \cdots < k_p$. Define the action of the operator $\mathcal{S}_N$ by

Definition 2.1.

$$(\mathcal{S}_N f)(q\,|\,b) \triangleq \sum_{p=0}^{N} (-1)^p \sum_{k_1<\cdots<k_p=1}^{N} f\big(q + b_0 + b_{k_1} + \cdots + b_{k_p}\big). \tag{2.1}$$

In other words, in (2.1) the action of $\mathcal{S}_N$ is defined as an alternating sum over all combinations of p elements for every $p = 0\cdots N$. Given a function $f(t)$ of Ruijsenaars class, confer Section 1, such that $f(t) > 0$ for $t \geq 0$, let $L_M(w)$ be the corresponding generalized log-gamma function defined in (1.4). The main example is the function $f(t)$ in (1.1) so that $L_M(w) = L_M(w\,|\,a)$ is the Barnes multiple log-gamma function. We can now define the main object that we will study in this paper.

Definition 2.2. *Given $q \in \mathbb{C} - (-\infty, -b_0]$, let*

$$\eta_{M,N}(q\,|\,b) \triangleq \exp\Big(\big(\mathcal{S}_N L_M\big)(q\,|\,b) - \big(\mathcal{S}_N L_M\big)(0\,|\,b)\Big). \tag{2.2}$$

The function $\eta_{M,N}(q\,|\,b)$ is holomorphic over $q \in \mathbb{C} - (-\infty, -b_0]$ and equals a product of ratios of generalized gamma functions by construction. Denoting $\Gamma_M(w) = \exp\big(L_M(w)\big)$, it is easy to write out examples of $\eta_{M,N}(q\,|\,b)$ for small N.

Example 2.3.

$$\eta_{M,0}(q\,|\,b) = \frac{\Gamma_M(q+b_0)}{\Gamma_M(b_0)},\ \eta_{M,1}(q\,|\,b) = \frac{\Gamma_M(q+b_0)}{\Gamma_M(b_0)}\frac{\Gamma_M(b_0+b_1)}{\Gamma_M(q+b_0+b_1)}, \tag{2.3}$$

$$\eta_{M,2}(q\,|\,b) = \frac{\Gamma_M(q+b_0)}{\Gamma_M(b_0)}\frac{\Gamma_M(b_0+b_1)}{\Gamma_M(q+b_0+b_1)}\frac{\Gamma_M(b_0+b_2)}{\Gamma_M(q+b_0+b_2)}\frac{\Gamma_M(q+b_0+b_1+b_2)}{\Gamma_M(b_0+b_1+b_2)}. \tag{2.4}$$

We now proceed to state our results.[4] We begin with the general case and then specialize to that of multiple gamma functions.

Theorem 2.4 (Existence)**.** *Given $M, N \in \mathbb{N}$ such that $M \leq N$, the function $\eta_{M,N}(q\,|\,b)$ is the Mellin transform of a probability distribution on $(0,1]$. Denote it by $\beta_{M,N}(b)$. Then,*

$$\mathbf{E}\big[\beta_{M,N}(b)^q\big] = \eta_{M,N}(q\,|\,b),\ \Re(q) > -b_0. \tag{2.5}$$

The distribution $-\log\beta_{M,N}(b)$ is infinitely divisible on $[0,\infty)$ and has the Lévy-Khinchine decomposition

$$\mathbf{E}\Big[\exp\big(-q\log\beta_{M,N}(b)\big)\Big] = \exp\Big(\int\limits_0^\infty (e^{tq}-1)e^{-b_0 t}\prod_{j=1}^{N}(1-e^{-b_j t})\frac{f(t)}{t^{M+1}}dt\Big),\ \Re(q) < b_0. \tag{2.6}$$

[4] Our results in the case of $M = 1$ correspond to a special case of the theory of Dufresne distributions (also known as G distributions) and were first obtained in [5]. The case of $M = N = 2$ first appeared in [17].

Corollary 2.5 (Structure). *If $M = N$, $\log \beta_{M,N}(b)$ is absolutely continuous. If $M < N$, $-\log \beta_{M,N}(b)$ is compound Poisson and*

$$\mathbf{P}\big[\beta_{M,N}(b) = 1\big] = \exp\Big(-\int\limits_0^\infty e^{-b_0 t} \prod_{j=1}^N (1-e^{-b_j t}) \frac{f(t)}{t^{M+1}} dt\Big). \tag{2.7}$$

Theorem 2.6 (Asymptotics). *If $M < N$ and $|\arg(q)| < \pi$,*

$$\lim_{q\to\infty} \eta_{M,N}(q\,|\,b) = \exp\big(-(\mathcal{S}_N L_M)(0\,|\,b)\big). \tag{2.8}$$

If $M = N$ and $|\arg(q)| < \pi$,

$$\eta_{N,N}(q\,|\,b) = \exp\big(-b_1\cdots b_N f(0)\log(q) + O(1)\big),\ q\to\infty. \tag{2.9}$$

Corollary 2.7 (Positivity). *If $M < N$,*

$$(\mathcal{S}_N L_M)(0\,|\,b) > 0. \tag{2.10}$$

From now on we restrict our attention to Barnes multiple gamma functions, *i.e.* $f(t)$ is as in (1.1), and write $\eta_{M,N}(q\,|\,a,\,b)$ to indicate dependence on $(a_1,\cdots,a_M)$ and $(b_0,\cdots,b_N)$. Also, $\hat{c}_i \triangleq (\cdots, c_{i-1},\, c_{i+1},\cdots)$ and $\eta_{M,N}(q\,|\,a,\,b_j{+}x) \triangleq \eta_{M,N}(q\,|\,a,\cdots b_{j-1}, b_j{+}x, b_{j+1},\cdots)$, $x>0$. Note that $\eta_{M,N}(q\,|\,a,\,b)$ is symmetric in $(a_1,\cdots,a_M)$ and $(b_1,\cdots,b_N)$.

Theorem 2.8 (Functional Equation). $1\le M\le N$, $q\in\mathbb{C}-(-\infty,-b_0]$, $i=1\cdots M$,

$$\eta_{M,N}(q+a_i\,|\,a,\,b) = \eta_{M,N}(q\,|\,a,\,b)\,\exp\big(-(\mathcal{S}_N L_{M-1})(q\,|\,\hat{a}_i, b)\big). \tag{2.11}$$

Corollary 2.9 (Symmetries). $1\le M\le N$, $q\in\mathbb{C}-(-\infty,-b_0]$, $i=1\cdots M$, $j=1\cdots N$,

$$\eta_{M,N}(q\,|\,a,\,b_0+x)\,\eta_{M,N}(x\,|\,a,\,b) = \eta_{M,N}(q+x\,|\,a,\,b), \tag{2.12}$$
$$\eta_{M,N}(q\,|\,a,\,b)\,\eta_{M,N-1}(q\,|\,a,\,b_0+b_j,\hat{b}_j) = \eta_{M,N-1}(q\,|\,a,\,\hat{b}_j), \tag{2.13}$$
$$\eta_{M,N}(q+a_i\,|\,a,\,b)\,\eta_{M-1,N}(q\,|\,\hat{a}_i,\,b) = \eta_{M,N}(q\,|\,a,\,b)\,\eta_{M,N}(a_i\,|\,a,\,b), \tag{2.14}$$
$$\eta_{M,N}(q\,|\,a,\,b_j+a_i)\,\eta_{M-1,N-1}(b_j\,|\,\hat{a}_i,\,\hat{b}_j) = \eta_{M,N}(q\,|\,a,\,b)\,\eta_{M-1,N-1}(q+b_j\,|\,\hat{a}_i,\,\hat{b}_j), \tag{2.15}$$
$$\eta_{M,N}(q+a_i\,|\,a,\,b)\,\eta_{M-1,N-1}(q\,|\,\hat{a}_i,\,\hat{b}_j) = \eta_{M,N}(q\,|\,a,\,b)\,\eta_{M-1,N-1}(q+b_j\,|\,\hat{a}_i,\,\hat{b}_j). \tag{2.16}$$

Corollary 2.10 (Moments). *Given $k\in\mathbb{N}$, the positive moments of $\beta_{M,N}(a,\,b)^{a_i}$ satisfy*

$$\begin{aligned}\mathbf{E}\big[\beta_{M,N}(a,b)^{ka_i}\big] &= \exp\Big(-\sum_{l=0}^{k-1}\big(\mathcal{S}_N L_{M-1}\big)(la_i\,|\,\hat{a}_i,b)\Big),\\ &= \prod_{l=0}^{k-1}\Big[\frac{\prod\limits_{j_1=1}^N \Gamma_{M-1}(la_i+b_0+b_{j_1}\,|\,\hat{a}_i)}{\Gamma_{M-1}(la_i+b_0\,|\,\hat{a}_i)}\star\\ &\quad\star\frac{\prod\limits_{j_1<j_2<j_3}^N \Gamma_{M-1}(la_i+b_0+b_{j_1}+b_{j_2}+b_{j_3}\,|\,\hat{a}_i)}{\prod\limits_{j_1<j_2}^N \Gamma_{M-1}(la_i+b_0+b_{j_1}+b_{j_2}\,|\,\hat{a}_i)}\cdots\Big].\end{aligned} \tag{2.17}$$

Given $k \in \mathbb{N}$ such that $ka_i < b_0$, the negative moments of $\beta_{M,N}(a,b)^{a_i}$ satisfy

$$\begin{aligned}\mathbf{E}\big[\beta_{M,N}(a,b)^{-ka_i}\big] &= \exp\Big(\sum_{l=0}^{k-1}\big(\mathcal{S}_N L_{M-1}\big)(-(l+1)a_i\,|\,\hat{a}_i,b)\Big),\\ &= \prod_{l=0}^{k-1}\Big[\frac{\Gamma_{M-1}(-(l+1)a_i+b_0\,|\,\hat{a}_i)}{\prod\limits_{j_1=1}^{N}\Gamma_{M-1}(-(l+1)a_i+b_0+b_{j_1}\,|\,\hat{a}_i)}\star\\ &\star\frac{\prod\limits_{j_1<j_2}^{N}\Gamma_{M-1}(-(l+1)a_i+b_0+b_{j_1}+b_{j_2}\,|\,\hat{a}_i)}{\prod\limits_{j_1<j_2<j_3}^{N}\Gamma_{M-1}(-(l+1)a_i+b_0+b_{j_1}+b_{j_2}+b_{j_3}\,|\,\hat{a}_i)}\cdots\Big].\end{aligned} \tag{2.18}$$

Corollary 2.11 (Laplace Transform)**.** *The power series*

$$\mathcal{L}^{(i)}_{M,N}(x\,|\,a,b) \triangleq \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\exp\Big(-\sum_{l=0}^{k-1}\big(\mathcal{S}_N L_{M-1}\big)(la_i\,|\,\hat{a}_i,b)\Big) \tag{2.19}$$

has infinite radius of convergence and gives the Laplace transform of $\beta_{M,N}(a,b)^{a_i}$.

$$\mathbf{E}\Big[\exp\big(-x\beta_{M,N}(a,b)^{a_i}\big)\Big] = \mathcal{L}^{(i)}_{M,N}(x\,|\,a,b),\; x>0. \tag{2.20}$$

In particular, for $x>0$,

$$\mathcal{L}^{(i)}_{M,N}(x\,|\,a,b) > 0. \tag{2.21}$$

Corollary 2.12 (Ramanujan Representation)**.** *The Laplace transform satisfies*

$$\int\limits_0^\infty x^{q-1}\mathcal{L}^{(i)}_{M,N}(x\,|\,a,b)\,dx = \Gamma(q)\,\eta_{M,N}(-qa_i\,|\,a,b),\; 0<\Re(q)<b_0/a_i. \tag{2.22}$$

Theorem 2.13 (Shintani Factorization)**.** *Given $1\le M\le N$ and $q\in\mathbb{C}-(-\infty,-b_0]$,*

$$\eta_{M,N}(q\,|\,a,b) = \prod_{k=0}^{\infty}\eta_{M-1,N}(q\,|\,\hat{a}_i,b_0+ka_i), \tag{2.23}$$

$$= \prod_{k=0}^{\infty}\frac{\eta_{M-1,N}(q+ka_i\,|\,\hat{a}_i,b)}{\eta_{M-1,N}(ka_i\,|\,\hat{a}_i,b)}, \tag{2.24}$$

$$= \prod_{k=0}^{\infty}\frac{\eta_{M-1,N-1}(q+ka_i\,|\,\hat{a}_i,\hat{b}_j)}{\eta_{M-1,N-1}(ka_i\,|\,\hat{a}_i,\hat{b}_j)}\,\frac{\eta_{M-1,N-1}(ka_i+b_j\,|\,\hat{a}_i,\hat{b}_j)}{\eta_{M-1,N-1}(q+ka_i+b_j\,|\,\hat{a}_i,\hat{b}_j)}. \tag{2.25}$$

Remark 2.14. *This factorization is the analogue of the factorization of the Barnes multiple gamma function $\Gamma_M(w\,|\,a)$ into the product of ratios of $\Gamma_{M-1}(w\,|\,\hat{a}_i)$ originally due to [20] for $M=2$ and, in general, due to [11].*

Corollary 2.15 (Solution to Functional Equations)**.** *The infinite product representation in Theorem 2.13 is the solution to the functional equation in Theorem 2.8.*

We conclude this section with two results that hold for special values of a and b.

Theorem 2.16 (Reduction to Independent Factors)**.** *Given i,j, if $b_j = n\,a_i$ for $n\in\mathbb{N}$,*

$$\beta_{M,N}(a,b) \overset{\text{in law}}{=} \prod_{k=0}^{n-1}\beta_{M-1,N-1}\big(\hat{a}_i, b_0+ka_i,\,\hat{b}_j\big). \tag{2.26}$$

Theorem 2.17 (Moments)**.** *Let $a_i = 1$ for all $i = 1 \cdots M$. Then, for any $n \in \mathbb{N}$,*

$$\mathbf{E}\big[\beta_{M,N}(a,b)^n\big] = \prod_{i=1}^{M-1} e^{(-1)^i \binom{n}{i} (\mathcal{S}_N L_{M-i})(0\,|\,b)} \prod_{i_1=0}^{n-1}\prod_{i_2=0}^{i_1-1}\cdots\prod_{i_M=0}^{i_{M-1}-1} e^{(-1)^M (\mathcal{S}_N L_0)(i_M\,|\,b)},$$

$$= \prod_{i=1}^{M-1} e^{(-1)^i \binom{n}{i} (\mathcal{S}_N L_{M-i})(0\,|\,b)} \prod_{i_1=0}^{n-1}\prod_{i_2=0}^{i_1-1}\cdots\prod_{i_M=0}^{i_{M-1}-1} \Big[\frac{\prod\limits_{j_1=1}^{N}(i_M+b_0+b_{j_1})}{(i_M+b_0)}\star$$

$$\star\,\frac{\prod\limits_{j_1<j_2<j_3}^{N}(i_M+b_0+b_{j_1}+b_{j_2}+b_{j_3})}{\prod\limits_{j_1<j_2}^{N}(i_M+b_0+b_{j_1}+b_{j_2})}\cdots\Big]^{(-1)^M}. \tag{2.27}$$

We note that the structure of $\beta_{M,N}(a)$ depends on rationality of $(a_1,\cdots,a_M)$ and $(b_1,\cdots,b_M)$. This is clear from Definition 2.2 as this structure is determined by ratios of multiple gamma functions that have poles specified in (1.11). This phenomenon was studied in a different context for the double gamma function in [9] and [12].

3 Examples

It is not difficult to compute $\beta_{M,N}(a,b)$ for small M and N. We give four examples that can be checked by direct inspection.

Example 3.1. *Let $\delta(x-1)$ be shorthand for an atom at $x=1$.*

$$\beta_{0,0} = b_0\, x^{b_0-1}\,dx, \tag{3.1}$$

$$\beta_{0,1} = \frac{b_1}{b_0+b_1} b_0\, x^{b_0-1}\,dx + \frac{b_0}{b_0+b_1}\delta(x-1)\,dx, \tag{3.2}$$

$$\beta_{1,1} = a_1 \frac{\Gamma\big((b_0+b_1)/a_1\big)}{\Gamma\big(b_0/a_1\big)\Gamma\big(b_1/a_1\big)} x^{b_0-1}(1-x^{a_1})^{b_1/a_1-1}\,dx, \tag{3.3}$$

$$\beta_{0,2} = \frac{b_0b_1b_2(b_0+b_1+b_2)}{(b_0+b_1)(b_0+b_2)(b_1+b_2)} x^{b_0-1}(1-x^{b_1+b_2})\,dx + \delta(x-1)\frac{b_0(b_0+b_1+b_2)}{(b_0+b_1)(b_0+b_2)}\,dx. \tag{3.4}$$

In the rest of this section we will focus on the special case of $M=N=2$ in order to illustrate the general theory with a concrete yet quite non-trivial example. In addition, this case is also of a particular interest in the probabilistic theory of the Selberg integral that we will review in Section 4. Let $a_1 = 1$ and $a_2 = \tau > 0$ and write $\beta_{2,2}(\tau,b)$, $\eta_{2,2}(q\,|\,\tau,b)$, and $\Gamma_2\big(w\,|\,(1,\tau)\big) = \Gamma_2(w\,|\,\tau)$ for brevity. From Definition 2.2 and Theorem 2.4 we have $\mathbf{E}\big[\beta_{2,2}(\tau,b)^q\big] = \eta_{2,2}(q\,|\,\tau,b)$ for $\Re(q) > -b_0$ and

$$\eta_{2,2}(q\,|\,\tau,b) = \frac{\Gamma_2(q+b_0\,|\,\tau)}{\Gamma_2(b_0\,|\,\tau)}\frac{\Gamma_2(b_0+b_1\,|\,\tau)}{\Gamma_2(q+b_0+b_1\,|\,\tau)}\frac{\Gamma_2(b_0+b_2\,|\,\tau)}{\Gamma_2(q+b_0+b_2\,|\,\tau)}\frac{\Gamma_2(q+b_0+b_1+b_2\,|\,\tau)}{\Gamma_2(b_0+b_1+b_2\,|\,\tau)}. \tag{3.5}$$

The asymptotic behavior of $\eta_{2,2}(q\,|\,\tau,b)$ follows from Theorem 2.6.

$$\eta_{2,2}(q\,|\,\tau,b) = \exp\Big(-\frac{b_1b_2}{\tau}\log(q) + O(1)\Big),\ q\to\infty,\ |\arg(q)|<\pi. \tag{3.6}$$

Using (1.12), the functional equation in Theorem 2.8 takes the form

$$\eta_{2,2}(q+1\,|\,\tau,b) = \eta_{2,2}(q\,|\,\tau,b)\,\frac{\Gamma\big((q+b_0+b_1)/\tau\big)\Gamma\big((q+b_0+b_2)/\tau\big)}{\Gamma\big((q+b_0)/\tau\big)\Gamma\big((q+b_0+b_1+b_2)/\tau\big)}, \tag{3.7}$$

$$\eta_{2,2}(q+\tau\,|\,\tau,b) = \eta_{2,2}(q\,|\,\tau,b)\,\frac{\Gamma(q+b_0+b_1)\Gamma(q+b_0+b_2)}{\Gamma(q+b_0)\Gamma\big(q+b_0+b_1+b_2\big)}. \tag{3.8}$$

The positive moments in Corollary 2.10 for $k \in \mathbb{N}$ are

$$\mathbf{E}\big[\beta_{2,2}(\tau,b)^k\big] = \prod_{l=0}^{k-1}\Big[\frac{\Gamma\big((l+b_0+b_1)/\tau\big)\,\Gamma\big((l+b_0+b_2)/\tau\big)}{\Gamma\big((l+b_0)/\tau\big)\,\Gamma\big((l+b_0+b_1+b_2)/\tau\big)}\Big], \tag{3.9}$$

$$\mathbf{E}\big[\beta_{2,2}(\tau,b)^{k\tau}\big] = \prod_{l=0}^{k-1}\Big[\frac{\Gamma(l\tau+b_0+b_1)\,\Gamma(l\tau+b_0+b_2)}{\Gamma(l\tau+b_0)\,\Gamma(l\tau+b_0+b_1+b_2)}\Big]. \tag{3.10}$$

The negative moments are

$$\mathbf{E}\big[\beta_{2,2}(\tau,b)^{-k}\big] = \prod_{l=0}^{k-1}\Big[\frac{\Gamma\big((-(l+1)+b_0)/\tau\big)\,\Gamma\big((-(l+1)+b_0+b_1+b_2)/\tau\big)}{\Gamma\big((-(l+1)+b_0+b_1)/\tau\big)\,\Gamma\big((-(l+1)+b_0+b_2)/\tau\big)}\Big],\ k<b_0, \tag{3.11}$$

$$\mathbf{E}\big[\beta_{2,2}(\tau,b)^{-k\tau}\big] = \prod_{l=0}^{k-1}\Big[\frac{\Gamma(-(l+1)\tau+b_0)\,\Gamma(-(l+1)\tau+b_0+b_1+b_2)}{\Gamma((-(l+1)\tau+b_0+b_1)\,\Gamma(-(l+1)\tau+b_0+b_2)}\Big],\ k\tau<b_0. \tag{3.12}$$

The positivity conditions in Corollary 2.11 for $x>0$ are

$$\mathbf{E}\Big[\exp\big(-x\beta_{2,2}(\tau,b)\big)\Big] = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\prod_{l=0}^{k-1}\Big[\frac{\Gamma\big((l+b_0+b_1)/\tau\big)\,\Gamma\big((l+b_0+b_2)/\tau\big)}{\Gamma\big((l+b_0)/\tau\big)\,\Gamma\big((l+b_0+b_1+b_2)/\tau\big)}\Big] > 0, \tag{3.13}$$

$$\mathbf{E}\Big[\exp\big(-x\beta_{2,2}(\tau,b)^{\tau}\big)\Big] = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\prod_{l=0}^{k-1}\Big[\frac{\Gamma(l\tau+b_0+b_1)\,\Gamma(l\tau+b_0+b_2)}{\Gamma(l\tau+b_0)\,\Gamma(l\tau+b_0+b_1+b_2)}\Big] > 0. \tag{3.14}$$

Corollary 2.12 gives for $0<\Re(q)<b_0$ and $0<\Re(q)<b_0/\tau$, respectively,

$$\Gamma(q)\,\eta_{2,2}(-q\,|\,\tau,b) = \int\limits_0^\infty x^{q-1}\Big\{\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\prod_{l=0}^{k-1}\Big[\frac{\Gamma\big((l+b_0+b_1)/\tau\big)\,\Gamma\big((l+b_0+b_2)/\tau\big)}{\Gamma\big((l+b_0)/\tau\big)\,\Gamma\big((l+b_0+b_1+b_2)/\tau\big)}\Big]\Big\}\,dx, \tag{3.15}$$

$$\Gamma(q)\,\eta_{2,2}(-q\tau\,|\,\tau,b) = \int\limits_0^\infty x^{q-1}\Big\{\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\prod_{l=0}^{k-1}\Big[\frac{\Gamma(l\tau+b_0+b_1)\,\Gamma(l\tau+b_0+b_2)}{\Gamma(l\tau+b_0)\,\Gamma(l\tau+b_0+b_1+b_2)}\Big]\Big\}\,dx. \tag{3.16}$$

Finally, the factorization equations in Theorem 2.13 are

$$\begin{aligned}\eta_{2,2}(q\,|\,\tau,b) = \prod_{k=0}^{\infty}\Big[&\frac{\Gamma((q+k+b_0)/\tau)}{\Gamma((k+b_0)/\tau)}\,\frac{\Gamma((k+b_0+b_1)/\tau)}{\Gamma((q+k+b_0+b_1)/\tau)}\,\frac{\Gamma((k+b_0+b_2)/\tau)}{\Gamma((q+k+b_0+b_2)/\tau)}\star\\ &\star\,\frac{\Gamma((q+k+b_0+b_1+b_2)/\tau)}{\Gamma((k+b_0+b_1+b_2)/\tau)}\Big],\end{aligned} \tag{3.17}$$

$$\begin{aligned}\eta_{2,2}(q\,|\,\tau,b) = \prod_{k=0}^{\infty}\Big[&\frac{\Gamma(q+k\tau+b_0)}{\Gamma(k\tau+b_0)}\,\frac{\Gamma(k\tau+b_0+b_1)}{\Gamma(q+k\tau+b_0+b_1)}\,\frac{\Gamma(k\tau+b_0+b_2)}{\Gamma(q+k\tau+b_0+b_2)}\star\\ &\star\,\frac{\Gamma(q+k\tau+b_0+b_1+b_2)}{\Gamma(k\tau+b_0+b_1+b_2)}\Big].\end{aligned} \tag{3.18}$$

The density of $\beta_{2,2}(\tau,b)$ can be computed by Laplace transform inversion. This computation requires a separate study similar to [9] as the structure of the residues of $\eta_{2,2}(q\,|\,\tau,b)$ depends on the rationality of τ, b_1, and b_2.

4 $\beta_{2,2}(\tau, b)$ and Selberg Integral

In this section we will review the application of $\beta_{2,2}(\tau, b)$ to the probabilistic structure of the celebrated Selberg integral that we developed in [17] using special properties of the Alexeiewsky-Barnes $G-$function (confer the appendix of [17] for a review of the $G-$function). The goal of re-formulating this structure here is to put it into the general framework of the Barnes beta distributions, which leads to a new interpretation of the Selberg integral.

The starting point of the probabilistic study of the Selberg integral is the following remarkable formula due to Selberg [19]. Given $0 < \mu < 2$, $\lambda_i > -\mu/2$, and $1 \leq l < 2/\mu$,

$$S_{\mu,l}\big[s^{\lambda_1}(1-s)^{\lambda_2}\big] = \prod_{k=0}^{l-1} \frac{\Gamma(1-(k+1)\mu/2)}{\Gamma(1-\mu/2)} \frac{\Gamma(1+\lambda_1-k\mu/2)\Gamma(1+\lambda_2-k\mu/2)}{\Gamma(2+\lambda_1+\lambda_2-(l+k-1)\mu/2)}, \tag{4.1}$$

$$S_{\mu,l}[\varphi] \triangleq \int\limits_{[0,\,1]^l} \prod_{i=1}^{l} \varphi(s_i) \prod_{i<j}^{l} |s_i - s_j|^{-\mu} ds_1 \cdots ds_l. \tag{4.2}$$

The reader who is familiar with the classical approach to the Selberg integral, confer [7], will notice that we have written Selberg's formula in a somewhat peculiar form. The reason for restricting $0 < \mu < 2$ is that in this case, given a general function $\varphi(s)$, $S_{\mu,l}[\varphi]$ equals the lth moment of the probability distribution constructed by integrating $\varphi(s)$ with respect to the limit lognormal stochastic measure,[5] confer [17]. In an attempt to compute this distribution for $\varphi(s) = s^{\lambda_1}(1-s)^{\lambda_2}$, in [17] we constructed[6] and factorized a probability distribution[7] having the moments given by Selberg's formula in (4.1).

Theorem 4.1. *Let $0 < \mu < 2$, $\lambda_i > -\mu/2$, and $\tau = 2/\mu$. Define the Mellin transform*

$$\begin{aligned}\mathbf{E}\big[M^q_{(\mu,\lambda_1,\lambda_2)}\big] \triangleq \tau^{\frac{q}{\tau}} (2\pi)^q\, \Gamma^{-q}\big(1-1/\tau\big) \frac{\Gamma_2(1-q+\tau(1+\lambda_1)\,|\,\tau)}{\Gamma_2(1+\tau(1+\lambda_1)\,|\,\tau)} \star \\ \star\, \frac{\Gamma_2(1-q+\tau(1+\lambda_2)\,|\,\tau)}{\Gamma_2(1+\tau(1+\lambda_2)\,|\,\tau)} \frac{\Gamma_2(-q+\tau\,|\,\tau)}{\Gamma_2(\tau\,|\,\tau)} \frac{\Gamma_2(2-q+\tau(2+\lambda_1+\lambda_2)\,|\,\tau)}{\Gamma_2(2-2q+\tau(2+\lambda_1+\lambda_2)\,|\,\tau)}\end{aligned} \tag{4.3}$$

for $\Re(q) < \tau$. Then, $M_{(\mu,\lambda_1,\lambda_2)}$ is a probability distribution on $(0,\infty)$ and

$$\mathbf{E}\big[M^l_{(\mu,\lambda_1,\lambda_2)}\big] = S_{\mu,l}\big[s^{\lambda_1}(1-s)^{\lambda_2}\big],\ 1 \leq l < \tau. \tag{4.4}$$

$\log M_{(\mu,\lambda_1,\lambda_2)}$ is absolutely continuous and infinitely divisible.

We can now relate the Selberg integral and $\beta_{2,2}(\tau, b)$. Let $\tau > 1$ and define

$$L \triangleq \exp\big(\mathcal{N}(0,\, 4\log 2/\tau)\big),\ Y \triangleq \tau\, y^{-1-\tau} \exp\big(-y^{-\tau}\big)\, dy,\ y > 0, \tag{4.5}$$

i.e. $\log L$ is a zero-mean normal with variance $4\log 2/\tau$ and Y is a power of the exponential. Given $\lambda_i > -1/\tau$, let X_1, X_2, X_3 have the $\beta^{-1}_{2,2}(\tau, b)$ distribution with the

[5] We will not attempt to quantify this statement here as it would take us too far afield and it is solely used to motivate Theorem 4.1 and our interpretation of the Selberg integral in Remark 4.3.

[6] In the special case of $\lambda_1 = \lambda_2 = 0$ an equivalent formula for the Mellin transform first appeared in [16]. The general case was first considered by [8], who gave an equivalent expression for the right-hand side of (4.3) and so matched the moments without proving that it corresponds to a probability distribution.

[7] If $M_{(\mu,\lambda_1,\lambda_2)}$ is the sought distribution as conjectured in [17], then the equality of their moments gives a probabilistic derivation of Selberg's formula by (4.2) and Theorem 4.1.

parameters[8]

$$X_1 \triangleq \beta_{2,2}^{-1}\Big(\tau, b_0 = 1+\tau+\tau\lambda_1,\, b_1 = \tau(\lambda_2-\lambda_1)/2,\, b_2 = \tau(\lambda_2-\lambda_1)/2\Big), \tag{4.6}$$

$$X_2 \triangleq \beta_{2,2}^{-1}\Big(\tau, b_0 = 1+\tau+\tau(\lambda_1+\lambda_2)/2,\, b_1 = 1/2,\, b_2 = \tau/2\Big), \tag{4.7}$$

$$X_3 \triangleq \beta_{2,2}^{-1}\Big(\tau, b_0 = 1+\tau,\, b_1 = (1+\tau+\tau\lambda_1+\tau\lambda_2)/2,\, b_2 = (1+\tau+\tau\lambda_1+\tau\lambda_2)/2\Big). \tag{4.8}$$

Theorem 4.2. *Let $\tau = 2/\mu$. $M_{(\mu,\lambda_1,\lambda_2)}$ decomposes into independent factors,*

$$M_{(\mu,\lambda_1,\lambda_2)} \overset{\text{in law}}{=} 2\pi\, 2^{-\big[3(1+\tau)+2\tau(\lambda_1+\lambda_2)\big]/\tau}\,\Gamma\big(1-1/\tau\big)^{-1}\, L\, X_1\, X_2\, X_3\, Y. \tag{4.9}$$

Remark 4.3 (Interpretation of Selberg Integral)**.** *The function $s^{\lambda_1}(1-s)^{\lambda_2}$ in* (4.1) *is, up to a constant, the density of $\beta_{1,1}\big(a_1 = 1, b_0 = 1+\lambda_1, b_1 = 1+\lambda_2\big)$, confer* (3.3)*. Selberg's formula and Theorems 4.1 and 4.2 extend the integral $S_{\mu,l}\big[$pdf of$\beta_{1,1}\big]$, viewed as a function of l, to the Mellin transform of $const\, L\, X_1\, X_2\, X_3\, Y$.*

$$\text{pdf of}\,\beta_{1,1} \xrightarrow{\text{Selberg}} S_{\mu,l}\big[\text{pdf of}\,\beta_{1,1}\big] \xrightarrow{\text{Ths. 4.1, 4.2}} const\, L\, X_1\, X_2\, X_3\, Y, \tag{4.10}$$

i.e. *the Selberg integral can be interpreted probabilistically as a transformation of $\beta_{1,1}$ into the product in* (4.9)*. It is an open question how to extend this mechanism to $\beta_{M,M}$,* i.e. *how to compute a probability distributions having $S_{\mu,l}\big[$pdf of$\beta_{M,M}\big]$ as its moments.*

5 Proofs

In this section we will give proofs of the results in Section 2. The proofs rely on Theorem 1.1, properties of infinitely divisible distributions, and the following lemma.

Lemma 5.1 (Main Lemma)**.** *Let $f(t)$ be of the Ruijsenaars class and the generalized Bernoulli polynomials be defined by* (1.2)*. Let $\{b_k\}$, $k \in \mathbb{N}$, be a sequence of real numbers, $n, r \in \mathbb{N}$, and $q \in \mathbb{C}$. Define the function $g(t)$*

$$g(t) \triangleq f(t)e^{-qt}\frac{d^r}{dt^r}\big[e^{-b_0 t}\prod_{j=1}^{N}(1-e^{-b_j t})\big]. \tag{5.1}$$

Then,

$$\begin{aligned} g^{(n)}(0) &= \sum_{m=0}^{n}\binom{n}{m}B^{(f)}_{n-m}(q)\frac{d^{m+r}}{dt^{m+r}}|_{t=0}\big[e^{-b_0 t}\prod_{j=1}^{N}(1-e^{-b_j t})\big], && (5.2)\\ &= (-1)^r\sum_{p=0}^{N}(-1)^p\sum_{k_1<\dots<k_p=1}^{N}\big(b_0+\sum b_{k_j}\big)^r\, B_n^{(f)}\big(q+b_0+\sum b_{k_j}\big), && (5.3)\\ &= 0, \text{ if } r+n<N, && (5.4)\\ &= f(0)\,N!\prod_{j=1}^{N}b_j, \text{ if } r+n=N. && (5.5)\end{aligned}$$

Proof. The expression in (5.2) follows from (1.2). Using the identity

$$\prod_{j=1}^{N}(1-e^{-b_j t}) = \sum_{p=0}^{N}(-1)^p\sum_{k_1<\dots<k_p=1}^{N}\exp\big(-(b_{k_1}+\dots+b_{k_p})t\big), \tag{5.6}$$

[8] The parameters of X_1 satisfy $b_1 b_2 > 0$, which is sufficient for Theorem 2.4 to hold by Theorem 3.5 in [17].

we can write

$$\frac{d^r}{dt^r}\big[e^{-b_0 t}\prod_{j=1}^N(1-e^{-b_j t})\big] = (-1)^r \sum_{p=0}^N (-1)^p \sum_{k_1<\cdots<k_p=1}^N \big(b_0+\sum b_{k_j}\big)^r \exp\big(-(b_0+\sum b_{k_j})t\big). \tag{5.7}$$

Substituting this expression into (5.1) and recalling (1.2), we obtain (5.3). (5.4) is immediate from the definition of $g(t)$ in (5.1), and (5.5) follows from (5.2). □

Corollary 5.2.

$$\big(\mathcal{S}_N\, B_n^{(f)}\big)(q) = 0,\; n = 0\cdots N-1, \tag{5.8}$$

$$\big(\mathcal{S}_N\, B_N^{(f)}\big)(q) = f(0)\, N! \prod_{j=1}^N b_j, \tag{5.9}$$

$$\big(\mathcal{S}_N\, x^n\big)(q) = 0,\; n = 0\cdots N-1, \tag{5.10}$$

$$\big(\mathcal{S}_N\, x^N\big)(q) = (-1)^N N! \prod_{j=1}^N b_j. \tag{5.11}$$

Proof. (5.8) and (5.9) follow from Lemma 5.1 by setting $r=0$ and recalling (2.1). (5.10) and (5.11) follows from (5.8) and (5.9) by letting $f(t)=1$ in Lemma 5.1 so that the corresponding Bernoulli polynomials are $B_n^{(f)}(x) = (-x)^n$. □

Proof of Theorem 2.4. Let $M \le N$ and $\Re(q) > -b_0$. We start with Definition 2.2 and substitute (1.5) for $L_M(w)$. By (5.8) in Corollary 5.2 and linearity of $\mathcal{S}_N$, we obtain

$$\eta_{M,N}(q\,|\,b) = \exp\Big(\int\limits_0^\infty \Big[\big(\mathcal{S}_N\, \exp(-xt)\big)(q) - \big(\mathcal{S}_N\, \exp(-xt)\big)(0)\Big] f(t) dt/t^{M+1}\Big). \tag{5.12}$$

Letting $r=0$ in (5.7), we have the identity

$$e^{-b_0 t} e^{-qt} \prod_{j=1}^N (1-e^{-b_j t}) = \big(\mathcal{S}_N \exp(-xt)\big)(q) \tag{5.13}$$

so that (5.12) can be simplified to

$$\eta_{M,N}(q\,|\,b) = \exp\Big(\int\limits_0^\infty (e^{-qt}-1) e^{-b_0 t} \prod_{j=1}^N (1-e^{-b_j t}) f(t)\, dt/t^{M+1}\Big). \tag{5.14}$$

This is the canonical representation of the Laplace transform of an infinitely divisible distribution on $[0,\infty)$, confer Theorem 4.3 in Chapter 3 of [21].

$$\eta_{M,N}(q\,|\,b) = \exp\Big(-\int\limits_0^\infty (1-e^{-tq}) dK_{M,N}^{(f)}(t\,|\,b)/t\Big), \tag{5.15}$$

$$dK_{M,N}^{(f)}(t\,|\,b) \triangleq e^{-b_0 t} \prod_{j=1}^N (1-e^{-b_j t}) f(t)\, dt/t^M. \tag{5.16}$$

It remains to note that $dK_{M,N}^{(f)}(t\,|\,b)$ satisfies the required integrability condition

$$\int\limits_0^\infty e^{-st} dK_{M,N}^{(f)}(t\,|\,b) = \int\limits_0^\infty e^{-st} e^{-b_0 t} \prod_{j=1}^N (1-e^{-b_j t}) f(t)\, dt/t^M < \infty,\; s>0. \tag{5.17}$$

Denote this non-negative distribution by $-\log\beta_{M,N}(b)$ so that $\beta_{M,N}(b)\in(0,\,1]$ and

$$\mathbf{E}\Big[\exp\big(q\log\beta_{M,N}(b)\big)\Big] = \eta_{M,N}(q\,|\,b),\ \Re(q) > -b_0. \tag{5.18}$$

This is equivalent to (2.6). □

Proof of Corollary 2.5. We note that

$$\int\limits_0^\infty dK^{(f)}_{M,N}(t\,|\,b)/t < \infty \text{ iff } M<N. \tag{5.19}$$

It follows from Proposition 4.13 in Chapter 3 of [21] that $\log\beta_{M,N}(b)$ is absolutely continuous if $M=N$. If $M<N$, $-\log\beta_{M,N}(b)$ is compound Poisson by Theorem 2.4 and Proposition 4.4 in Chapter 3 of [21]. In particular,

$$\mathbf{P}\big[\log\beta_{M,N}(b) = 0\big] = \exp\Big(-\int\limits_0^\infty dK^{(f)}_{M,N}(t\,|\,b)/t\Big). \tag{5.20}$$

The result follows from (5.16). □

Proof of Theorem 2.6. The starting point of the proof is (1.6). Substituting (1.6) into (2.2) and using linearity of $\mathcal{S}_N$, we can write in the limit of $q\rightarrow\infty$, $|\arg(q)|<\pi$,

$$\begin{aligned}\eta_{M,N}(q\,|\,b) = \exp\Big(-\big(\mathcal{S}_N L_M\big)(0\,|\,b)\Big)\exp\Big(-\frac{1}{M!}\mathcal{S}_N\big(B^{(f)}_M(w)\,\log(w)\big)(q\,|\,b)+ \\ +\sum_{k=0}^{M}\frac{B^{(f)}_k(0)\big(\mathcal{S}_N(-w)^{M-k}\big)(q\,|\,b)}{k!(M-k)!}\sum_{l=1}^{M-k}\frac{1}{l}+O(q^{-1})\Big).\end{aligned} \tag{5.21}$$

Now, to compute $\mathcal{S}_N\big(B^{(f)}_M(w)\,\log(w)\big)(q\,|\,b)$, we expand the logarithm in powers of $1/q$, resulting in terms of the form (5.3) with $n=M$. By (5.2) in Lemma 5.1, if $r+m>M$, then such terms are of order $O(1/q)$. If $r+m\leq M$ and $M<N$, they are all zero by (5.2). If $r+m\leq M$ and $M=N$, the only non-zero terms satisfy $r+m=N$ so that they have degree zero in q. Hence, we have the estimate

$$\mathcal{S}_N\big(B^{(f)}_M(w)\,\log(w)\big)(q\,|\,b) = \log(q)\,\mathcal{S}_N\big(B^{(f)}_M(w)\big)(q\,|\,b)+O(q^{-1}),\text{ if } M<N, \tag{5.22}$$

$$= \log(q)\,\mathcal{S}_N\big(B^{(f)}_M(w)\big)(q\,|\,b)+O(1),\text{ if } M=N. \tag{5.23}$$

If $M<N$, the expression in (5.22) is zero by (5.8) and the sum in (5.21) is zero by (5.10) so that (2.8) follows from (5.21). If $M=N$, the result follows from (5.9) and (5.11). □

Proof of Corollary 2.7. The result follows from (2.8) by letting $q\rightarrow+\infty$ in (5.15) and recalling (5.20). □

Proof of Theorem 2.8. It is sufficient to substitute (1.10), written in the form

$$L_M\big(w+a_i\,|\,a\big) = L_M(w\,|\,a) - L_{M-1}(w\,|\,\hat{a}_i), \tag{5.24}$$

into (2.2) and recall the definition of $\eta_{M-1,N}(q\,|\,\hat{a}_i, b)$. □

Proof of Corollary 2.9. To prove (2.12), note that Definition 2.1 implies the identity

$$(\mathcal{S}_N f)(q\,|\,b_0+x) = (\mathcal{S}_N f)(q+x\,|\,b), \tag{5.25}$$

and the result follows from Definition 2.2. (2.13) is immediate from Definition 2.2. (2.14) is equivalent to (2.11) due to the special case of $q = 0$ in (2.11),

$$\eta_{M,N}(a_i\,|\,a,b) = \exp\bigl(-(\mathcal{S}_N L_{M-1})(0\,|\,\hat{a}_i,b)\bigr). \tag{5.26}$$

The proof of (2.15) follows from (2.13), (2.12), and (2.14), in this order.

$$\begin{aligned}\eta_{M,N}(q\,|\,a,\,b_j+a_i) &= \frac{\eta_{M,N-1}(q\,|\,a,\,\hat{b}_j)}{\eta_{M,N-1}(q\,|\,a,\,b_0+b_j+a_i,\hat{b}_j)},\\ &= \eta_{M,N-1}(q\,|\,a,\,\hat{b}_j)\frac{\eta_{M,N-1}(a_i\,|\,a,\,b_0+b_j,\hat{b}_j)}{\eta_{M,N-1}(q+a_i\,|\,a,\,b_0+b_j,\hat{b}_j)},\\ &= \eta_{M-1,N-1}(q\,|\,\hat{a}_i,\,b_0+b_j,\hat{b}_j)\frac{\eta_{M,N-1}(q\,|\,a,\,\hat{b}_j)}{\eta_{M,N-1}(q\,|\,a,\,b_0+b_j,\hat{b}_j)}.\end{aligned} \tag{5.27}$$

The result follows by yet another application of (2.12) and (2.13). Finally, to verify (2.16), we combine (2.13) and (2.14) to obtain

$$\begin{aligned}\eta_{M,N}(q+a_i\,|\,a,\,b) &= \eta_{M,N}(q\,|\,a,\,b)\,\eta_{M,N}(a_i\,|\,a,\,b)\frac{\eta_{M-1,N-1}(q\,|\,\hat{a}_i,\,b_0+b_j,\hat{b}_j)}{\eta_{M-1,N-1}(q\,|\,\hat{a}_i,\,\hat{b}_j)},\\ &= \eta_{M,N}(q\,|\,a,\,b)\,\frac{\eta_{M,N}(a_i\,|\,a,\,b)}{\eta_{M-1,N-1}(b_j\,|\,\hat{a}_i,\hat{b}_j)}\frac{\eta_{M-1,N-1}(q+b_j\,|\,\hat{a}_i,\hat{b}_j)}{\eta_{M-1,N-1}(q\,|\,\hat{a}_i,\,\hat{b}_j)}\end{aligned} \tag{5.28}$$

by (2.12). It remains to notice that (2.13) and (5.26) imply

$$\eta_{M,N}(a_i\,|\,a,\,b) = \eta_{M-1,N-1}(b_j\,|\,\hat{a}_i,\hat{b}_j). \tag{5.29}$$

□

Proof of Corollary 2.10. Repeated application of (2.11) gives the identity

$$\eta_{M,N}(q+ka_i\,|\,a,b) = \eta_{M,N}(q\,|\,a,b)\exp\Bigl(-\sum_{l=0}^{k-1}\bigl(\mathcal{S}_N L_{M-1}\bigr)(q+la_i\,|\,\hat{a}_i,b)\Bigr). \tag{5.30}$$

Equations (2.17) and (2.18) now follow by letting $q = 0$ and $q = -ka_i$, respectively. □

Proof of Corollary 2.11. The absolute convergence of the series in (2.19) follows from Theorem 2.6 and (2.17). Its equality to the Laplace transform is the general property of the power series of positive integral moments, confer Section 7.6 of [6]. □

Proof of Corollary 2.12. The proof is a direct corollary of Ramanujan's Master Theorem, confer [1]. It is only sufficient to note that $\eta_{M,N}(q\,|\,a,b)$ is analytic over $\Re(q) > -b_0$ and, by Theorem 2.6, satisfies Hardy's growth conditions there. □

Proof of Theorem 2.13. It is sufficient to verify (2.23) as (2.24) and (2.25) are equivalent by (2.12) and (2.13). Let $\Re(q) > -b_0$. Consider the product on the right-hand side of (2.23) and reduce it by means of the Lévy-Khinchine representation for $\eta_{M-1,N}(q\,|\,\hat{a}_i,b)$.

$$\prod_{k=0}^{L}\eta_{M-1,N}(q\,|\,\hat{a}_i,b_0+ka_i) = \exp\Bigl(\int\limits_0^\infty\frac{dt}{t}(e^{-qt}-1)\frac{e^{-b_0t}\prod_{j=1}^N(1-e^{-b_jt})}{\prod_{j\neq i}^M(1-e^{-a_jt})}\frac{1-e^{-a_it(L+1)}}{1-e^{-a_it}}\Bigr). \tag{5.31}$$

Letting $L\to\infty$, we obtain (2.6) by dominated convergence. As $\eta_{M,N}(q\,|\,a,b)$ is holomorphic over $q\in\mathbb{C}-(-\infty,-b_0]$, (2.23) holds there by analytic continuation. □

Proof of Corollary 2.15. By the infinite product representation in (2.24), we have

$$\begin{aligned}
\eta_{M,N}(q+a_i\,|\,a) &= \lim_{K\to\infty}\Big[\prod_{k=0}^{K}\frac{\eta_{M-1,N}(q+(k+1)a_i\,|\,\hat{a}_i,b)}{\eta_{M-1,N}(ka_i\,|\,\hat{a}_i,b)}\Big],\\
&= \lim_{K\to\infty}\Big[\frac{\eta_{M-1,N}\big((K+1)a_i\,|\,\hat{a}_i,b\big)}{\eta_{M-1,N}(q\,|\,\hat{a}_i,b)}\prod_{k=0}^{K+1}\frac{\eta_{M-1,N}(q+ka_i\,|\,\hat{a}_i,b)}{\eta_{M-1,N}(ka_i\,|\,\hat{a}_i,b)}\Big],\\
&= \eta_{M,N}(q\,|\,a,b)\exp\big(-(\mathcal{S}_N L_{M-1})(q\,|\,\hat{a}_i,b)\big) \qquad (5.32)
\end{aligned}$$

by (2.8). Incidently, the same argument shows that (2.25) is the solution to (2.15). □

Proof of Theorem 2.16. It is sufficient to show that we have the identity

$$\eta_{M,N}(q\,|\,a,b_j=n\,a_i)=\prod_{k=0}^{n-1}\eta_{M-1,N-1}\big(q\,|\,\hat{a}_i,b_0+ka_i,\,\hat{b}_j\big). \qquad (5.33)$$

This is done by induction on n. If $n=1$, then the result follows from (2.6). Assume (5.33) holds for $n-1$, *i.e.* $b_j=(n-1)\,a_i$. By (2.15), we have

$$\begin{aligned}
\eta_{M,N}(q\,|\,a,\,b_j=(n-1)\,a_i+a_i) &= \eta_{M,N}(q\,|\,a,\,b_j=(n-1)\,a_i)\,\frac{\eta_{M-1,N-1}(q+b_j\,|\,\hat{a}_i,\,\hat{b}_j)}{\eta_{M-1,N-1}(b_j\,|\,\hat{a},\,\hat{b}_j)},\\
&= \prod_{k=0}^{n-1}\eta_{M-1,N-1}\big(\hat{a}_i,b_0+ka_i,\,\hat{b}_j\big), \qquad (5.34)
\end{aligned}$$

by the induction assumption and (2.12). □

Proof of Theorem 2.17. In the case of $a_i=1$ we can write (2.17) in the form

$$\big(\mathcal{S}_N L_M\big)(k\,|\,b)=\big(\mathcal{S}_N L_M\big)(0\,|\,b)-\sum_{l=0}^{k-1}\big(\mathcal{S}_N L_{M-1}\big)(l\,|\,b). \qquad (5.35)$$

By repeated application of (5.35) and the identity

$$\sum_{i_1=0}^{n-1}\sum_{i_2=0}^{i_1-1}\cdots\sum_{i_k=0}^{i_{k-1}-1}1=\binom{n}{k}, \qquad (5.36)$$

we obtain by induction on $k=0\cdots M-1$,

$$\begin{aligned}
\big(\mathcal{S}_N L_M\big)(n\,|\,b)-\big(\mathcal{S}_N L_M\big)(0\,|\,b) &= \sum_{i=1}^{k}(-1)^i\binom{n}{i}\big(\mathcal{S}_N L_{M-i}\big)(0\,|\,b)+\\
&\quad+(-1)^{k+1}\sum_{i_1=0}^{n-1}\sum_{i_2=0}^{i_1-1}\cdots\sum_{i_{k+1}=0}^{i_k-1}\big(\mathcal{S}_N L_{M-k-1}\big)(i_{k+1}\,|\,b). \qquad (5.37)
\end{aligned}$$

The result follows by letting $k=M-1$ and recalling that $L_0(w)=-\log(w)$. □

6 Conclusions

We constructed and studied the main properties of a novel class of what we called Barnes beta probability distributions $\beta_{M,N}(a,b)$. $\beta_{M,N}(a,b)$ is a distribution on $(0,1]$ that is parameterized by two sets of positive real numbers $a=(a_1,\cdots,a_M)$ and $b=(b_0,\cdots,b_N)$ and defined by its Mellin transform $\eta_{M,N}(q\,|\,a,b)$. We gave four different representations of $\eta_{M,N}(q\,|\,a,b)$. The defining representation is in the form of a product of ratios of Barnes multiple gamma functions $\Gamma_M(w\,|\,a)$, thereby generalizing the classic beta distribution. We used Malmstén-type formula of Ruijsenaars for $\log\Gamma_M(w\,|\,a)$ to show that $-\log\beta_{M,N}(a,b)$ is infinitely divisible on $[0,\infty)$ by deriving its Lévy-Khinchine form and thus giving the 2nd representation of $\eta_{M,N}(q\,|\,a,b)$. The Lévy-Khinchine form allowed us to show that $-\log\beta_{M,N}(a,b)$ is compound Poisson if $M<N$ and absolutely continuous if $M=N$. We used the functional equation of $\Gamma_M(w\,|\,a)$ to derive a functional equation for $\eta_{M,N}(q\,|\,a,b)$ and thus to compute the integral moments of $\beta_{M,N}(a,b)$ in the form of Selberg-type products of $\Gamma_{M-1}(w\,|\,a)$. The Ruijsenaars form of the asymptotic expansion of $\log\Gamma_M(w\,|\,a)$ in the limit $w\to\infty$ gave us the asymptotic of $\eta_{M,N}(q\,|\,a,b)$ in the limit $q\to\infty$. We used this asymptotic in the case of $M<N$ to give a probabilistic proof of an inequality involving multiple gamma functions. We also used it to show the convergence of the power series of moments of $\beta_{M,N}(a,b)$ to the Laplace transform. The resulting series of Selberg-type products of $\Gamma_{M-1}(w\,|\,a)$ is therefore positive, giving an interesting application of the general theory. We related this series to $\eta_{M,N}(q\,|\,a,b)$ by Ramanujan's Master Theorem, giving the 3rd representation of $\eta_{M,N}(q\,|\,a,b)$. We solved the functional equation of $\eta_{M,N}(q\,|\,a,b)$ in the form of Shintani-type infinite products of $\eta_{M-1,N}(q\,|\,a,b)$ and $\eta_{M-1,N-1}(q\,|\,a,b)$, resulting in the 4th representation. Finally, we established several symmetries of the Mellin transform in the form of functional equations relating $\eta_{M,N}(q\,|\,a,b)$ to $\eta_{M,N-1}(q\,|\,a,b)$, $\eta_{M-1,N}(q\,|\,a,b)$, and $\eta_{M-1,N-1}(q\,|\,a,b)$.

We illustrated our theory of Barnes beta distributions with several examples. First, we considered two special cases of a and b. If b_N is an integer multiple of a_M, $\beta_{M,N}(a,b)$ decomposes into a product of $\beta_{M-1,N-1}(a,b)s$. If $a_i=1$ for all i, the moments of $\beta_{M,N}(a,b)$ are given by a multiple product generalizing the moments of the classic beta distribution. Second, in some elementary cases of small M and N we computed the density and weight at 1 of $\beta_{M,N}(a,b)$ exactly. Our main non-elementary example is $\beta_{2,2}(a,b)$, in which case our formulas simplify to expressions involving Euler's gamma function.

The main area of applications that we considered in this paper is the probabilistic theory of the Selberg integral. We constructed a distribution with the property that its moments are given by Selberg's formula and decomposed it into a product of $\beta^{-1}_{2,2}(a,b)s$. This construction leads to a probabilistic interpretation of the Selberg integral.

References

[1] Amdeberhan, T., Espinosa, O., Gonzalez, I., Harrison, M., Moll, V., Straub, A.: Ramanujan's Master Theorem. *Ramanujan J.* **29**, (2012), 103–120. MR-2994092

[2] Barnes, E. W.: On the theory of the multiple gamma function. *Trans. Camb. Philos. Soc.* **19**, (1904), 374–425.

[3] Biane, P., Pitman, J., Yor, M.: Probability laws related to the Jacobi theta and Riemann zeta functions, and brownian excursions. *Bulletin of the American Mathematical Society* **38**, (2001), 435–465. MR-1848256

[4] Chamayou, J-F. and Letac, G.: Additive properties of the Dufresne laws and their multivariate extension. *J. Theoret. Probab.* **12**, (1999), 1045–1066. MR-1729469

[5] Dufresne, D.: G distributions and the beta-gamma algebra. *Elect. J. Probab.* **15**, (2010), 2163–2199. MR-2745729

[6] Feller, W. An Introduction to Probability Theory and Its Applications, vol. II, 2nd ed. *John Wiley and Sons*, New York, 1971. MR-0270403

[7] Forrester, P. J.: Log-Gases and Random Matrices. *Princeton University Press*, Princeton, 2010. MR-2641363

[8] Fyodorov, Y. V., Le Doussal, P., Rosso, A.: Statistical mechanics of logarithmic REM: duality, freezing and extreme value statistics of 1/f noises generated by gaussian free fields. *J. Stat. Mech. Theory Exp*, (2009), no. 10, P10005. MR-2882779

[9] Hubalek, F. and Kuznetsov, A.: A convergent series representation for the density of the supremum of a stable process, *Elect. Comm. in Probab.* **16**, (2011), 84–95. MR-2763530

[10] Jacod, J., Kowalski, E., Nikeghbali, A.: Mod-gaussian convergence: new limit theorems in probability and number theory. *Forum Mathematicum* **23**, (2011), 835–873. MR-2820392

[11] Katayama, K. and Ohtsuki, M. On the multiple gamma-functions. *Tokyo J. Math.* **21**, (1998), 159–182. MR-1630167

[12] Kuznetsov, A. On extrema of stable processes. *Ann. Probab.* **39**, (2011), 1027–1060. MR-2789582

[13] Kuznetsov, A. and Pardo, J. C. Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. *Acta Applicandae Mathematicae* May (2012). MR-3010227

[14] Lagarias, J. and Rains, E. On a two-variable zeta function for number fields. *Ann. Inst. Fourier, Grenoble* **53**, (2003), 1–68. MR-1973068

[15] Nikeghbali, A. and Yor, M.: The Barnes G function and its relations with sums and products of generalized gamma convolutions variables. *Elect. Comm. in Prob.* **14**, (2009), 396–411. MR-2545290

[16] Ostrovsky, D.: Mellin transform of the limit lognormal distribution. *Comm. Math. Phys.* **288**, (2009), 287–310. MR-2491625

[17] Ostrovsky, D. Selberg integral as a meromorphic function. *Int. Math. Res. Notices*, (2012), DOI: 10.1093/imrn/rns170.

[18] Ruijsenaars, S. N. M.: On Barnes' multiple zeta and gamma functions. *Advances in Mathematics* **156**, (2000), 107–132. MR-1800255

[19] Selberg, A.: Remarks on a multiple integral. *Norske Mat. Tidsskr.* **26**, (1944), 71–78. MR-0018287

[20] Shintani, T.: A proof of the classical Kronecker limit formula. *Tokyo J. Math.* **3**, (1980), 191–199. MR-0605088

[21] Steutel, F. W. and van Harn, K.: Infinite Divisibility of Probability Distributions on the Real Line. *Marcel Dekker*, New York, 2004. MR-2011862

[22] Yor, M.: A Note about Selberg's integrals in relation to the beta-gamma algebra, in Advances in Mathematical Finance, Fu, M., Jarrow, R., Yen, J., editors, *Birkhauser*, Boston, 2007, 49–58. MR-2359362