Mixing time bounds for
oriented kinetically constrained spin models[*]

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Abstract

We analyze the mixing time of a class of oriented kinetically constrained spin models
(KCMs) on a d-dimensional lattice of n^d sites. A typical example is the North-East
model, a 0-1 spin system on the two-dimensional integer lattice that evolves according
to the following rule: whenever a site's southerly and westerly nearest neighbours
have spin 0, with rate one it resets its own spin by tossing a p-coin, at all other times
its spin remains frozen. Such models are very popular in statistical physics because,
in spite of their simplicity, they display some of the key features of the dynamics of
real glasses. We prove that the mixing time is $O(n \log n)$ whenever the relaxation
time is $O(1)$. Our study was motivated by the "shape" conjecture put forward by G.
Kordzakhia and S.P. Lalley.

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1 Introduction

Kinetically constrained spin models (KCMs) are interacting 0-1 particle systems, on
general graphs, which evolve with a simple Glauber dynamics described as follows. At
every site x the system tries to update the occupancy variable (or spin) at x to the
value 1 or 0 with probability p and q respectively. However the update at x is accepted
only if the *current* local configuration satisfies a certain constraint, hence the models
are "kinetically constrained". It is always assumed that the constraint at site x does
not to depend on the spin at x and therefore the product Bernoulli(p) measure π is the
reversible measure. Constraints may require, for example, that a certain *number* of
the neighbouring spins are in state 0, or more restrictively, that certain *preassigned*
neighbouring spins are in state 0 (e.g. the children of x when the underlying graph is a
rooted tree).

The main interest in the physical literature for KCMs (see e.g. [20] for a review)
stems from the fact that they display many key dynamical features of real glassy ma-
terials: ergodicity breaking transition at some critical value q_c, huge relaxation time
for q close to q_c, dynamic heterogeneity (non-trivial spatio-temporal fluctuations of the

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local relaxation to equilibrium) and aging, just to mention a few. Mathematically, despite their simple definition, KCMs pose very challenging and interesting problems because of the hardness of the constraint, with ramifications towards bootstrap percolation problems [23], combinatorics [7, 24], coalescence processes [10, 11] and random walks on upper triangular matrices [18]. Some of the mathematical tools developed for the analysis of the relaxation process of KCMs [4] proved to be quite powerful also in other contexts such as card shuffling problems [2] and random evolution of surfaces [5].

In this paper we focus on *oriented* KCMs on a d-dimensional lattice, $d \geq 2$, with n^d sites, in particular on their mixing time. A prototypical model belonging to the above class of KCMs is the North-East model in two dimensions (see e.g. [12] and [4]) for which the constraint at any given site x requires the south *and* west neighbours of x to be empty in order for a flip at x to occur. In order to avoid trivial irreducibility issues the south-westerly most spin is unconstrained and sites outside the upper quadrant are treated as fixed zeros.

With p_c the percolation threshold for oriented percolation in two dimensions (see e.g. [9]), it was proved in [4] that for all $p < p_c$ the relaxation time of the North-East process is $O(1)$ while it becomes $\Omega(e^{cn})$[1], for some $c > 0$, when $p > p_c$. At $p = p_c$ the relaxation time is expected to have a poly(n) growth. Consider now the North-East model in the first quadrant of $\mathbb{Z}^2$. In [12] it was conjectured that, for $p < p_c$ and starting from all 1's, the *influence region* R_t, defined as the union of all unit squares around those sites which have flipped at least once by time t, has a definite limiting shape $S \subset \mathbb{R}^2$ in the sense that a.s. $\frac{R_t}{t} \to S$ as $t \to \infty$. Since the North-East process is neither monotone or additive (see [14]), the usual tools to prove a shape theorem do not apply in this case.

The above conjecture implies that, for $p < p_c$, the influence coming from the unconstrained spin at the South-West corner propagates at a definite linear rate as it does in the East model [3], the one dimensional analog of the model (for background see [1, 4, 22]). In particular the mixing time of the model should grow linearly in n (the linear size of the system). However in dimension $d \geq 2$ the analysis of the propagation of influence is quite delicate because of the many paths along which it can occur (see [24] for combinatorial results in this direction).

In this paper we prove that the mixing time is $O(n \log n)$ as long as the spectral gap of the process is $\Omega(1)$. Our technique bares some similarities to those employed in [6] to analyse the Glauber dynamics of biased plane partitions.

2 Models and Results

2.1 Setting and notation

We consider a class of 0-1 interacting particle systems on finite subsets Λ of the integer lattice $\mathbb{Z}^d$, reversible with respect to the product measure $\pi := \prod_{x \in \Lambda} \pi_x$, where π_x is the Bernoulli(p) measure.

The d-dimensional cube of linear size n (which contain n^d points) will be denoted by

$$\Lambda_n := ([1,n] \times \ldots \times [1,n]) \cap \mathbb{Z}^d .$$

The standard basis vectors in $\mathbb{Z}^d$ are denoted $\mathbf{e}_1 = (1, \ldots, 0)$, $\mathbf{e}_1 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_d = (0, \ldots, 1)$. For $x \in \Lambda_n$ we write x_j for the component of x in the direction $\mathbf{e}_j$.

The set of probability measures on the finite state space $\Omega_n = \{0, 1\}^{\Lambda_n}$ is denoted by $\mathcal{P}(\Omega_n)$. Elements of Ω_n will be denoted by the small greek letters $\sigma, \eta, \ldots$ and σ_x will denote the spin at the vertex x.

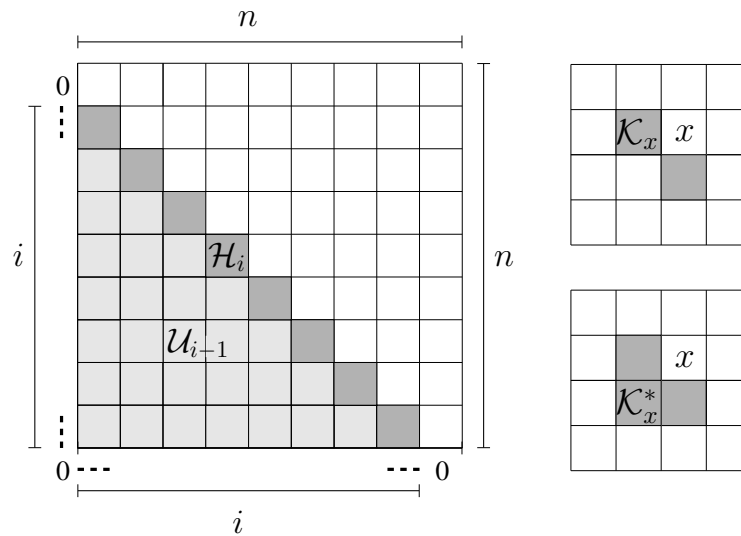[1]We recall that $f = \Omega(g)$ if $|f| \geq cg$ for some $c > 0$.

Figure 1: Left: The the lattice Λ_n in two dimensions, $d = 2$. The sets $\mathcal{H}_i$ and $\mathcal{U}_{i-1}$ are shown in dark grey and light grey respectively. Right: The constraining neighbourhoods, $\mathcal{K}_x$ and $\mathcal{K}_x^*$, of a vertex x in two dimensions, shown in dark grey.

We denote the i-th level hyperplane in Λ_n by $\mathcal{H}_i = \{x \in \Lambda_n : \sum_{j=1}^d x_j = i\}$ (see Fig. 1). The set of sites on and below this hyperplane will be written $\mathcal{U}_i = \{x \in \Lambda_n : \sum_{j=1}^d x_j \leq i\}$. For each $\sigma \in \Omega_n$ we write $\sigma^{(i)}$ for the restriction of σ to $\mathcal{U}_i$, and $\sigma_{\mathcal{H}_i}$ for the restriction of σ to $\mathcal{H}_i$. Similarly, for any probability measure $\nu \in \mathcal{P}(\Omega_n)$, we write $\nu^{(i)}$ for the marginal of ν on $\Omega^{(i)} := \{0,1\}^{\mathcal{U}_i} \subset \Omega_n$.

For any vertex $x \in \Lambda_n$ we also let (see Fig. 1 Right)

$$\mathcal{K}_x^* = \left\{ y \in \mathbb{Z}^d : y = x - \sum_{i=1}^d \alpha_i \mathbf{e}_i, \ \alpha_i \in \{1,0\} \right\} \setminus \{x\}$$

$$\mathcal{K}_x = \left\{ y \in \mathbb{Z}^d : \exists \ i = 1, \ldots, d \text{ such that } y = x - \mathbf{e}_i \right\}.$$

Definition 2.1 (constraints)**.** *Let $x_* = (1, \ldots, 1)$ be the south-west corner of Λ_n. Consider a collection $\{\mathcal{C}_x\}_{x \in \Lambda_n}$ of constraining neighborhoods such that*

$$\mathcal{C}_{x_*} = \emptyset, \qquad \text{and} \qquad \emptyset \neq \mathcal{C}_x \subset K_x^* \cap \Lambda_n, \quad x \neq x_*.$$

Let

$$c_x(\sigma) := \begin{cases} \prod_{y \in \mathcal{C}_x} (1 - \sigma_y) & \text{if } x \neq x_* \\ 1 & x = x_*. \end{cases} \tag{2.1}$$

We then say that the constraint at site x is satisfied by the configuration σ if $c_x(\sigma) = 1$. In words, the constraint at x is satisfied if all the spin in $\mathcal{C}_x$ are 0. Note that x_* is unconstrained.

2.2 Oriented KCMs and main result

We give a general definition of the process to include a large class of directed KCMs, such as the North-East model and higher dimensional analogues. For constraining

neighborhoods $\{\mathcal{C}_x\}_{x \in \Lambda_n}$ we define the associated directed KCM by the following graphical construction. To each $x \in \Lambda_n$ we associate a mean one Poisson process and, independently, a family of independent Bernoulli(p) random variables $\{s_{x,k} : k \in \mathbb{N}\}$. The occurrences of the Poisson process associated to x will be denoted by $\{t_{x,k} : k \in \mathbb{N}\}$. We assume independence as x varies in Λ_n. The probability measure will be denoted by $\mathbb{P}^{\Lambda_n}$. Notice that $\mathbb{P}^{\Lambda_n}$-almost surely all the occurrences $\{t_{x,k}\}_{k \in \mathbb{N}, x \in \Lambda_n}$ are different.

Given $\eta \in \Omega_n$ we construct a continuous time Markov chain $\{\eta(s)\}_{s \geq 0}$ on the probability space above, starting from η at $t = 0$, according to the following rules. At each time $t_{x,n}$ the site x queries the state of its own constraint c_x (see (2.1)). If and only if the constraint is satisfied ($c_x = 1$) then $t_{x,n}$ is called a *legal ring* and the configuration resets its value at site x to the value of the corresponding Bernoulli variable $s_{x,n}$.

The above construction gives rise to an irreducible, continuous time Markov chain, reversible w.r.t. π, with generator

$$\mathcal{L}_n f(\sigma) = \sum_{x \in \Lambda_n} c_x(\sigma)[\pi_x(f) - f(\sigma)], \tag{2.2}$$

where π_x denotes the conditional mean $\pi(f \mid \{\sigma_y\}_{y \neq x})$. Irreducibility follows because we can invade Λ_n with 0's starting from the unconstrained corner x_*.

Using a standard percolation argument [15, 8] together with the fact that the constraints $\{c_x\}_{x \in \Lambda_n}$ are uniformly bounded and of finite range, it is not difficult to see that the graphical construction can be extended without problems also to the infinite volume case.

For an initial distribution ν at $t = 0$ the law and expectation of the process will be denoted by $\mathbb{P}_\nu$ and $\mathbb{E}_\nu$ respectively. In the sequel, we will write ν_t for the distribution of the chain at time t,

$$\nu_t[\eta] := \mathbb{P}_\nu(\eta(t) = \eta).$$

If ν is concentrated on a single configuration η we will write $\mathbb{P}_\eta(\cdot)$. Note that, for each $i = 1, \ldots, n$, the same graphical construction can be used to define the process on $\Omega^{(i)}$ whose law is denoted by $\mathbb{P}_\nu^{(i)}(\cdot)$.

It follows from the graphical construction, and the fact that the constraints are oriented, that given an $i \in \{1 \ldots, n\}$ the evolution in $\mathcal{U}_i$ is not influenced by the evolution above $\mathcal{U}_i$. In particular, for any $\eta \in \Omega_n$ and any event $\mathcal{A}$ in the σ-algebra generated by $\{\eta_x(s)\}_{s \leq t, x \in \mathcal{U}_i}$,

$$\mathbb{P}_\eta(\mathcal{A}) = \mathbb{P}_{\eta^{(i)}}^{(i)}(\mathcal{A}). \tag{2.3}$$

In fact the same holds for any subset $U \subset \Lambda_n$ with monotone surface, *i.e.* $x - \mathbf{e}_i \in U$ whenever $x \in U$ and $x - \mathbf{e}_i \in \Lambda_n$, for each $i \in \{1, \ldots, d\}$.

We finish this section with definitions of the spectral gap and mixing time of the process.

Definition 2.2 (spectral gap). *The spectral gap, $\lambda^{(n)}$, of the infinitesimal generator (2.2) is the smallest positive eigenvalue of $-\mathcal{L}_n$, and is given by the variational principle*

$$\lambda^{(n)} := \inf_{\substack{f : \Omega_n \to \mathbb{R} \\ f \neq const}} \frac{\mathcal{D}_n(f)}{\mathrm{Var}_n(f)}, \tag{2.4}$$

where $\mathcal{D}_n(f) = -\pi(f\mathcal{L}_n f)$ is the Dirichlet form of the process.

Definition 2.3 (Mixing time). *The mixing time is defined in the usual way as*

$$T_{\mathrm{mix}}^{(n)} = \inf\{t > 0 : \sup_{\nu \in \mathcal{P}(\Omega_n)} \|\nu_t - \pi\|_{\mathrm{TV}} \leq 1/4\}, \tag{2.5}$$

where $\| \cdot \|_{\mathrm{TV}}$ denotes the total variation distance.

Remark 2.4. *It follows from [4, Theorem 4.1] that there exists $0 < p_0 < 1$ such that $\inf_n \lambda^{(n)} > 0$ whenever $p < p_0$. For the North-East model p_0 coincides with the oriented percolation threshold. Moreover, for any $p \in (0,1)$, one has $1/\lambda^{(n)} \leq T_{\mathrm{mix}}^{(n)} \leq c\, n^d/\lambda^{(n)}$ (see e.g. [13]).*

Our main result reads as follows.

Theorem 2.5. *Assume $\inf_n \lambda^{(n)} > 0$, then*

$$C\, n \leq T_{\mathrm{mix}}^{(n)} \leq C'\, n \log n \tag{2.6}$$

for some $C, C' > 0$ independent of n.

Remark 2.6. *In the case of maximal constraints, i.e. $\mathcal{C}_x = \mathcal{K}_x^* \cap \Lambda_n$, inserting the indicator of the configuration identically equal to 1 as a test function in the logarithmic Sobolev inequality [21] shows that the logarithmic Sobolev constant $\alpha^{(n)}$ of the generator (2.2) is $O(1/n^d)$. Therefore, if*

$$T_2^{(n)} := \inf\{t \geq 0 : \ \sup_\nu \mathrm{Var}\left(\frac{\nu_t}{\pi}\right) \leq 1/4\},$$

then for a universal constants $c > 0$ and a constant $c' > 0$ depending only on p,

$$T_2^{(n)} \geq \frac{c}{\alpha^{(n)}} \geq c' n^d,$$

where the first inequality follows from [21, Corollary 2.2.7]. This observation shows that oriented models in $\mathbb{Z}^d$ can be quite different from oriented models on rooted k-regular trees. In the latter case it was recently proved [16] that both the mixing time $T_{\mathrm{mix}}^{(n)}$ and $T_2^{(n)}$ grow linearly in the depth n of the tree whenever the relaxation time $1/\lambda^{(n)}$ is $O(1)$.

3 Proof of theorem 2.5

Proof of the lower bound. The lower bound on the mixing time is quite straightforward using finite speed of propagation. Define τ^* as the first time the spin at site $x^* := (n, \ldots, n)$ is 0, and denote the configuration of all 1's by $\mathbb{1}$. Using results in [17, 19] there exists $c > 0$ such that

$$T_{\mathrm{mix}}^{(n)} \geq c\mathbb{E}_{\mathbb{1}}\left[\tau^*\right] \geq \frac{c\, n}{2}\mathbb{P}_{\mathbb{1}}\left(\tau^* \geq n/2\right). \tag{3.1}$$

Clearly the event $\tau^* < n/2$ requires the existence of a path $\gamma = \{x_* = x^{(0)}, x^{(1)}, \ldots, x^{(\ell)} = x^*\}$ and times $t_0 < t_1 < \cdots < t_\ell < n/2$ such that, for any $i \leq \ell$, $x^{(i)} \in \mathcal{C}_{x^{(i+1)}}$ and t_i is a legal ring for $x^{(i)}$. Standard Poisson large deviations show that the probability of the above event is $o(1)$ as $n \to \infty$. $\qquad\square$

Proof of the upper bound. The proof of the upper bound is based on an iterative scheme. For $i \leq n$, we find some time, $\Delta_i = O(\log(i))$, such that if the initial measure ν has marginal $\nu^{(i-1)}$ equal to $\pi^{(i-1)}$ on $\Omega^{(i-1)}$ (i.e. below the hyperplane $\mathcal{H}_i$), then after time Δ_i the marginal $\nu^{(i)}$ is very close to the equilibrium marginal $\pi^{(i)}$. This is the content of Lemma 3.1. Then, starting from an arbitrary initial measure, we can iterate the above result using the triangle inequality for the variation distance and propagate the error.

Lemma 3.1 (Mixing time for a single diagonal.). *There exists $c = c(q, d) > 0$ such that, for any initial measure ν with marginal on $\Omega^{(i-1)}$ equal to $\pi^{(i-1)}$,*

$$\left\|\nu_t^{(i)} - \pi^{(i)}\right\|_{\mathrm{TV}} \leq \epsilon \quad \text{for all} \quad t \geq c \log(i/\epsilon).$$

Before proving Lemma 3.1 let us recall a useful characterisation of the total variation distance (see for example [13]),

$$\|\nu - \pi\|_{\mathrm{TV}} = \frac{1}{2} \sup_{\|f\|_\infty \leq 1} |\nu(f) - \pi(f)| \,,$$

where $\|f\|_\infty$ denotes the sup-norm.

Proof of Lemma 3.1. Fix $i \leq n$ and a function f depending only on the spin configuration in $\mathcal{U}_i$ and such that $\|f\|_\infty \leq 1$. Without loss of generality assume $\pi(f) = 0$. Given an initial measure ν with marginal $\pi^{(i-1)}$ on $\Omega^{(i-1)}$ it follows from (2.3) that $\nu_t^{(i-1)} = \pi^{(i-1)}$ for all $t \geq 0$. Moreover, conditioned on the history $\{\sigma^{(i-1)}(s)\}_{s \leq t}$ in $\mathcal{U}_{i-1}$, the spins $\{\sigma_x(t)\}_{x \in \mathcal{H}_i}$ on the hyperplane $\mathcal{H}_i$ evolve independently from each other. Each one goes to equilibrium with rate one during the intervals of time in which its constraint c_x is satisfied and stays fixed otherwise. In particular, conditioned on having had a legal ring at each $x \in \mathcal{H}_i$ before time t, the distribution of $\sigma_{\mathcal{H}_i}(t)$ is π. These simple observations gives rise to an upper-bound on the expectation of f at time t as follows.

Let $\tau^{(i)}$ be the first time there has been at least one legal ring on each $x \in \mathcal{H}_i$. Following the argument above we have,

$$\left| \mathbb{E}_\nu \left[f \left(\sigma^{(i)}(t) \right) \right] \right| = \left| \mathbb{E}_\nu \left(\mathbb{E}_\nu \left[f \left(\sigma^{(i)}(t) \right) \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right] \right) \right|$$

$$\leq \left| \mathbb{E}_\nu \left(\mathbb{E}_\nu \left[f \left(\sigma^{(i)}(t) \right) \mathbb{1}_{\{\tau^i < t\}} \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right] \right) \right| +$$

$$+ \left| \mathbb{E}_\nu \left(\mathbb{E}_\nu \left[f \left(\sigma^{(i)}(t) \right) \mathbb{1}_{\{\tau^i \geq t\}} \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right] \right) \right|$$

$$\leq \left| \mathbb{E}_\nu \left(\pi_{\mathcal{H}_i}(f) \, \mathbb{P}_\nu \left(\tau^i < t \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right) \right) \right| +$$

$$+ \left| \mathbb{E}_\nu \left(\mathbb{E}_\nu \left[f \left(\sigma^{(i)}(t) \right) \mathbb{1}_{\{\tau^i \geq t\}} \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right] \right) \right|$$

$$\leq 2\mathbb{E}_\pi \left[\mathbb{P}_\nu \left(\tau^i \geq t \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right) \right].$$

In the second inequality above we used the strong Markov property together with the observation that the distribution of $\sigma_{\mathcal{H}_i}(t)$ conditioned on $\{\tau^{(i)} < t\}$ and $\{\sigma^{(i-1)}(s)\}_{s \leq t}$ is π. In the third inequality we used the fact that $\mathbb{E}_\nu(\pi_{\mathcal{H}_i}(f)) = \pi(f) = 0$.

To bound the final term above we denote the number of rings of the Poisson clock at site x during the set $B \subset [0,t]$ by $N_x(B) = |\{t_{x,k}\}_{k \geq 0} \cap B|$, and we define the set of *legal times* at site x before t as $\mathcal{G}(x,t) := \{s < t : c_x(\sigma(s)) = 1\}$. By construction, for $x \in \mathcal{H}_i$ the set $\mathcal{G}(x,t)$ depends only on $\{\sigma_y(s)\}_{s \leq t}$, $y \in \mathcal{C}_x$. Using the notation above,

$$\mathbb{P}_\nu \left(\tau^{(i)} \geq t \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right) \leq \sum_{x \in \mathcal{H}_i} \mathbb{P} \left(N_x(\mathcal{G}(x,t)) = 0 \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right).$$

By construction $N_x(\mathcal{G}(x,t))$ is a Poisson random variable of mean $|\mathcal{G}(x,t)|$. Thus

$$\sum_{x \in \mathcal{H}_i} \mathbb{P} \left(N_x(\mathcal{G}(x,t)) = 0 \mid \{\sigma^{(i-1)}(s)\}_{s \leq t} \right) = \sum_{x \in \mathcal{H}_i} e^{-|\mathcal{G}(x,t)|}.$$

In conclusion,

$$\left| \mathbb{E}_\nu \left[f \left(\sigma^{(i)}(t) \right) \right] \right| \leq 2 \sum_{x \in \mathcal{H}_i} \mathbb{E}_\pi \left[e^{-|\mathcal{G}(x,t)|} \right] \leq 2i^{d-1} \max_{y \in \Lambda_n} \mathbb{E}_\pi \left[e^{-|\mathcal{G}(y,t)|} \right].$$

We estimate $\mathbb{E}_\pi \left[e^{-|\mathcal{G}(y,t)|} \right]$ using a Feynman-Kac approach (a similar method has been used to bound the persistence function in [4]). The total time that site y satisfies its

constraints is

$$|\mathcal{G}(y,t)| = \int_0^t c_y\left(\sigma(s)\right) ds.$$

On $L^2(\pi)$ we define the self adjoint operator $H := \mathcal{L} - V$ with $V(\sigma) := c_y(\sigma)$. The Feynman-Kac formula allows us to rewrite the expectation $\mathbb{E}_\pi\left[e^{-\int_0^t V}\right]$ as $\left\langle \mathbf{1}, e^{tH}\mathbf{1}\right\rangle_\pi$ (where $\langle \cdot, \cdot \rangle_\pi$ denotes the inner-product in $L^2(\pi)$). Thus, if β_y is the supremum of the spectrum of H we have

$$\mathbb{E}_\pi\left[e^{-|\mathcal{G}(y,t)|}\right] \le e^{t\beta_y} . \tag{3.2}$$

In order to complete the proof of the Lemma it remains to show that $\beta_y < 0$.

We decompose each $L^2(\pi)$-norm one function ϕ in the domain of H as $\phi = \alpha\mathbf{1} + g$, for some mean zero function g. So $\langle \mathbf{1}, g\rangle_\pi = 0$ and $\alpha^2 + \langle g, g\rangle_\pi = 1$. Then,

$$\langle \phi, H\phi\rangle = \langle g, \mathcal{L}g\rangle_\pi - \alpha^2 \langle \mathbf{1}, V\mathbf{1}\rangle_\pi - 2\alpha \langle \mathbf{1}, Vg\rangle_\pi - \langle g, Vg\rangle_\pi \tag{3.3}$$

We proceed by bounding $\langle \phi, H\phi\rangle_\pi$. Using (3.3) together with the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\langle \phi, H\phi\rangle &\le -\lambda^{(n)} \langle g, g\rangle_\pi - q^d\left(\alpha^2 + 2\mathbb{E}_\pi\left(g \mid V = 1\right)\alpha + \mathbb{E}_\pi\left(g^2 \mid V = 1\right)\right) \\
&\le -\delta\,\lambda^{(n)} - q^d\left(|\alpha| - \mathbb{E}_\pi\left(g^2 \mid V = 1\right)^{1/2}\right)^2 \\
&\le -\delta\,\lambda^{(n)}
\end{aligned}$$

where $\delta = 1 - \alpha^2$. Again from (3.3), dropping the last term on the right hand side and applying the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\langle \phi, H\phi\rangle &\le -\lambda^{(n)} \langle g, g\rangle_\pi - \alpha^2 q^d + 2|\alpha|\left(\langle g, g\rangle_\pi q^d(1 - q^d)\right)^{1/2} \\
&\le (\delta - 1)q^d + 2\left(\delta q^d(1 - q^d)\right)^{1/2} .
\end{aligned}$$

Thus

$$\beta_y \le -c_0(q) := \min\left(-\delta\,\lambda^{(n)}, (\delta - 1)q^d + 2\left(\delta q^d(1 - q^d)\right)^{1/2}\right).$$

Finally we observe that $c_0(q) > 0$ because

$$(\delta - 1)q^d + 2\left(\delta q^d(1 - q^d)\right)^{1/2} \le -\frac{q^d}{2} \quad \text{if} \quad \delta \le \frac{2(1 - \sqrt{1 - q^d}) - q^d}{4q^d}.$$

In conclusion

$$\left|\mathbb{E}_\nu\left[f\left(\sigma^{(i)}(t)\right)\right]\right| \le i^d e^{-c_0(q)t} \le \epsilon,$$

for $t \ge c\log(i/\epsilon)$ for some $c = c(d, q)$. $\qquad\square$

We now prove Theorem 2.5 by iterating the previous Lemma and propagating the error. We use the following iterative scheme

$$t_i = t_{i-1} + \Delta_i, \quad t_0 = 0, \tag{3.4}$$

where $\Delta_i(\epsilon) = \frac{1}{c}\log(i^3/\epsilon)$ with c as in the Lemma. We now show by induction that starting from an arbitrary initial distribution ν

$$2\left\|\nu_t^{(i)} - \pi^{(i)}\right\|_{\text{TV}} \le \hat{\epsilon}_i := \epsilon\sum_{j=1}^i \frac{1}{j^2} \quad \text{for all} \quad t \ge t_i . \tag{3.5}$$

The case $i = 1$ (3.5) is an immediate consequence of the fact that the corner $x_* = (1, \ldots, 1)$ goes to equilibrium with rate one. Assume (3.5) holds for all $i \leq k - 1$. It is clear that $\min_\sigma \nu_t(\sigma) > 0$ for any $t > 0$, so that we may define an auxiliary probability measure μ with marginal $\mu^{(k-1)} = \pi^{(k-1)}$ by

$$\mu[\sigma] := \pi^{(k-1)} \left[\sigma^{(k-1)} \right] \nu_{t_{k-1}} \left[\sigma \,|\, \sigma^{(k-1)} \right].$$

Fix a function f depending only on the configuration in $\mathcal{U}_k$ with $\|f\|_\infty \leq 1$. Again, without loss of generality, assume $\pi(f) = 0$. Then

$$\left| \mathbb{E}_\nu \left[f\left(\sigma(t_k)\right) \right] \right| = \left| \mathbb{E}_{\nu_{t_{k-1}}} \left[f\left(\sigma(\Delta_k)\right) \right] \right|$$
$$\leq \left| \mathbb{E}_{\nu_{t_{k-1}}} \left[f\left(\sigma(\Delta_k)\right) \right] - \mathbb{E}_\mu \left[f\left(\sigma(\Delta_k)\right) \right] \right| + \frac{\varepsilon}{k^2} . \qquad (3.6)$$

where we used the Markov property in the first line and Lemma 3.1 together with the triangle inequality in the second line. The two measures μ and $\nu_{t_{k-1}}$ have the same marginal on $\mathcal{H}_k$, so we may reduce the remaining term to something that can be dealt with using the inductive hypothesis as follows. For any $\eta \in \Omega^{(k-1)}$ let

$$F(\eta) := \mathbb{E}_{\nu_{t_{k-1}}} \left[f\left(\sigma(\Delta_k)\right) \,|\, \sigma^{(k-1)} = \eta \right].$$

Clearly $\|F\|_\infty \leq 1$, so by the definition of μ and the inductive hypothesis (3.5),

$$\left| \mathbb{E}_{\nu_{t_{k-1}}} \left[f\left(\sigma(\Delta_k)\right) \right] - \mathbb{E}_\mu \left[f\left(\sigma(\Delta_k)\right) \right] \right| = \left| \nu_{t_{k-1}}(F) - \pi(F) \right| \leq \hat{\varepsilon}_{k-1} .$$

In conclusion the r.h.s. of (3.6) is bounded from above by

$$\left| \mathbb{E}_\nu \left[f\left(\sigma(t_k)\right) \right] \right| = \hat{\varepsilon}_{k-1} + \frac{\varepsilon}{k^2} = \hat{\varepsilon}_k .$$

Since $\mathcal{U}_{2n-1} \supseteq \Lambda_n$, it follows that

$$2 \left\| \nu_t - \pi \right\|_{\mathrm{TV}} \leq \hat{\epsilon}_{2n-1} = \epsilon \sum_{j=1}^{2n-1} \frac{1}{j^2} \leq 2\varepsilon \quad \text{for all} \quad t \geq t_{2n-1}$$

and

$$t_{2n-1} = \sum_{j=1}^{2n-1} \Delta_j \leq \frac{1}{c} \int_0^{2n} \log(x^3/\epsilon) dx \leq c' n \log(n)$$

for some $c' > 0$. $\qquad \square$

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