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Complex Brownian motion representation of the Dyson model

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Abstract

Dyson's Brownian motion model with the parameter $\beta = 2$, which we simply call the Dyson model in the present paper, is realized as an h-transform of the absorbing Brownian motion in a Weyl chamber of type A. Depending on initial configuration with a finite number of particles, we define a set of entire functions and introduce a martingale for a system of independent complex Brownian motions (CBMs), which is expressed by a determinant of a matrix with elements given by the conformal transformations of CBMs by the entire functions. We prove that the Dyson model can be represented by the system of independent CBMs weighted by this determinantal martingale. From this CBM representation, the Eynard-Mehta-type correlation kernel is derived and the Dyson model is shown to be determinantal. The CBM representation is a useful extension of h-transform, since it works also in infinite particle systems. Using this representation, we prove the tightness of a series of processes, which converges to the Dyson model with an infinite number of particles, and the noncolliding property of the limit process.

Keywords: Dyson model; h-transform; complex Brownian motions; entire functions; conformal martingales; determinantal process; tightness; noncolliding property.
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1 Introduction and Results

Dyson's Brownian motion model is a one-parameter family of the systems of one-dimensional Brownian motions with long-ranged repulsive interactions, whose strength is represented by a parameter $\beta > 0$. The system solves the following stochastic differential equations (SDEs),

$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{\substack{1 \le j \le n: \\ j \neq i}} \frac{dt}{X_i(t) - X_j(t)}, \quad 1 \le i \le n, \quad t \in [0, \infty), \tag{1.1}$$

where $B_i(t)$'s are independent one-dimensional standard Brownian motions [3, 19]. In the present paper we consider the case with $\beta = 2$, since in this special case the system is realized by the following three processes [8, 9],

(i) the process of eigenvalues of Hermitian matrix-valued diffusion process in the Gaussian unitary ensemble (GUE) [3, 16, 5],

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(ii) the system of one-dimensional Brownian motions conditioned never to collide with each other [6],

(iii) the *harmonic transform* of the absorbing Brownian motion in a Weyl chamber of type A_{n-1} [6],

$$\mathbb{W}_n^{\mathrm{A}} = \{\boldsymbol{x} \in \mathbb{R}^n : x_1 < x_2 < \cdots < x_n\},$$

with a harmonic function given by the Vandermonde determinant

$$h(\boldsymbol{x}) = \prod_{1 \le i < j \le n} (x_j - x_i) = \det_{1 \le i,j \le n} \left[x_j^{i-1} \right]. \tag{1.2}$$

In the family of particle systems (1.1), the case with $\beta = 2$ plays the role which is similar to that of the three-dimensional ($\nu = 1/2$) Bessel process in the family of Bessel processes with parameter $\nu > -1$ [13]. We call the case with $\beta = 2$ of Dyson's Brownian motion model simply *the Dyson model* in this paper.

Let $\mathscr{M}$ be the space of nonnegative integer-valued Radon measures on $\mathbb{R}$, which is a Polish space with the vague topology. Any element ξ of $\mathscr{M}$ can be represented as $\xi(\cdot) = \sum_{i \in \mathbb{I}} \delta_{x_i}(\cdot)$ with a countable index set $\mathbb{I}$ and a sequence of points in $\mathbb{R}$, $\boldsymbol{x} = (x_i)_{i \in \mathbb{I}}$ satisfying $\xi(K) = \sharp\{x_i : x_i \in K\} < \infty$ for any compact subset $K \subset \mathbb{R}$. In this paper the cardinality of a finite set S is denoted by $\sharp S$. We call an element ξ of $\mathscr{M}$ an unlabeled configuration, and a sequence $\boldsymbol{x}$ a labeled configuration. We write the restriction of configuration in $A \subset \mathbb{R}$ as $(\xi \cap A)(\cdot) = \sum_{i \in \mathbb{I}: x_i \in A} \delta_{x_i}(\cdot)$, a shift of configuration by $u \in \mathbb{R}$ as $\tau_u \xi(\cdot) = \sum_{i \in \mathbb{I}} \delta_{x_i + u}(\cdot)$, and a square of configuration as $\xi^{\langle 2 \rangle}(\cdot) = \sum_{i \in \mathbb{I}} \delta_{x_i^2}(\cdot)$, respectively. The set of $\mathscr{M}$-valued continuous functions defined on $[0, \infty)$ is denoted by $\mathrm{C}([0,\infty) \to \mathscr{M})$, which has the topology of uniform convergence on any compact sets. For $\xi \in \mathscr{M}$ with $\xi(\mathbb{R}) \in \mathbb{N} \equiv \{1, 2, \dots\}$, we introduce a one-parameter family of *entire functions* of $z \in \mathbb{C}$ [15] parameterized by $u \in \mathbb{C}$, $\{\Phi_\xi^u(z) : u \in \mathbb{C}\}$, in which

$$\Phi_\xi^u(z) = \prod_{r \in \mathrm{supp}\, \xi \cap \{u\}^{\mathrm{c}}} \left(1 - \frac{z-u}{r-u} \right)^{\xi(\{r\})} \tag{1.3}$$

with $\mathrm{supp}\, \xi = \{r \in \mathbb{R} : \xi(\{r\}) > 0\}$. The zero set of the function (1.3) is $\mathrm{supp}\, \xi \cap \{u\}^{\mathrm{c}}$.

With the SDEs (1.1), we consider the diffusion process $\Xi(t) = \sum_{i \in \mathbb{I}} \delta_{X_i(t)}$ in $\mathscr{M}$ and the process under the initial configuration $\xi = \sum_{i \in \mathbb{I}} \delta_{x_i} \in \mathscr{M}$ is denoted by $(\Xi(t), \mathbb{P}_\xi)$. We write the expectation with respect to $\mathbb{P}_\xi$ as $\mathbb{E}_\xi$. We introduce a filtration $\{\mathcal{F}(t)\}_{t \in [0,\infty)}$ on the space $\mathrm{C}([0,\infty) \to \mathscr{M})$ defined by $\mathcal{F}(t) = \sigma(\Xi(s), s \in [0,t])$. Let $\mathrm{C}_0(\mathbb{R})$ be the set of all continuous real-valued functions with compact supports. For any integer $M \in \mathbb{N}$, a sequence of times $\boldsymbol{t} = (t_1, t_2, \dots, t_M)$ with $0 < t_1 < \cdots < t_M \le T < \infty$, and a sequence of functions $\boldsymbol{f} = (f_{t_1}, f_{t_2}, \dots, f_{t_M}) \in \mathrm{C}_0(\mathbb{R})^M$, the moment generating function of multitime distribution of $(\Xi(t), \mathbb{P}_\xi)$ is defined by

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] \equiv \mathbb{E}_\xi \left[\exp \left\{ \sum_{m=1}^M \int_{\mathbb{R}} f_{t_m}(x) \Xi(t_m, dx) \right\} \right]. \tag{1.4}$$

We put $\mathscr{M}_0 = \{\xi \in \mathscr{M} : \xi(\{x\}) \le 1 \text{ for any } x \in \mathbb{R}\}$. Since any element ξ of $\mathscr{M}_0$ is determined uniquely by its support, it is identified with a countable subset $\{x_i\}_{i \in \mathbb{I}}$ of $\mathbb{R}$. When $\xi = \sum_{i \in \mathbb{I}} \delta_{u_i} \in \mathscr{M}_0$, (1.3) gives

$$\Phi_\xi^{u_i}(u_j) = \delta_{ij}, \quad i, j \in \mathbb{I}. \tag{1.5}$$

For a finite index set $\mathbb{I}$ and $\boldsymbol{u} = (u_i)_{i \in \mathbb{I}}, u_i \in \mathbb{R}$, let $Z_i(t), t \ge 0$, $i \in \mathbb{I}$ be a sequence of independent complex Brownian motions (CBMs) on a probability space $(\Omega, \mathcal{F}, \mathbf{P}_{\boldsymbol{u}})$ with

$Z_i(0) = u_i$. We write the expectation with respect to $\mathbf{P}_{\boldsymbol{u}}$ as $\mathbf{E}_{\boldsymbol{u}}$. The real part and the imaginary part of $Z_i(t)$ are denoted by $V_i(t) = \mathrm{Re} Z_i(t)$ and $W_i(t) = \mathrm{Im} Z_i(t)$, respectively, $i \in \mathbb{I}$, which are independent one-dimensional standard Brownian motions. For any sequences $(u_i)_{i \in \mathbb{I}}$ and $x \in \mathbb{R}$, if we set $\xi = \sum_{i \in \mathbb{I}} \delta_{u_i}$,

$$\Phi_\xi^x(Z_i(\cdot)),\ i \in \mathbb{I} \text{ are independent conformal local martingales,} \tag{1.6}$$

since Φ_ξ^x is an entire function. Each of them is a time change of a CBM [18]. When $\xi = \sum_{i \in \mathbb{I}} \delta_{u_i} \in \mathscr{M}_0$, combination of (1.5) and (1.6) gives, for $0 < t \le T < \infty$,

$$\mathbf{E}_{\boldsymbol{u}}[\Phi_\xi^{u_i}(Z_j(t))] = \mathbf{E}_{\boldsymbol{u}}[\Phi_\xi^{u_i}(Z_j(0))] = \Phi_\xi^{u_i}(u_j) = \delta_{ij}, \quad i, j \in \mathbb{I}. \tag{1.7}$$

A key observation for the present study is that the equality

$$\det_{1 \le i,j \le \xi(\mathbb{R})} \left[\Phi_\xi^{u_i}(z_j) \right] = \frac{h(\boldsymbol{z})}{h(\boldsymbol{u})} \tag{1.8}$$

holds for any $\xi = \sum_{i=1}^{\xi(\mathbb{R})} \delta_{u_i}$ with $\xi(\mathbb{R}) \in \mathbb{N}$, $\boldsymbol{u} = (u_1, \dots, u_{\xi(\mathbb{R})}) \in \mathbb{W}^{\mathrm{A}}_{\xi(\mathbb{R})}$ and $\boldsymbol{z} = (z_1, \dots, z_{\xi(\mathbb{R})}) \in \mathbb{C}^{\xi(\mathbb{R})}$. This is proved as follows. Let $\xi(\mathbb{R}) = n$ and

$$H(\boldsymbol{u}, \boldsymbol{z}) = \det_{1 \le i,j \le n} \Big[\prod_{1 \le k \le n: k \ne i} (u_k - z_j) \Big], \quad \boldsymbol{u}, \boldsymbol{z} \in \mathbb{C}^n.$$

Since H is a polynomial function with degree $n(n-1)$ satisfying the conditions that $H(\boldsymbol{u}, \boldsymbol{u}) = (-1)^{n(n-1)/2} h(\boldsymbol{u})^2$, and $H(\boldsymbol{u}, \boldsymbol{z}) = 0$, if $u_i = u_j$ or $z_i = z_j$ for some i, j with $1 \le i < j \le n$, we find $H(\boldsymbol{u}, \boldsymbol{z}) = (-1)^{n(n-1)/2} h(\boldsymbol{u}) h(\boldsymbol{z})$. (It is a special case of the determinantal identity given as Lemma 2.2 in [14].) Since the LHS of (1.8) is equal to $(-1)^{n(n-1)/2} H(\boldsymbol{u}, \boldsymbol{z})/h(\boldsymbol{u})^2$ by definition (1.3) for $\xi \in \mathscr{M}_0$, we obtain (1.8). This equality implies that from a harmonic function $h(\cdot)$ given by (1.2), we have a martingale for a system of independent CBMs $\{Z_i(\cdot) : 1 \le i \le \xi(\mathbb{R})\}$ in the determinantal form, $\det_{1 \le i,j \le \xi(\mathbb{R})} [\Phi_\xi^{u_i}(Z_j(\cdot))]$.

Let $\mathbb{1}(\omega)$ be the indicator function of a condition ω; $\mathbb{1}(\omega) = 1$ if ω is satisfied and $\mathbb{1}(\omega) = 0$ otherwise, and $\mathbb{I}_p = \{1, 2, \dots, p\}$ for $p \in \mathbb{N}$. The main theorem of the present paper is the following.

Theorem 1.1. *Suppose that* $\xi = \sum_{i=1}^{\xi(\mathbb{R})} \delta_{u_i} \in \mathscr{M}_0$ *with* $\xi(\mathbb{R}) \in \mathbb{N}$. *Let* $0 < t \le T < \infty$. *For any* $\mathcal{F}(t)$*-measurable bounded function* F *we have*

$$\mathbb{E}_\xi \left[F\big(\Xi(\cdot)\big) \right] = \boldsymbol{E}_{\boldsymbol{u}} \left[F \left(\sum_{i=1}^{\xi(\mathbb{R})} \delta_{V_i(\cdot)} \right) \det_{1 \le i,j \le \xi(\mathbb{R})} \left[\Phi_\xi^{u_i}(Z_j(T)) \right] \right]. \tag{1.9}$$

In particular, the moment generating function (1.4) is given by

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \sum_{p=0}^{\xi(\mathbb{R})} \sum_{(\mathbb{J}_m)_{m=1}^M} \int_{\mathbb{W}_p^{\mathrm{A}}} \xi^{\otimes p}(d\boldsymbol{v}) \boldsymbol{E}_{\boldsymbol{v}} \left[\prod_{m=1}^M \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{i,j \in \mathbb{I}_p} \left[\Phi_\xi^{v_i}(Z_j(T)) \right] \right], \tag{1.10}$$

where $\chi_{t_m}(\cdot) = e^{f_{t_m}(\cdot)} - 1$, $1 \le m \le M$, *and the second summation in the right hand side of the above equation runs over all* $\mathbb{J}_1, \mathbb{J}_2, \dots, \mathbb{J}_M \subset \mathbb{I}_p$ *with* $\bigcup_{m=1}^M \mathbb{J}_m = \mathbb{I}_p$.

We call the above results *the complex Brownian motion (CBM) representations* of the Dyson model. In order to show the simplest application of this representation, we consider the density function at a single time for $(\Xi(t), \mathbb{P}_\xi)$ denoted by $\rho_\xi(t,x)$. It is defined as a continuous function of $x \in \mathbb{R}$ for $0 < t \le T < \infty$ such that for any $\chi \in \mathrm{C}_0(\mathbb{R})$

$$\mathbb{E}_\xi \left[\int_{\mathbb{R}} \chi(x) \Xi(t, dx) \right] = \int_{\mathbb{R}} dx\, \chi(x) \rho_\xi(t,x). \tag{1.11}$$

By (1.7), the equality (1.9) gives the following expression for (1.11)

$$\int_{\mathbb{R}} \xi(dv) \mathbf{E}_v \left[\chi(V(t)) \Phi_\xi^v(Z(t)) \right] = \int_{\mathbb{R}} \xi(dv) \int_{\mathbb{R}} dx\, p_{0,t}(v,x) \int_{\mathbb{R}} dw\, p_{0,t}(0,w) \chi(x) \Phi_\xi^v(x + \sqrt{-1}w),$$

where $p_{s,t}(x,y) = \dfrac{e^{-(y-x)^2/2(t-s)}}{\sqrt{2\pi(t-s)}}$, $0 \le s < t, x, y \in \mathbb{R}$, since $V(0) = \mathrm{Re} Z(0) = v \in \text{supp}\, \xi$ and $W(0) = \mathrm{Im} Z(0) = 0$. Then, if we define the function

$$\mathscr{G}_{s,t}(x,y) = \int_{\mathbb{R}} \xi(dv) p_{0,s}(v,x) \int_{\mathbb{R}} dw\ p_{0,t}(0,w) \Phi_\xi^v(y + \sqrt{-1}w) \tag{1.12}$$

for $(x,y) \in \mathbb{R}^2, (s,t) \in (0,T]^2$, we obtain the expression for the density function

$$\rho_\xi(t,x) = \mathscr{G}_{t,t}(x,x), \quad x \in \mathbb{R}, \quad 0 < t \le T < \infty$$

for any initial configuration $\xi \in \mathscr{M}_0$. The above calculation will be fully generalized and the following formula can be derived from our CBM representations.

Corollary 1.2. *Suppose that* $\xi \in \mathscr{M}_0$ *with* $\xi(\mathbb{R}) \in \mathbb{N}$. *Let*

$$\mathbb{K}_\xi(s,x;t,y) = \mathscr{G}_{s,t}(x,y) - \mathbb{1}(s > t) p_{t,s}(y,x). \tag{1.13}$$

Then the moment generating function (1.10) for the multitime distribution is given by a Fredholm determinant

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \mathop{\mathit{Det}}_{\substack{(s,t) \in \{t_1, t_2, \dots, t_M\}^2, \\ (x,y) \in \mathbb{R}^2}} \left[\delta_{st} \delta_x(y) + \mathbb{K}_\xi(s,x;t,y) \chi_t(y) \right]. \tag{1.14}$$

By definition of Fredholm determinant (see, for example, [5]) the moment generating function (1.14) can be expanded with respect to $\chi_{t_m}(\cdot), 1 \le m \le M$, as

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \sum_{\substack{N_m \ge 0, \\ 1 \le m \le M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}^{\mathrm{A}}} \prod_{m=1}^M \left\{ d\boldsymbol{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_{t_m}\left(x_i^{(m)}\right) \right\} \rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right), \tag{1.15}$$

with

$$\rho_\xi\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right) = \det_{\substack{1 \le i \le N_m, 1 \le j \le N_n, \\ 1 \le m,n \le M}} \left[\mathbb{K}_\xi(t_m, x_i^{(m)}; t_n, x_j^{(n)}) \right], \tag{1.16}$$

where $\boldsymbol{x}_{N_m}^{(m)}$ denotes $(x_1^{(m)}, \dots, x_{N_m}^{(m)})$ and $d\boldsymbol{x}_{N_m}^{(m)} = \prod_{i=1}^{N_m} dx_i^{(m)}$, $1 \le m \le M$. The functions ρ_ξ's are multitime correlation functions, and $\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}]$ can be regarded as a generating function of them. The function $\mathbb{K}_\xi$ given by (1.13) with (1.12) is thus called the *correlation kernel* [11]. In general, when the moment generating function for the multitime distribution is given by a Fredholm determinant, the process is said to be *determinantal* [9, 11]. The results by Eynard and Mehta reported in [4] for a multi-layer matrix model can be regarded as the theorem that the Dyson model is determinantal for the special initial configuration $\xi = \xi(\mathbb{R})\delta_0$, *i.e.*, all particles are put at the origin, for any

$\xi(\mathbb{R}) \in \mathbb{N}$. The correlation kernel is expressed by using the Hermite orthogonal polynomials [17]. The present authors proved that, for any fixed initial configuration $\xi \in \mathscr{M}$ with $\xi(\mathbb{R}) \in \mathbb{N}$, the Dyson model $(\Xi(t), \mathbb{P}_\xi)$ is determinantal, in which the correlation kernel is given by

$$\begin{aligned}\mathbb{K}_\xi(s,x;t,y) &= \frac{1}{2\pi\sqrt{-1}} \oint_{\Gamma(\xi)} dz\, p_{0,s}(z,x) \int_{\mathbb{R}} dw\, p_{0,t}(w, -\sqrt{-1}y) \\ &\times \frac{1}{\sqrt{-1}w - z} \prod_{r \in \text{supp}\, \xi} \left(1 - \frac{\sqrt{-1}w - z}{r - z}\right) - \mathbb{1}(s > t) p_{t,s}(y,x), \quad (1.17)\end{aligned}$$

where $\Gamma(\xi)$ is a closed contour on the complex plane $\mathbb{C}$ encircling the points in supp ξ on the real line $\mathbb{R}$ once in the positive direction (Proposition 2.1 in [11]). In order to derive (1.17), we used the integral representations of multiple Hermite polynomials given by Bleher and Kuijlaars [2].

In the present paper, we assume $\xi \in \mathscr{M}_0$ preventing the initial configuration from having any multiple points. This restriction is only for simplicity of calculation. (Note that, if $\xi \in \mathscr{M}_0$, the Cauchy integrals in (1.17) can be readily performed and the expression (1.13) with (1.12) is obtained.) The fact that we would like to report here is that, although the theory of (multiple-)orthogonal functions are very useful to analyze determinantal processes [11, 10, 12], it is not necessary to deriving the Eynard-Mehta-type determinantal expressions for multitime correlation functions. The essential point may be the extension of h-transform to the conformal martingale of CBMs in the determinantal form $\det_{1 \le i,j \le \xi(\mathbb{R})}[\Phi_\xi^{u_i}(Z_j(\cdot))]$, which we have named the CBM representation. In other words, the proof of Corollary 1.2 will provide a probability-theoretical derivation of the *Eynard-Mehta-type correlation kernel*.

We gave useful sufficient conditions of ξ in [11] so that the Dyson model is well defined as a determinantal process even if $\xi(\mathbb{R}) = \infty$. For $L > 0, \alpha > 0$ and $\xi \in \mathscr{M}$ we put

$$M(\xi, L) = \int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{x}, \qquad M_\alpha(\xi, L) = \left(\int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{|x|^\alpha} \right)^{1/\alpha},$$

and $M(\xi) = \lim_{L\to\infty} M(\xi, L)$, $M_\alpha(\xi) = \lim_{L\to\infty} M_\alpha(\xi, L)$, if the limits finitely exist. Then

(**C.1**) there exists $C_0 > 0$ such that $|M(\xi, L)| < C_0$, $L > 0$,

(**C.2**) (i) there exist $\alpha \in (1,2)$ and $C_1 > 0$ such that $M_\alpha(\xi) \le C_1$,
(ii) there exist $\beta > 0$ and $C_2 > 0$ such that

$$M_1(\tau_{-a^2}\xi^{\langle 2 \rangle}) \le C_2(\max\{|a|, 1\})^{-\beta} \quad \forall a \in \text{supp}\, \xi.$$

It was shown that, if $\xi \in \mathscr{M}_0$ satisfies the conditions (**C.1**) and (**C.2**), then for $a \in \mathbb{R}$ and $z \in \mathbb{C}$, $\Phi_\xi^a(z) \equiv \lim_{L\to\infty} \Phi^a_{\xi\cap[a-L,a+L]}(z)$ finitely exists, and

$$|\Phi_\xi^a(z)| \le C \exp\left\{ c(|a|^\theta + |z|^\theta) \right\} \left| \frac{z}{a} \right|^{\xi(\{0\})} \left| \frac{a}{a-z} \right|, \quad a \in \text{supp}\, \xi, \ z \in \mathbb{C}, \qquad (1.18)$$

for some $c, C > 0$ and $\theta \in (\max\{\alpha, (2-\beta)\}, 2)$, which are determined by the constants C_0, C_1, C_2 and the indices α, β in the conditions (Lemma 4.4 in [11]). We have noted that in the case that $\xi \in \mathscr{M}_0$ satisfies the conditions (**C.1**) and (**C.2**) with constants C_0, C_1, C_2 and indices α and β, then $\xi \cap [-L, L], \forall L > 0$ does as well. Hence we can obtain the convergence of moment generating functions $\Psi^{\boldsymbol{t}}_{\xi\cap[-L,L]}[\boldsymbol{f}] \to \Psi^{\boldsymbol{t}}_\xi[\boldsymbol{f}]$ as $L \to \infty$, which implies the convergence of the probability measures $\mathbb{P}_{\xi\cap[-L,L]} \to \mathbb{P}_\xi$ in $L \to \infty$ in the sense of finite dimensional distributions. Moreover, even if $\xi(\mathbb{R}) = \infty$, $\mathbb{K}_\xi$ given by (1.13) with (1.12) is well-defined as a correlation kernel and dynamics of the Dyson model with an infinite number of particles $(\Xi(t), \mathbb{P}_\xi)$ exists as a determinantal process [11].

The CBM representation is indeed a non-trivial extension of h-transforms, since it works also in infinite particle systems.

Corollary 1.3. *Suppose that the initial configuration $\xi \in \mathscr{M}_0$ satisfies the conditions (**C.1**) and (**C.2**). Then the expression (1.9) is valid also in the case with $\xi(\mathbb{R}) = \infty$, if F is represented as*

$$F(\Xi(\cdot)) = G\left(\int_{\mathbb{R}} \phi_1(x)\Xi(t_1, dx), \int_{\mathbb{R}} \phi_2(x)\Xi(t_2, dx), \dots, \int_{\mathbb{R}} \phi_k(x)\Xi(t_k, dx)\right)$$

with $\phi_i \in C_0(\mathbb{R})$, $1 \le i \le k$ and a polynomial function G on $\mathbb{R}^k$, $k \in \mathbb{N}$.

In order to demonstrate the usefulness of the CBM representations to characterize infinite particle systems, we show that the following estimate is readily obtained from the expression (1.9). Let $\mathrm{C}_0^\infty(\mathbb{R})$ be the set of all infinitely differentiable real functions with compact supports.

Proposition 1.4. *Suppose that the initial configuration $\xi \in \mathscr{M}_0$ satisfies the conditions (**C.1**) and (**C.2**) with constants C_0, C_1, C_2 and indices α, β. Then for any $T > 0$ and $\varphi \in C_0^\infty(\mathbb{R})$ there exists a positive constant $C = C(C_0, C_1, C_2, \alpha, \beta, T, \varphi)$, which is independent of s, t, such that*

$$\mathbb{E}_\xi\left[\left|\int_{\mathbb{R}} \varphi(x)\Xi(t, dx) - \int_{\mathbb{R}} \varphi(x)\Xi(s, dx)\right|^4\right] \le C|t-s|^2, \quad \forall s, t \in [0, T]. \tag{1.19}$$

By a criterion of Kolmogorov (see, for example, [7, 1]), Proposition 1.4 implies that the sequence of the process $(\Xi(t), \mathbb{P}_{\xi \cap [-L, L]})$, $L \in \mathbb{N}$ is tight in $\mathrm{C}([0, \infty) \to \mathscr{M})$. Then we can conclude the following.

Theorem 1.5. *Suppose that $\xi \in \mathscr{M}_0$ satisfies the conditions (**C.1**) and (**C.2**). Then the process $(\Xi(t), \mathbb{P}_{\xi \cap [-L, L]})$ converges to the process $(\Xi(t), \mathbb{P}_\xi)$ weakly on $C([0, \infty) \to \mathscr{M})$ as $L \to \infty$. In particular, the process $(\Xi(t), \mathbb{P}_\xi)$ has a modification which is almost-surely continuous on $[0, \infty)$ with $\Xi(0) = \xi$.*

Finally in the present paper, we show that the noncolliding property of the Dyson model with an infinite number of particles is obtained by using the CBM representations.

Proposition 1.6. *Suppose that the initial configuration $\xi \in \mathscr{M}_0$ satisfies the conditions (**C.1**) and (**C.2**). Then $\mathbb{P}_\xi\big[\Xi(t) \in \mathscr{M}_0, \, t > 0\big] = 1$.*

In the following five sections, we give the proofs of Theorem 1.1, Corollary 1.2, Corollary 1.3, Proposition 1.4, and Proposition 1.6, respectively.

2 Proof of Theorem 1.1

For the proof of Theorem 1.1, it is sufficient to consider the case that F is given as $F(\Xi(\cdot)) = \prod_{i=1}^M g_i(\boldsymbol{X}(t_i))$ for $M \in \mathbb{N}$, $0 < t_1 < \dots < t_M \le T < \infty$, with symmetric bounded measurable functions g_i on $\mathbb{R}^{\xi(\mathbb{R})}$, $1 \le i \le M$. We give the proof for the case with $M = 2$, *i.e.*, for $\xi = \sum_{i=1}^{\xi(\mathbb{R})} \delta_{u_i}, \boldsymbol{u} = (u_1, \dots, u_{\xi(\mathbb{R})}) \in \mathbb{W}_{\xi(\mathbb{R})}^{\mathrm{A}}$

$$\mathbb{E}_\xi\left[g_1(\boldsymbol{X}(t_1))g_2(\boldsymbol{X}(t_2))\right] = \mathbf{E}_{\boldsymbol{u}}\left[g_1(\boldsymbol{V}(t_1))g_2(\boldsymbol{V}(t_2)) \det_{1 \le i,j \le \xi(\mathbb{R})}\left[\Phi_\xi^{u_i}(Z_j(T))\right]\right]. \tag{2.1}$$

The generalization for $M > 2$ is straightforward.

We use the fact that the Dyson model is obtained as an h-transform of the absorbing Brownian motion in the Weyl chamber $\mathbb{W}^{\mathrm{A}}_{\xi(\mathbb{R})}$ [6]. Put $\tau = \inf\{t > 0 : \boldsymbol{V}(t) \notin \mathbb{W}^{\mathrm{A}}_{\xi(\mathbb{R})}\}$, then the LHS of (2.1) is given by

$$\mathbf{E}_{\boldsymbol{u}}\left[\mathbb{1}(\tau > t_2) g_1(\boldsymbol{V}(t_1)) g_2(\boldsymbol{V}(t_2)) \frac{h(\boldsymbol{V}(t_2))}{h(\boldsymbol{u})}\right]. \tag{2.2}$$

For a finite set S, we write the collection of all permutations of elements in S as $\mathbb{S}(S)$. In particular, we express $\mathbb{S}(\mathbb{I}_p)$ simply by $\mathbb{S}_p, p \in \mathbb{N}$. We put $\sigma(\boldsymbol{u}) = (u_{\sigma(1)}, \dots, u_{\sigma(\xi(\mathbb{R}))})$ for each permutation $\sigma \in \mathbb{S}_{\xi(\mathbb{R})}$. We introduce the stopping times

$$\tau_{ij} = \inf\{t > 0 : V_i(t) = V_j(t)\}, \quad 1 \le i < j \le \xi(\mathbb{R}). \tag{2.3}$$

Let $\sigma_{ij} \in \mathbb{S}_{\xi(\mathbb{R})}$ be the permutation of (i, j). Note that if $\sigma_{ij}(\boldsymbol{u}) = \boldsymbol{u}$, the processes $\boldsymbol{V}(t)$ and $\sigma_{ij}(\boldsymbol{V}(t))$ are identical in distribution under the probability measure $\mathbf{P}_{\boldsymbol{u}}$. Then by the strong Markov property of the process $\boldsymbol{V}(t)$ and by the fact that h is anti-symmetric and g_1, g_2 are symmetric,

$$\mathbf{E}_{\boldsymbol{u}}\Big[\mathbb{1}(\tau = \tau_{ij} \le t_2) g_1(\boldsymbol{V}(t_1)) g_2(\boldsymbol{V}(t_2)) \frac{h(\boldsymbol{V}(t_2))}{h(\boldsymbol{u})}\Big] = 0.$$

Since $\mathbf{P}_{\boldsymbol{u}}(\tau_{ij} = \tau_{i'j'}) = 0$ if $(i, j) \neq (i', j')$, and $\tau = \min_{1 \le i < j \le \xi(\mathbb{R})} \tau_{ij}$,

$$\mathbf{E}_{\boldsymbol{u}}\Big[\mathbb{1}(\tau \le t_2) g_1(\boldsymbol{V}(t_1)) g_2(\boldsymbol{V}(t_2)) \frac{h(\boldsymbol{V}(t_2))}{h(\boldsymbol{u})}\Big] = 0.$$

Hence, (2.2) equals

$$\mathbf{E}_{\boldsymbol{u}}\left[g_1(\boldsymbol{V}(t_1)) g_2(\boldsymbol{V}(t_2)) \frac{h(\boldsymbol{V}(t_2))}{h(\boldsymbol{u})}\right]. \tag{2.4}$$

Then we use the equality (1.8) in (2.4). Note that $V_i(t), 1 \le i \le \xi(\mathbb{R})$ and $W_i(t), 1 \le i \le \xi(\mathbb{R})$ are independent. We can regard the probability space $(\Omega, \mathcal{F}, \mathbf{P}_{\boldsymbol{v}})$ as a product of two probability spaces $(\Omega_1, \mathcal{F}_1, \mathrm{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathrm{P}_2)$, and $V_i(t), 1 \le i \le \xi(\mathbb{R})$ are $\mathcal{F}_1$-measurable and $W_i(t), 1 \le i \le \xi(\mathbb{R})$ are $\mathcal{F}_2$-measurable. We write E_α for the expectation with respect to P_α, $\alpha = 1, 2$. We see

$$\mathrm{E}_2\big[h(\boldsymbol{Z}(t))\big] = \mathrm{E}_2\left[\det_{1 \le i,j \le \xi(\mathbb{R})}\big[Z_j(t)^{i-1}\big]\right] = \det_{1 \le i,j \le \xi(\mathbb{R})}\big[\mathrm{E}_2\big[Z_j(t)^{i-1}\big]\big],$$

where the independence of $Z_j(t)$, $1 \le j \le \xi(\mathbb{R})$, is used in the last equality. By binomial expansion, $\mathrm{E}_2[Z_j(t)^{i-1}] = G(V_j(t))$ with $G(x) = \sum_{p=0}^{i-1} \binom{i-1}{p} \mathrm{E}_2\Big[(\sqrt{-1} W_j(t))^{i-1-p}\Big] x^p$. Since $G(x)$ is a monic polynomial with degree $i - 1$, $\mathrm{E}_2\big[h(\boldsymbol{Z}(t))\big] = h(\boldsymbol{V}(t))$. Combining the above results and the fact (1.6), we have (2.1).

For the proof of (1.10) with $M = 2$, we first prove that for any $N_1, N_2 \in \mathbb{N}$

$$\begin{aligned}
&\sum_{\substack{\mathbb{J}_1, \mathbb{J}_2 \subset \mathbb{I}_{\xi(\mathbb{R})}: \\ \sharp\mathbb{J}_1 = N_1, \sharp\mathbb{J}_2 = N_2}} \mathbb{E}_\xi\left[\prod_{m=1}^{2} \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(X_{j_m}(t_m))\right] \\
&= \sum_{p=1}^{N_1+N_2} \sum_{\substack{\mathbb{J}_1, \mathbb{J}_2 \subset \mathbb{I}_p: \\ \sharp\mathbb{J}_1 = N_1, \sharp\mathbb{J}_2 = N_2 \\ \mathbb{J}_1 \cup \mathbb{J}_2 = \mathbb{I}_p}} \int_{\mathbb{W}^{\mathrm{A}}_p} \xi^{\otimes p}(d\boldsymbol{v}) \mathbf{E}_{\boldsymbol{v}}\left[\prod_{m=1}^{2} \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{i,j \in \mathbb{I}_p}\Big[\Phi_\xi^{v_i}(Z_j(T))\Big]\right].
\end{aligned} \tag{2.5}$$

Applying (2.1) with $g_m(\boldsymbol{x}) = \displaystyle\sum_{\mathbb{J}_m \subset \mathbb{I}_{\xi(R)} : \sharp\mathbb{J}_m = N_m} \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(x_{j_m})$, $m = 1, 2$, we see that the LHS of (2.5) equals

$$\sum_{\substack{\mathbb{J}_1, \mathbb{J}_2 \subset \mathbb{I}_{\xi(\mathbb{R})}: \\ \sharp\mathbb{J}_1 = N_1, \sharp\mathbb{J}_2 = N_2}} \mathbf{E}_{\boldsymbol{u}} \left[\prod_{m=1}^{2} \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{i,j \in \mathbb{I}_{\xi(\mathbb{R})}} \left[\Phi_\xi^{u_i}(Z_j(T)) \right] \right]$$
$$= \sum_{p=1}^{N_1+N_2} \sum_{\substack{\mathbb{J}_1, \mathbb{J}_2 \subset \mathbb{I}_{\xi(\mathbb{R})}: \\ \sharp(\mathbb{J}_1 \cup \mathbb{J}_2) = p, \\ \sharp\mathbb{J}_1 = N_1, \sharp\mathbb{J}_2 = N_2}} \mathbf{E}_{\boldsymbol{u}} \left[\prod_{m=1}^{2} \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{i,j \in \mathbb{J}_1 \cup \mathbb{J}_2} \left[\Phi_\xi^{u_i}(Z_j(T)) \right] \right],$$

where we have used (1.7). We see the RHS of the last equation coincides with the RHS of (2.5). By using relation

$$\exp\left\{ \sum_{m=1}^{2} \sum_{j_m=1}^{\xi(\mathbb{R})} f_{t_m}(x_{j_m}) \right\} = \prod_{m=1}^{2} \prod_{j_m=1}^{\xi(\mathbb{R})} \left\{ \chi_{t_m}(x_{j_m}) + 1 \right\} = \sum_{\mathbb{J}_1, \mathbb{J}_2 \subset \mathbb{I}_{\xi(\mathbb{R})}} \prod_{m=1}^{2} \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(x_{j_m}),$$

the equality (1.10) with $M = 2$ is readily derived form (2.5). By the similar argument, (1.10) is concluded from (1.9) for any $M > 2$. □

3 Proof of Corollary 1.2

Since the Fredholm determinant (1.14) is explicitly given by (1.15) with (1.16), (1.10) in Theorem 1.1 implies that, for proof of Corollary 1.2, it is enough to show that the following equality is established for any $M \in \mathbb{N}, (N_1, \dots, N_M) \in \mathbb{N}^M$

$$\int_{\prod_{m=1}^M \mathbb{W}_{N_m}^{\mathbb{A}}} \prod_{m=1}^{M} \left\{ d\boldsymbol{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_{t_m}\left(x_i^{(m)}\right) \right\} \det_{\substack{1 \le i \le N_m, 1 \le j \le N_n, \\ 1 \le m, n \le M}} \left[\mathbb{K}_\xi(t_m, x_i^{(m)}; t_n, x_j^{(n)}) \right]$$
$$= \sum_{p=1}^{N} \sum_{\substack{\sharp\mathbb{J}_m = N_m, \\ 1 \le m \le M: \\ \bigcup_{m=1}^M \mathbb{J}_m = \mathbb{I}_p}} \int_{\mathbb{W}_p^{\mathbb{A}}} \xi^{\otimes p}(d\boldsymbol{v}) \mathbf{E}_{\boldsymbol{v}} \left[\prod_{m=1}^{M} \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{i,j \in \mathbb{I}_p} \left[\Phi_\xi^{v_i}(Z_j(T)) \right] \right]. \tag{3.1}$$

If we take the summation of (3.1) over all $0 \le N_m \le \xi(\mathbb{R}), 1 \le m \le M$, the LHS gives (1.15) with (1.16) and the RHS does (1.10). In this section we will prove (3.1). So in the following, we fix $M \in \mathbb{N}$, $(N_1, \dots, N_M) \in \mathbb{N}^M$.

Let $\mathbb{I}^{(1)} = \mathbb{I}_{N_1}$ and $\mathbb{I}^{(m)} = \mathbb{I}_{\sum_{k=1}^m N_k} \setminus \mathbb{I}_{\sum_{k=1}^{m-1} N_k}, 2 \le m \le M$. Put $N = \sum_{m=1}^M N_m$ and $\tau_i = \sum_{m=1}^M t_m \mathbb{1}(i \in \mathbb{I}^{(m)}), 1 \le i \le N$. Then the integrand in the LHS of (3.1) is simply written as $\displaystyle\prod_{i=1}^{N} \chi_{\tau_i}(x_i) \det_{1 \le i,j \le N} [\mathbb{K}_\xi(\tau_i, x_i; \tau_j, x_j)]$, and the integral $\int_{\prod_{m=1}^M \mathbb{W}_{N_m}^{\mathbb{A}}} \prod_{m=1}^M d\boldsymbol{x}_{N_m}^{(m)}(\cdot)$ can be replaced by $\{\prod_{m=1}^M N_m!\}^{-1} \int_{\mathbb{R}^N} d\boldsymbol{x}\,(\cdot)$. The determinant is defined using the notion of permutations and we note that any permutation $\sigma \in \mathbb{S}_N$ can be decomposed into a product of cycles. Let the number of cycles in the decomposition be $\ell(\sigma)$ and express σ by $\sigma = c_1 c_2 \cdots c_{\ell(\sigma)}$, where c_λ denotes a cyclic permutation $c_\lambda = (c_\lambda(1) c_\lambda(2) \cdots c_\lambda(q_\lambda)), 1 \le q_\lambda \le N, 1 \le \lambda \le \ell(\sigma)$. For each $1 \le \lambda \le \ell(\sigma)$, we write the set of entries $\{c_\lambda(i)\}_{i=1}^{q_\lambda}$ of c_λ simply as $\{c_\lambda\}$, in which the periodicity $c_\lambda(i + q_\lambda) = c_\lambda(i), 1 \le i \le q_\lambda$ is assumed. By definition, for each $1 \le \lambda \le \ell(\sigma)$, $c_\lambda(i), 1 \le i \le q_\lambda$ are distinct indices chosen from $\mathbb{I}_N$, $\{c_\lambda\} \cap \{c_{\lambda'}\} = \emptyset$ for $1 \le \lambda \ne \lambda' \le \ell(\sigma)$, and $\sum_{\lambda=1}^{\ell(\sigma)} q_\lambda = N$. The determinant $\det_{1\le i,j\le N}[\mathbb{K}_\xi(\tau_i, x_i; \tau_j, x_j)]$ is written as

$$\begin{aligned}
&\sum_{\sigma\in\mathbb{S}_N} (-1)^{N-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{i=1}^{q_\lambda} \mathbb{K}_\xi(\tau_{c_\lambda(i)}, x_{c_\lambda(i)}; \tau_{c_\lambda(i+1)}, x_{c_\lambda(i+1)}) \\
= &\sum_{\sigma\in\mathbb{S}_N} (-1)^{N-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{i=1}^{q_\lambda} \Big\{ \mathscr{G}_{\tau_{c_\lambda(i)}, \tau_{c_\lambda(i+1)}}(x_{c_\lambda(i)}, x_{c_\lambda(i+1)}) \\
&\qquad -\mathbb{1}(\tau_{c_\lambda(i)} > \tau_{c_\lambda(i+1)}) p_{\tau_{c_\lambda(i+1)}, \tau_{c_\lambda(i)}}(x_{c_\lambda(i+1)}, x_{c_\lambda(i)}) \Big\},
\end{aligned} \tag{3.2}$$

where the definition (1.13) of the correlation kernel $\mathbb{K}_\xi$ is used. In order to express binomial expansions for (3.2), we introduce the following notations: For each cyclic permutation c_λ, we consider a subset of $\{c_\lambda\}$, $C(c_\lambda) = \Big\{ c_\lambda(i) \in \{c_\lambda\} : \tau_{c_\lambda(i)} > \tau_{c_\lambda(i+1)} \Big\}$. Choose $\mathbf{M}_\lambda$ such that $\{c_\lambda\} \setminus C(c_\lambda) \subset \mathbf{M}_\lambda \subset \{c_\lambda\}$, and define $\mathbf{M}_\lambda^{\rm c} = \{c_\lambda\} \setminus \mathbf{M}_\lambda$. Therefore if we put

$$\begin{aligned}
G(c_\lambda, \mathbf{M}_\lambda) \quad = \quad &\int_{\mathbb{R}^{\{c_\lambda\}}} \prod_{i=1}^{q_\lambda} \Big\{ dx_{c_\lambda(i)} \ \chi_{\tau_{c_\lambda(i)}}(x_{c_\lambda(i)}) p_{\tau_{c_\lambda(i+1)}, \tau_{c_\lambda(i)}}(x_{c_\lambda(i+1)}, x_{c_\lambda(i)})^{\mathbb{1}(c_\lambda(i) \in \mathbf{M}_\lambda^{\rm c})} \\
&\qquad \times \mathscr{G}_{\tau_{c_\lambda(i)}, \tau_{c_\lambda(i+1)}}(x_{c_\lambda(i)}, x_{c_\lambda(i+1)})^{\mathbb{1}(c_\lambda(i) \in \mathbf{M}_\lambda)} \Big\},
\end{aligned} \tag{3.3}$$

the LHS of (3.1) is expanded as

$$\frac{1}{\prod_{m=1}^M N_m!} \sum_{\sigma\in\mathbb{S}_N} (-1)^{N-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \sum_{\substack{\mathbf{M}_\lambda: \\ \{c_\lambda\}\setminus C(c_\lambda) \subset \mathbf{M}_\lambda \subset \{c_\lambda\}}} (-1)^{\sharp \mathbf{M}_\lambda^{\rm c}} G(c_\lambda, \mathbf{M}_\lambda). \tag{3.4}$$

From now on, we will explain how to rewrite $G(c_\lambda, \mathbf{M}_\lambda)$ until (3.8). We note that if we set

$$\begin{aligned}
&F(\{x_{c_\lambda(j)} : c_\lambda(j) \in \mathbf{M}_\lambda^{\rm c}\}) \\
&\quad = \int_{\mathbb{R}^{\mathbf{M}_\lambda}} \prod_{i: c_\lambda(i) \in \mathbf{M}_\lambda} \Big\{ dx_{c_\lambda(i)} \ \chi_{\tau_{c_\lambda(i)}}(x_{c_\lambda(i)}) \mathscr{G}_{\tau_{c_\lambda(i)}, \tau_{c_\lambda(i+1)}}(x_{c_\lambda(i)}, x_{c_\lambda(i+1)}) \Big\} \\
&\qquad \times \prod_{j: c_\lambda(j) \in \mathbf{M}_\lambda^{\rm c}} p_{\tau_{c_\lambda(j+1)}, \tau_{c_\lambda(j)}}(x_{c_\lambda(j+1)}, x_{c_\lambda(j)}),
\end{aligned} \tag{3.5}$$

which is the integral over $\mathbb{R}^{\mathbf{M}_\lambda}$, then (3.3) is obtained by performing the integral of it over $\mathbb{R}^{\mathbf{M}_\lambda^{\rm c}} = \mathbb{R}^{\{c_\lambda\}} \setminus \mathbb{R}^{\mathbf{M}_\lambda}$,

$$G(c_\lambda, \mathbf{M}_\lambda) = \int_{\mathbb{R}^{\mathbf{M}_\lambda^{\rm c}}} \prod_{j: c_\lambda(j) \in \mathbf{M}_\lambda^{\rm c}} \Big\{ dx_{c_\lambda(j)} \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) \Big\} F(\{x_{c_\lambda(j)} : c_\lambda(j) \in \mathbf{M}_\lambda^{\rm c}\}). \tag{3.6}$$

In (3.5), use the integral representation (1.12) for $\mathscr{G}_{\tau_{c_\lambda(i)}, \tau_{c_\lambda(i+1)}}(x_{c_\lambda(i)}, x_{c_\lambda(i+1)})$ by putting the integral variables to be $v = v_{c_\lambda(i)}$ and $w = w_{c_\lambda(i+1)}$. We obtain

$$\begin{aligned}
& F(\{x_{c_\lambda(j)} : c_\lambda(j) \in \mathbf{M}_\lambda^{\rm c}\}) \\
= & \int_{\mathbb{R}^{\mathbf{M}_\lambda}} \prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda} \xi(dv_{c_\lambda(i)}) \int_{\mathbb{R}^{\mathbf{M}_\lambda}} \prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda} \Big\{ dx_{c_\lambda(i)} p_{0,\tau_{c_\lambda(i)}}(v_{c_\lambda(i)}, x_{c_\lambda(i)}) \chi_{\tau_{c_\lambda(i)}}(x_{c_\lambda(i)}) \Big\} \\
\times & \int_{\mathbb{R}^{\mathbf{M}_\lambda}} \prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda} \Big\{ dw_{c_\lambda(i+1)} p_{0,\tau_{c_\lambda(i+1)}}(0, w_{c_\lambda(i+1)}) \Phi_\xi^{v_{c_\lambda(i)}}(x_{c_\lambda(i+1)} + \sqrt{-1} w_{c_\lambda(i+1)}) \Big\} \\
& \quad \times \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda^{\rm c}} p_{\tau_{c_\lambda(j+1)},\tau_{c_\lambda(j)}}(x_{c_\lambda(j+1)}, x_{c_\lambda(j)}). \\
= & \int_{\mathbb{R}^{\mathbf{M}_\lambda}} \prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda} \xi(dv_{c_\lambda(i)}) \mathbf{E}_{\boldsymbol{v}} \Bigg[\prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda} \Big\{ \chi_{\tau_{c_\lambda(i)}}(V_{c_\lambda(i)}(\tau_{c_\lambda(i)})) \\
& \qquad \times \Phi_\xi^{v_{c_\lambda(i)}}(Z_{c_\lambda(i+1)}(\tau_{c_\lambda(i+1)}))^{\mathbb{1}(c_\lambda(i+1)\in\mathbf{M}_\lambda)} \\
& \qquad \times \Phi_\xi^{v_{c_\lambda(i)}}(x_{c_\lambda(i+1)} + \sqrt{-1} W_{c_\lambda(i+1)}(\tau_{c_\lambda(i+1)}))^{\mathbb{1}(c_\lambda(i+1)\in\mathbf{M}_\lambda^{\rm c})} \Big\} \\
& \quad \times \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda^{\rm c}} \Big\{ p_{\tau_{c_\lambda(j+1)},\tau_{c_\lambda(j)}}(V_{c_\lambda(j+1)}(\tau_{c_\lambda(j+1)}), x_{c_\lambda(j)})^{\mathbb{1}(c_\lambda(j+1)\in\mathbf{M}_\lambda)} \\
& \qquad \times p_{\tau_{c_\lambda(j+1)},\tau_{c_\lambda(j)}}(x_{c_\lambda(j+1)}, x_{c_\lambda(j)})^{\mathbb{1}(c_\lambda(j+1)\in\mathbf{M}_\lambda^{\rm c})} \Big\} \Bigg].
\end{aligned}$$

Using Fubini's theorem, (3.6) is given by

$$\begin{aligned}
& \int_{\mathbb{R}^{\mathbf{M}_\lambda}} \prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda} \xi(dv_{c_\lambda(i)}) \mathbf{E}_{\boldsymbol{v}} \Bigg[\prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda} \chi_{\tau_{c_\lambda(i)}}(V_{c_\lambda(i)}(\tau_{c_\lambda(i)})) \\
& \qquad \times \prod_{i:c_\lambda(i),c_\lambda(i+1)\in\mathbf{M}_\lambda} \Phi_\xi^{v_{c_\lambda(i)}}(Z_{c_\lambda(i+1)}(\tau_{c_\lambda(i+1)})) \\
& \quad \times \int_{\mathbb{R}^{\mathbf{M}_\lambda^{\rm c}}} \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda^{\rm c}} \Big\{ dx_{c_\lambda(j)} \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) \Big\} \\
& \qquad \times \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda^{\rm c}, c_\lambda(j+1)\in\mathbf{M}_\lambda} p_{\tau_{c_\lambda(j+1)},\tau_{c_\lambda(j)}}(V_{c_\lambda(j+1)}(\tau_{c_\lambda(j+1)}), x_{c_\lambda(j)}) \\
& \qquad \times \prod_{j:c_\lambda(j),c_\lambda(j+1)\in\mathbf{M}_\lambda^{\rm c}} p_{\tau_{c_\lambda(j+1)},\tau_{c_\lambda(j)}}(x_{c_\lambda(j+1)}, x_{c_\lambda(j)}) \\
& \qquad \times \prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda, c_\lambda(i+1)\in\mathbf{M}_\lambda^{\rm c}} \Phi_\xi^{v_{c_\lambda(i)}}(x_{c_\lambda(i+1)} + \sqrt{-1} W_{c_\lambda(i+1)}(\tau_{c_\lambda(i+1)})) \Bigg]. \qquad (3.7)
\end{aligned}$$

For each $1 \le i \le q_\lambda$ with $c_\lambda(i) \in \mathbf{M}_\lambda$, we define $\bar{i} = \min\{j > i : c_\lambda(j) \in \mathbf{M}_\lambda\}$ and $\underline{i} = \max\{j < i : c_\lambda(j) \in \mathbf{M}_\lambda\}$. Then we perform integration over $x_{c_\lambda(j)}$'s for $c_\lambda(j) \in \mathbf{M}_\lambda^{\rm c}$ before taking the expectation $\mathbf{E}_{\boldsymbol{v}}$. That is, integrals over $x_{c_\lambda(j)}$'s with indices in intervals $\underline{i} < j < i$ for all i, s.t. $c_\lambda(i) \in \mathbf{M}_\lambda$ are done. For each i, s.t. $c_\lambda(i) \in \mathbf{M}_\lambda$, if $\underline{i} < i-1$,

$$\begin{aligned}
& \chi_{\tau_{c_\lambda(i)}}(V_{c_\lambda(i)}(\tau_{c_\lambda(i)})) \prod_{j=\underline{i}+1}^{i-1} \int_{\mathbb{R}} dx_{c_\lambda(j)} \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) p_{\tau_{c_\lambda(i)},\tau_{c_\lambda(i-1)}}(V_{c_\lambda(i)}(\tau_{c_\lambda(i)}), x_{c_\lambda(i-1)}) \\
& \quad \times \prod_{k=\underline{i}+2}^{i-1} p_{\tau_{c_\lambda(k)},\tau_{c_\lambda(k-1)}}(x_{c_\lambda(k)}, x_{c_\lambda(k-1)}) \Phi_\xi^{v_{c_\lambda(\underline{i})}}(x_{c_\lambda(\underline{i}+1)} + \sqrt{-1} W_{c_\lambda(\underline{i}+1)}(\tau_{c_\lambda(\underline{i}+1)}))
\end{aligned}$$

coincides with the conditional expectation of

$$\prod_{j=\underline{i}+1}^{i} \chi_{\tau_{c_\lambda(j)}}(V_{c_\lambda(i)}(\tau_{c_\lambda(j)}))\Phi_\xi^{v_{c_\lambda(\underline{i})}}(V_{c_\lambda(i)}(\tau_{c_\lambda(\underline{i}+1)})+\sqrt{-1}W_{c_\lambda(\underline{i}+1)}(\tau_{c_\lambda(\underline{i}+1)})).$$

with respect to $\mathbf{E}_{\boldsymbol{v}}[\cdot|V_{c_\lambda(i)}, W_{c_\lambda(\underline{i}+1)}]$. Since $W_i(\cdot), i \in \{c_\lambda\}$ are i.i.d. random variables which are independent of $V_i(\cdot), i \in \{c_\lambda\}$, $V_{c_\lambda(i)}(\tau_{c_\lambda(\underline{i}+1)})+\sqrt{-1}W_{c_\lambda(\underline{i}+1)}\ (\tau_{c_\lambda(\underline{i}+1)})$ has the same distribution as $V_{c_\lambda(i)}(\tau_{c_\lambda(\underline{i}+1)})+\sqrt{-1}W_{c_\lambda(i)}(\tau_{c_\lambda(\underline{i}+1)}) = Z_{c_\lambda(i)}(\tau_{c_\lambda(\underline{i}+1)})$. Since $\prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda}\Phi_\xi^{v_{c_\lambda(\underline{i})}}\ (Z_{c_\lambda(i)}(\tau_{c_\lambda(\underline{i}+1)})) = \prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda}\Phi_\xi^{v_{c_\lambda(i)}}(Z_{c_\lambda(\bar{i})}(\tau_{c_\lambda(i+1)}))$, (3.7) is equal to

$$\int_{\mathbb{R}^{\mathbf{M}_\lambda}}\prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda}\xi(dv_{c_\lambda(i)})\mathbf{E}_{\boldsymbol{v}}\left[\prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda}\prod_{j=\underline{i}+1}^{i}\chi_{\tau_{c_\lambda(j)}}(V_{c_\lambda(i)}(\tau_{c_\lambda(j)}))\Phi_\xi^{v_{c_\lambda(i)}}(Z_{c_\lambda(\bar{i})}(\tau_{c_\lambda(i+1)}))\right].$$

Using only the entries of $\mathbf{M}_\lambda$, we can define a subcycle $\widehat{c_\lambda}$ of c_λ uniquely as follows: Since c_λ is a cyclic permutation, $\widehat{q_\lambda} \equiv \sharp\mathbf{M}_\lambda \geq 1$. Let $i_1 = \min\{1 \leq i \leq q_\lambda : c_\lambda(i) \in \mathbf{M}_\lambda\}$. If $\widehat{q_\lambda} \geq 2$, define $i_{j+1} = \bar{i}_j, 1 \leq j \leq \widehat{q_\lambda}-1$. Then $\widehat{c_\lambda} = (\widehat{c_\lambda}(1)\widehat{c_\lambda}(2)\cdots\widehat{c_\lambda}(\widehat{q_\lambda})) \equiv (c_\lambda(i_1)c_\lambda(i_2)\cdots c(i_{\widehat{q_\lambda}}))$. Moreover, we decompose the set $\mathbf{M}_\lambda$ into M subsets, $\mathbf{M}_\lambda = \bigcup_{m=1}^M \mathbf{J}_m^\lambda$, by letting

$$\mathbf{J}_m^\lambda = \mathbf{J}_m^\lambda(c_\lambda, \mathbf{M}_\lambda) = \Big\{c_\lambda(i) \in \mathbf{M}_\lambda : \underline{i} <^{\exists} j \leq i, \text{ s.t. } c_\lambda(j) \in \mathbb{I}^{(m)}\Big\}, \quad 1 \leq m \leq M.$$

Note that by definition $\mathbf{J}_m^\lambda \cap \mathbf{J}_{m'}^\lambda \neq \emptyset, m \neq m'$ in general, and $\mathbf{J}_1^\lambda = \mathbb{I}_{N_1} \cap \mathbf{M}_\lambda = \mathbb{I}_{N_1} \cap \{c_\lambda\}$, $\mathbf{J}_m^\lambda \subset \mathbb{I}_{\sum_{k=1}^m N_k}$ for $2 \leq m \leq M$, $\mathbf{J}_m^\lambda \cap \mathbb{I}^{(k)} \subset \mathbf{J}_k^\lambda$ for $1 \leq k < m \leq M$. Finally we arrive at the following expression of $G(c_\lambda, \mathbf{M}_\lambda)$,

$$\int_{\mathbb{R}^{\mathbf{M}_\lambda}}\prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda}\xi(dv_{c_\lambda(i)})\ \mathbf{E}_{\boldsymbol{v}}\left[\prod_{m=1}^M\prod_{j_m\in\mathbf{J}_m^\lambda}\chi_{t_m}(V_{j_m}(t_m))\prod_{i=1}^{\widehat{q_\lambda}}\Phi_\xi^{v_{\widehat{c_\lambda}(i)}}(Z_{\widehat{c_\lambda}(i+1)}(T))\right], \tag{3.8}$$

where the martingale property (1.6) is used. Let $\mathbf{M} \equiv \bigcup_{\lambda=1}^{\ell(\sigma)}\mathbf{M}_\lambda$. Since $N - \sum_{\lambda=1}^{\ell(\sigma)}\sharp\mathbf{M}_\lambda^{\rm c} = \sharp\mathbf{M}$, the LHS of (3.1), which is written as (3.4), becomes now

$$\begin{aligned}\frac{1}{\prod_{m=1}^M N_m!}\sum_{\sigma\in\mathbb{S}_N}\sum_{\substack{\mathbf{M}:\\ \mathbb{I}_N\setminus\bigcup_{\lambda=1}^{\ell(\sigma)}C(c_\lambda)\subset\mathbf{M}\subset\mathbb{I}_N}}(-1)^{\sharp\mathbf{M}-\ell(\sigma)}\int_{\mathbb{R}^{\mathbf{M}}}\prod_{\lambda=1}^{\ell(\sigma)}\prod_{i:c_\lambda(i)\in\mathbf{M}_\lambda}\xi(dv_{c_\lambda(i)})\\ \times\mathbf{E}_{\boldsymbol{v}}\left[\prod_{\lambda=1}^{\ell(\sigma)}\left\{\prod_{m=1}^M\prod_{j_m\in\mathbf{J}_m^\lambda}\chi_{t_m}(V_{j_m}(t_m))\prod_{i=1}^{\widehat{q_\lambda}}\Phi_\xi^{v_{\widehat{c_\lambda}(i)}}(Z_{\widehat{c_\lambda}(i+1)}(T))\right\}\right].\end{aligned} \tag{3.9}$$

We define $\widehat{\sigma} \equiv \widehat{c_1}\widehat{c_2}\cdots\widehat{c_{\ell(\sigma)}}$ and $\mathbf{J}_m \equiv \bigcup_{\lambda=1}^{\ell(\sigma)}\mathbf{J}_m^\lambda, 1 \leq m \leq M$. Note that $\ell(\widehat{\sigma}) = \ell(\sigma)$. The obtained $(\mathbf{J}_m)_{m=1}^M$'s form a collection of series of index sets satisfying the following conditions, which we write as $\mathcal{J}(\{N_m\}_{m=1}^M)$:

(C.J) $\mathbf{J}_1 = \mathbb{I}_{N_1}, \mathbf{J}_m \subset \mathbb{I}_{\sum_{k=1}^m N_k}$ for $2 \leq m \leq M$, $\mathbf{J}_m \cap \mathbb{I}^{(k)} \subset \mathbf{J}_k$ for $1 \leq k < m \leq M$, and $\sharp\mathbf{J}_m = N_m$ for $1 \leq m \leq M$.

For each $(\mathbf{J}_m)_{m=1}^M \in \mathcal{J}(\{N_m\}_{m=1}^M)$, we put $A_1 = 0$ and $A_m = \sharp(\mathbf{J}_m \cap \mathbb{I}_{\sum_{k=1}^{m-1}N_k}) = \sharp(\mathbf{J}_m \cap \bigcup_{k=1}^{m-1}\mathbf{J}_k), 2 \leq m \leq M$. Then, if we put $\mathbf{M} = \bigcup_{m=1}^M \mathbf{J}_m$, $\sharp\mathbf{M} = \sum_{m=1}^M(N_m - A_m)$, which means that from the original index set $\mathbb{I}_N = \bigcup_{m=1}^M \mathbb{I}^{(m)}$ with $\sharp\mathbb{I}^{(m)} = N_m, 1 \leq m \leq M$, we obtain a subset $\mathbf{M}$ by eliminating A_m elements at each level $1 \leq m \leq M$. By this reduction, we obtain $\widehat{\sigma} \in \mathbb{S}(\mathbf{M})$ from $\sigma \in \mathbb{S}_N$. It implies that, for all $\widehat{\sigma} \in \mathbb{S}(\mathbf{M})$, the number of σ's in $\mathbb{S}_N$ which give the same $\widehat{\sigma}$ and $(\mathbf{J}_m)_{m=1}^M$ by this reduction is given by $\prod_{m=1}^M A_m!$, where $0! \equiv 1$. Then (3.9) is equal to

$$\begin{aligned}
&\sum_{\substack{\mathbf{M}:\\ \max_m\{N_m\}\le\sharp\mathbf{M}\le N}} \sum_{\substack{(\mathbf{J}_m)_{m=1}^M\subset\mathcal{J}(\{N_m\}_{m=1}^M):\\ \bigcup_{m=1}^M\mathbf{J}_m=\mathbf{M}}} \frac{\prod_{m=1}^M A_m!}{\prod_{m=1}^M N_m!} \sum_{\widehat{\sigma}\in\mathbb{S}(\mathbf{M})} (-1)^{\sharp\mathbf{M}-\ell(\widehat{\sigma})} \\
&\quad\times\sharp\mathbf{M}! \int_{\mathbb{W}^{\mathbb{A}}_{\sharp\mathbf{M}}} \xi^{\otimes\mathbf{M}}(d\boldsymbol{v})\mathbf{E}_{\boldsymbol{v}}\left[\prod_{m=1}^M\prod_{j_m\in\mathbf{J}_m}\chi_{t_m}(V_{j_m}(t_m))\prod_{\lambda=1}^{\ell(\widehat{\sigma})}\prod_{i=1}^{\widehat{q_\lambda}}\Phi_\xi^{v_{\widehat{c_\lambda}(i)}}(Z_{\widehat{c_\lambda}(i+1)}(T))\right] \\
&=\sum_{\substack{\mathbf{M}:\\ \max_m\{N_m\}\le\sharp\mathbf{M}\le N}} \sum_{\substack{(\mathbf{J}_m)_{m=1}^M\subset\mathcal{J}(\{N_m\}_{m=1}^M):\\ \bigcup_{m=1}^M\mathbf{J}_m=\mathbf{M}}} \sharp\mathbf{M}!\prod_{m=1}^M\frac{A_m!}{N_m!} \\
&\quad\times\int_{\mathbb{W}^{\mathbb{A}}_{\sharp\mathbf{M}}} \xi^{\otimes\mathbf{M}}(d\boldsymbol{v})\mathbf{E}_{\boldsymbol{v}}\left[\prod_{m=1}^M\prod_{j_m\in\mathbf{J}_m}\chi_{t_m}(V_{j_m}(t_m))\det_{i,j\in\mathbf{M}}\left[\Phi_\xi^{v_i}(Z_j(T))\right]\right]. \qquad (3.10)
\end{aligned}$$

Assume $\max_m\{N_m\} \le p \le N$, $0 \le A_m \le N_m, 2 \le m \le M$ and set $A_1 = 0$. Consider

$$\Lambda_1 = \left\{(\mathbf{J}_m)_{m=1}^M \subset \mathcal{J}(\{N_m\}_{m=1}^M) : \sharp(\bigcup_{m=1}^M \mathbf{J}_m) = p, \sharp(\mathbf{J}_m \cap \bigcup_{k=1}^{m-1}\mathbf{J}_k) = A_m, 2 \le m \le M\right\},$$

$$\Lambda_2 = \left\{(\mathbb{J}_m)_{m=1}^M : \sharp\mathbb{J}_m = N_m, 1 \le m \le M, \bigcup_{m=1}^M \mathbb{J}_m = \mathbb{I}_p, \sharp(\mathbb{J}_m \cap \bigcup_{k=1}^{m-1}\mathbb{J}_k) = A_m, 2 \le m \le M\right\}.$$

Since the CBMs are i.i.d. in $\mathbf{P}_{\boldsymbol{v}}$, the integral in (3.10) has the same value for all $(\mathbf{J}_m)_{m=1}^M \in \Lambda_1$ with $\bigcup_{m=1}^M \mathbf{J}_m = \mathbf{M}$ and it is also equal to

$$\int_{\mathbb{W}^{\mathbb{A}}_p} \xi^{\otimes p}(d\boldsymbol{v})\mathbf{E}_{\boldsymbol{v}}\left[\prod_{m=1}^M\prod_{j_m\in\mathbb{J}_m}\chi_{t_m}(V_{j_m}(t_m))\det_{i,j\in\mathbb{I}_p}\left[\Phi_\xi^{v_i}(Z_j(T))\right]\right]$$

for all $(\mathbb{J}_m)_{m=1}^M \in \Lambda_2$. In Λ_1, for each $2 \le m \le M$, A_m elements in $\mathbf{J}_m$ are chosen from $\bigcup_{k=1}^{m-1}\mathbf{J}_k$, in which $\sharp(\bigcup_{k=1}^{m-1}\mathbf{J}_k) = \sum_{k=1}^{m-1}(N_k - A_k)$, and the remaining $N_m - A_m$ elements in $\mathbf{J}_m$ are from $\mathbb{I}^{(m)}$ with $\sharp\mathbb{I}^{(m)} = N_m$. Then

$$\sharp\Lambda_1 = \prod_{m=2}^M \binom{\sum_{k=1}^{m-1}(N_k - A_k)}{A_m}\binom{N_m}{N_m - A_m}.$$

In Λ_2, on the other hand, N_1 elements in $\mathbb{J}_1$ is chosen from $\mathbb{I}_p$, and then for each $2 \le m \le M$, A_m elements in $\mathbb{J}_m$ are chosen from $\bigcup_{k=1}^{m-1}\mathbb{J}_k$ with $\sharp(\bigcup_{k=1}^{m-1}\mathbb{J}_k) = \sum_{k=1}^{m-1}(N_k - A_k)$ and the remaining $N_m - A_m$ elements in $\mathbb{J}_m$ are from $\mathbb{I}_p\setminus\bigcup_{k=1}^{m-1}\mathbb{J}_k$ with $\sharp(\mathbb{I}_p\setminus\bigcup_{k=1}^{m-1}\mathbb{J}_k) = p - \sum_{k=1}^{m-1}(N_k - A_k)$. Then

$$\sharp\Lambda_2 = \binom{p}{N_1}\prod_{m=2}^M \binom{\sum_{k=1}^{m-1}(N_k - A_k)}{A_m}\binom{p - \sum_{k=1}^{m-1}(N_k - A_k)}{N_m - A_m}.$$

Since $\sum_{m=1}^M (N_m - A_m) = p$, we see $\sharp\Lambda_2/\sharp\Lambda_1 = p!\prod_{m=1}^M A_m!/N_m!$. Then (3.10) is equal to the RHS of (3.1) and the proof is completed. □

4 Proof of Corollary 1.3

We consider the case that $k=1$ and $G(x)=x^q$, $q\in\mathbb{N}$. (The argument will be easily extended to general polynomials of order q.) We introduce a map π form $\bigoplus_{n=1}^{\infty}\mathbb{R}^n$ to $\bigoplus_{p=1}^{\infty}\mathbb{W}_p^{\mathrm{A}}$ such that

$$\pi(w_1,w_2,\dots,w_n)=(v_1,v_2,\dots,v_p), \quad \text{where } \{w_i\}_{i=1}^n=\{v_i\}_{i=1}^p.$$

We also introduce the functions $p(\boldsymbol{w})=\sharp\{w_i\}_{i=1}^n$, and $a_i(\boldsymbol{w})=\sharp\{j: w_j=(\pi\boldsymbol{w})_i\}, i\in\mathbb{I}_{p(\boldsymbol{w})}$. Then in case $\xi(\mathbb{R})<\infty$ the CBM representation (1.9) gives

$$\begin{aligned}
&\mathbb{E}_\xi\left[F\big(\Xi(\cdot)\big)\right]=\mathbf{E}_{\boldsymbol{u}}\left[\left(\sum_{i=1}^{\xi(\mathbb{R})}\phi_1(V_i(t_1))\right)^q \det_{1\le i,j\le\xi(\mathbb{R})}\left[\Phi_\xi^{u_i}(Z_j(T))\right]\right]\\
&=\int_{\mathbb{R}^q}\xi^{\otimes q}(d\boldsymbol{w})\mathbf{E}_{\pi\boldsymbol{w}}\left[\prod_{i=1}^{p(\boldsymbol{w})}\phi_1(V_i(t_1))^{a_i(\boldsymbol{w})}\det_{i,j\in\mathbb{I}_{p(\boldsymbol{w})}}\left[\Phi_\xi^{(\pi\boldsymbol{w})_i}(Z_j(T))\right]\right]\\
&=\sum_{p=1}^{q}\sum_{\substack{a_i\in\mathbb{N},i\in\mathbb{I}_p:\\ \sum_{i=1}^p a_i=q}}\binom{q}{a_1,a_2,\dots,a_p}\int_{\mathbb{W}_p^{\mathrm{A}}}\xi^{\otimes p}(d\boldsymbol{v})\mathbf{E}_{\boldsymbol{v}}\left[\prod_{i=1}^{p}\phi_1(V_i(t_1))^{a_i}\det_{i,j\in\mathbb{I}_p}\left[\Phi_\xi^{v_i}(Z_j(T))\right]\right].
\end{aligned} \tag{4.1}$$

Here we used the fact that for $\boldsymbol{v}\in\mathbb{W}_p^{\mathrm{A}}$, $a_i\in\mathbb{N}, 1\le i\le p, \sum_{i=1}^p a_i=q$,

$$\sharp\Big\{\boldsymbol{w}\in\mathbb{R}^q:\pi\boldsymbol{w}=\boldsymbol{v},\ a_i(\boldsymbol{w})=a_i, 1\le i\le p\Big\}=\frac{q!}{a_1!a_2!\cdots a_p!}\equiv\binom{q}{a_1,a_2,\dots,a_p},$$

and the equality in the measure $\xi^{\otimes p}(d\boldsymbol{v})$

$$\begin{aligned}
&\mathbf{E}_{\boldsymbol{v}}\left[\prod_{i=1}^{p}\phi_1(V_i(t_1))^{a_i}\det_{1\le i,j\le\xi(\mathbb{R})}\left[\Phi_\xi^{v_i}(Z_j(T))\right]\right]\\
&\qquad=\mathbf{E}_{\boldsymbol{v}}\left[\prod_{i=1}^{p}\phi_1(V_i(t_1))^{a_i}\det_{i,j\in\mathbb{I}_p}\left[\Phi_\xi^{v_i}(Z_j(T))\right]\right],
\end{aligned} \tag{4.2}$$

which holds for any $p\le\xi(\mathbb{R})$ by (1.7). Note that the equality (4.2) is valid also in the case that $\xi(\mathbb{R})=\infty$ under the conditions (**C.1**) and (**C.2**).

By the bound (1.18) obtained from the conditions (**C.1**) and (**C.2**), we can prove the uniform integrability of the functions $\prod_{i=1}^{p}\phi_1(V_i(t_1))^{a_i}\mathbb{1}(|v_i|\le L)\det_{i,j\in\mathbb{I}_p}\left[\Phi_\xi^{v_i}(Z_j(T))\right]$, $L\in\mathbb{N}$, with respect to the measure $\xi^{\otimes p}(d\boldsymbol{v})\mathbf{P}_{\boldsymbol{v}}$. Then from (4.1) we conclude that

$$\begin{aligned}
\lim_{L\to\infty}\mathbb{E}_{\xi\cap[-L,L]}\left[F\big(\Xi(\cdot)\big)\right]&=\sum_{p=1}^{q}\sum_{\substack{a_i\in\mathbb{N},i\in\mathbb{I}_p:\\ \sum_{i=1}^p a_i=q}}\binom{q}{a_1,a_2,\dots,a_p}\\
&\times\int_{\mathbb{W}_p^{\mathrm{A}}}\xi^{\otimes p}(d\boldsymbol{v})\mathbf{E}_{\boldsymbol{v}}\left[\prod_{i=1}^{p}\phi_1(V_i(t_1))^{a_i}\det_{i,j\in\mathbb{I}_p}\left[\Phi_\xi^{v_i}(Z_j(T))\right]\right].
\end{aligned} \tag{4.3}$$

This is a realization of the RHS of (1.9), when $F(\sum_{i=1}^{\xi(\mathbb{R})}\delta_{V_i(\cdot)})=(\sum_{i=1}^{\xi(\mathbb{R})}\phi_1(V_i(t_1)))^q$. If q is finite, $q\in\mathbb{N}$, the sizes of matrices for the determinants in (1.9) can be reduced from $\xi(\mathbb{R})$ to p with $1\le p\le q$ as in (4.3). Then, even if $\xi(\mathbb{R})=\infty$, we needn't deal with infinite-dimensional determinants. Generalization for $k\ge 2$ is straightforward. □

5 Proof of Proposition 1.4

Suppose that the initial configuration $\xi \in \mathscr{M}_0$ satisfies the conditions (**C.1**) and (**C.2**) with constants C_0, C_1, C_2 and indices α, β. From the proof of Corollary 1.3 given in the previous section, we see that the LHS of (1.19) is given by

$$\sum_{p=1}^{4} \int_{\mathbb{W}_p^{\mathbb{A}}} \xi^{\otimes p}(d\boldsymbol{v}) \mathbf{E}_{\boldsymbol{v}} \left[F_p\left(\Big(\varphi(V_i(t)) - \varphi(V_i(s)) \Big)_{i \in \mathbb{I}_p} \right) \det_{1 \le i,j \le p} \Big[\Phi_\xi^{v_i}(Z_j(T)) \Big] \right]$$

with $F_1(x_1) = x_1^4$, $F_2(\boldsymbol{x}_2) = 6x_1^2 x_2^2 + 4x_1 x_2(x_1^2 + x_2^2)$, $F_3(\boldsymbol{x}_3) = 12 x_1 x_2 x_3 (x_1 + x_2 + x_3)$ and $F_4(\boldsymbol{x}_4) = 24 x_1 x_2 x_3 x_4$. Then Proposition 1.4 is concluded from the following estimate.

Lemma 5.1. *Let $\{a_i\}_{i=1}^p$ be a sequence of positive integers with length $p \in \mathbb{N}$. Then for any $T > 0$ and $\varphi \in C_0^\infty(\mathbb{R})$ there exists a positive constant $C = C(C_0, C_1, C_2, \alpha, \beta, T, \varphi)$, which is independent of s, t, such that*

$$\int_{\mathbb{W}_p^{\mathbb{A}}} \xi^{\otimes p}(d\boldsymbol{v}) \boldsymbol{E}_{\boldsymbol{v}} \left[\prod_{i=1}^{p} \Big(\varphi(V_i(t)) - \varphi(V_i(s)) \Big)^{a_i} \det_{1 \le i,j \le p} \Big[\Phi_\xi^{v_i}(Z_j(T)) \Big] \right]$$
$$\le C |t-s|^{\sum_{i=1}^p a_i/2}, \quad \forall s, t \in [0, T]. \tag{5.1}$$

Proof. Choose $L \in \mathbb{N}$ so that $\operatorname{supp} \varphi \subset [-L, L]$, and put $\mathbb{1}_L(x,y) = 0$, if $|x| > L$ and $|y| > L$, and $\mathbb{1}_L(x,y)$=1, otherwise. By the Schwartz inequality the LHS of (5.1) is bounded from the above by

$$\int_{\mathbb{W}_p^{\mathbb{A}}} \xi^{\otimes p}(d\boldsymbol{v}) \mathbf{E}_{\boldsymbol{v}} \left[\prod_{i=1}^{p} \Big(\varphi(V_i(t)) - \varphi(V_i(s)) \Big)^{2a_i} \right]^{1/2} \mathbf{E}_{\boldsymbol{v}} \left[\prod_{i=1}^{p} \mathbb{1}_L(V_i(s), V_i(t)) \right]^{1/4}$$
$$\times \mathbf{E}_{\boldsymbol{v}} \left[\left(\det_{1 \le i,j \le p} \Big[\Phi_\xi^{v_i}(Z_j(T)) \Big] \right)^4 \right]^{1/4}.$$

Since $V_i(t)$, $i \in \mathbb{N}$ are independent Brownian motions, $\mathbf{E}_{\boldsymbol{v}}[\prod_{i=1}^p (\varphi(V_i(t)) - \varphi(V_i(s)))^{2a_i}] \le c_1 |t-s|^{\sum_{i=1}^p a_i}$ and $\mathbf{E}_{\boldsymbol{v}}[\prod_{i=1}^p \mathbb{1}_L(V_i(s), V_i(t))] \le c_2 e^{-c_2' \sum_{i=1}^p |v_i|^2}, \forall s, t \in [0,T]$, with positive constants c_1, c_2, c_2', which are independent of s, t. And from the estimate (1.18) we have

$$\mathbf{E}_{\boldsymbol{v}} \left[\left(\det_{1 \le i,j \le p} \Big[\Phi_\xi^{v_i}(Z_j(T)) \Big] \right)^4 \right]^{1/4} \le c_3 \exp \left\{ c_3' \sum_{i=1}^p |v_i|^\theta \right\}$$

with positive constants c_3, c_3' and $\theta \in (\max\{\alpha, (2-\beta)\}, 2)$. Combining the above estimates with the fact that, for any $c, c' > 0$, $\int_{\mathbb{R}} \xi(dv) e^{c|v|^\theta - c'|v|^2} < \infty$, which is derived from the condition (**C.2**) (i) and the fact $\theta < 2$, we obtain the lemma. □

6 Proof of Proposition 1.6

Let $\tau = \inf\{t > 0 : \Xi(t) \notin \mathscr{M}_0\}$ and τ_{ij} be defined by (2.3). From the CBM representation (1.9), for any $\xi \in \mathscr{M}_0$ with $\xi(\mathbb{R}) \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}_\xi \Big[\tau \le T \Big] &\le \mathbf{E}_{\boldsymbol{u}} \left[\sum_{1 \le i < j \le \xi(\mathbb{R})} \mathbb{1}(\tau_{ij} \le T) \det_{1 \le i,j \le \xi(\mathbb{R})} \Big[\Phi_\xi^{u_i}(Z_j(T)) \Big] \right] \\ &= \int_{\mathbb{W}_2^{\mathbb{A}}} \xi^{\otimes 2}(d\boldsymbol{v}) \ \mathbf{E}_{\boldsymbol{v}} \left[\mathbb{1}(\tau_{12} \le T) \det_{1 \le i,j \le 2} \Big[\Phi_\xi^{v_i}(Z_j(T)) \Big] \right]. \end{aligned} \tag{6.1}$$

Since θ in (1.18) is strictly less than 2, and the constants and the indices in the conditions for the configurations $\xi \cap [-L, L]$ can be taken to be independent of $L > 0$,

$$\lim_{L\to\infty} \Phi^a_{\xi\cap[-L,L]}(Z_1(t)) = \Phi^a_\xi(Z_1(t)) \quad \text{in } L^k(\Omega, \mathbf{P}_v)$$

holds for any $k \in \mathbb{N}$. Hence, the inequality (6.1) holds for $\xi \in \mathscr{M}_0$ under the conditions (**C.1**) and (**C.2**). By the strong Markov property of CBM started at $\boldsymbol{v} \in \mathbb{W}_2^{\rm A}$

$$\begin{aligned}\mathbf{E}_{\boldsymbol{v}} &\left[\mathbb{1}(\tau_{12} \le T) \det_{1\le i,j\le 2} \left[\Phi^{v_i}_\xi (Z_j(T)) \right] \right] \\ &= \mathbf{E}_{\boldsymbol{v}} \left[\mathbb{1}(\tau_{12} \le T) \mathbf{E}_{\boldsymbol{Z}(\tau_{12})} \left[\det_{1\le i,j\le 2} \left[\Phi^{v_i}_\xi (Z_j(T - \tau_{12})) \right] \right] \right].\end{aligned}$$

By the martingale property of $\Phi^{v_i}_\xi(Z_j(T))$ we can apply the optional stopping theorem and show that the RHS of the above equation coincides with

$$\begin{aligned}&\mathbf{E}_{\boldsymbol{v}} \left[\mathbb{1}(\tau_{12} \le T) \det_{1\le i,j\le 2} \left[\Phi^{v_i}_\xi (Z_j(\tau_{12})) \right] \right] = \mathrm{E}_1 \left[\mathbb{1}(\tau_{12} \le T) \mathrm{E}_2 \left[\det_{1\le i,j\le 2} \left[\Phi^{v_i}_\xi (Z_j(\tau_{12})) \right] \right] \right] \\ &= \frac{\sqrt{-1}}{v_1 - v_2} \prod_{r\in \text{supp } \xi\setminus\{v_1,v_2\}} \frac{1}{(r - v_1)(r - v_2)} \\ &\quad \times \mathrm{E}_1 \left[\mathbb{1}(\tau_{12} \le T) \mathrm{E}_2 \left[\big(W_1(\tau_{12}) - W_2(\tau_{12})\big) \prod_{k=1,2} G\big(V_k(\tau_{12}), W_k(\tau_{12})\big) \right] \right],\end{aligned}$$

where $G(v, w) = \prod_{r\in\text{supp } \xi\setminus\{v_1,v_2\}} (r - v - \sqrt{-1}w)$, and the fact that $V_1(\tau_{12}) = V_2(\tau_{12})$ almost surely was used in the last equality. Since $W_k(\tau_{12}), k = 1, 2$ are i.i.d. under P_2, we have $\mathrm{E}_2 \left[\big(W_1(\tau_{12}) - W_2(\tau_{12})\big) \prod_{k=1,2} G\big(V_k(\tau_{12}), W_k(\tau_{12})\big) \right] = 0$. This completes the proof. □

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