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The probability law of the Brownian motion divided by its range

Florin Spinu*

Abstract

In the present paper we deduce explicit formulas for the probability laws of the quotients X_t/R_t and m_t/R_t, where X_t is the standard Brownian motion and m_t, M_t, R_t are its running minimum, maximum and range, respectively. The computation makes use of standard techniques from analytic number theory and the theory of the Hurwitz zeta function.

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1 Introduction

The connection between the Riemann zeta function and its allies (Jacobi theta function, Hurwitz zeta function) on the one hand, and the probability laws of various processes associated to the standard Brownian motion, on the other, is well established. (See [3] for a comprehensive survey.) In the present paper we add two new results to this theme.

Let X_t be the standard one-dimensional Brownian motion: this is a Wiener-Levy process with mean zero and covariance $\mathrm{cov}(X_s, X_t) = s \wedge t$. We use the following notations for the max, min, and range of X_t:

$$M_t = \max_{0\le s\le t} X_s, \quad m_t = -\min_{0\le s\le t} X_s, \quad R_t = M_t + m_t \ . \tag{1.1}$$

For a fixed $t > 0$, we define the following quotients:

$$\bar{X} = X_t/R_t, \quad Q = m_t/R_t \ . \tag{1.2}$$

The random variable $\bar{X}$ is bounded between -1 and 1 and, by the scaling property of the Brownian motion, its distribution is independent of t. The following theorem gives an explicit formula for its probability law.

Theorem 1.1. *The distribution of* $\bar{X}$ *is supported in the interval* $[-1, 1]$*, symmetric around zero and, for* $0 < v < 1$*,*

$$P(\bar{X} \le v) = \frac{1+v}{2} + \frac{v^2(1-v)}{2} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-v)^2} - \frac{1}{(2n+v)^2} \right] \ . \tag{1.3}$$

*OMERS Capital Markets, 200 Bay St., Toronto, Canada. E-mail: fspinu@gmail.com

The probability law of Q was given in [5, eq. (2.5)]. We state it here as well and provide a new proof, which is similar to that of Theorem 1.1.

Theorem 1.2. *[5, Csáki] The distribution of Q is supported in the interval $[0,1]$, symmetric about $1/2$ and, for $0 < v < 1$,*

$$\begin{aligned} P(Q \le v) &= v(1-v)\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n-v}+\frac{1}{n+v}\right) \\ &= 1-v+\frac{1}{2}v(1-v)\left(\psi(\frac{v}{2})+\psi(1-\frac{v}{2})-\psi(\frac{1-v}{2})-\psi(\frac{1+v}{2})\right) \ , \end{aligned} \tag{1.4}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

2 Proof of Theorem 1.1

2.1 An identity of Feller

It suffices to consider the case $\bar{X} = X_1/R_1$. Let $w(x,y,z)$ be the density of the event $\{X_1 = x, m_1 \le y, M_1 \le z\}$. Its explicit expression is given in [6] (as well as [4, 1.15.8]):

$$w(x,y,z) = \sum_{k=-\infty}^{\infty} \phi\left(2ky+2kz-x\right) - \phi\left(2ky+2(k-1)z+x\right) \ , \tag{2.1}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$ is the probability density function of the standard normal distribution. From this, we can compute, when $x > 0$,

$$\frac{P(R_1 \le r, X_1 \in dx)}{dx} = \int_x^{r-x} dz \int_0^{r-z} \frac{\partial^2 w}{\partial y \partial z}(x,y,z)\, dy = \int_x^{r-x} \frac{\partial w}{\partial z}(x, r-z, z)\, dx \ ,$$

since $w(x,0,z) \equiv 0$. To evaluate this integral, we differentiate term-by-term (2.1)

$$\frac{\partial w}{\partial z}(x,r-z,z) = \sum_{k=-\infty}^{\infty} \left[2k\phi'(2kr-x) - 2(k-1)\phi'(2kr-2z+x)\right] \ ,$$

and then integrate from $z = x$ to $z = r - x$ to obtain (cf. [4, 1.15.8(2)])

$$P(R_1 \le r, X_1 \in dx) = \sum_{k=-\infty}^{\infty} [(2k+1) - 2k(r-x)(2kr+x)]\phi(2kr+x)\cdot dx, \quad x \le r \ .$$

We now turn to the quantity $\bar{X} = X_1/R_1$. For a fixed $v \in (0,1)$ we have, by definition,

$$P(\bar{X} > v) = P(R_1 \le X_1/v) = \int_0^{\infty} P(R_1 < x/v, X_1 \in dx) = \int_0^{\infty} f(x,v)\, dx \ , \tag{2.2}$$

where

$$f(x,v) := \sum_{k=-\infty}^{\infty} \left[(2k+1) - 2kx^2(1/v-1)(2k/v+1)\right]\phi((2k/v+1)x) \ . \tag{2.3}$$

2.2 The Mellin Transform

Our strategy for computing $\int_0^{\infty} f(x,v)$ (eq. 2.2) relies on the observation that, although we cannot integrate the series (2.3) term by term from 0 to ∞, we can integrate it against x^s, when s is a complex number with real part $\Re(s) > 2$. In other words, we consider the Mellin transform

$$M(s) := \int_0^{\infty} f(x,v)\, x^s\, \frac{dx}{x} \ , \quad \Re(s) > 2 \ . \tag{2.4}$$

We prove in the Appendix (Proposition 4.2) that as a function of x, $f(x,v)$ is smooth and rapidly decreasing in x at both ends of the interval $(0,\infty)$. This implies that $M(s)$ is defined everywhere as an entire function in the complex argument $s\in\mathbb{C}$. We then go through the following steps:

- *Step 1.* Express $M(s)$ in terms of well-known Dirichlet series when $\Re(s)>2$.
- *Step 2.* Identify $\int_0^\infty f(x,v)\,dx = M(1)$ by analytic continuation.

Step 1. When $\Re(s)>2$, we use (2.3) to integrate term by term in (2.4) and obtain, through a change of variable,

$$\begin{aligned} M(s) &= \sum_{k=-\infty}^{\infty}(2k+1)\int_0^\infty \phi((2k/v+1)x)\,x^s\,\frac{dx}{x} \\ &\quad -(1/v-1)\sum_{k=-\infty}^{\infty}2k(2k/v+1)\int_0^\infty \phi\big((2k/v+1)x\big)\,x^{s+2}\,\frac{dx}{x} \\ &= v^s M_\phi(s)\sum_{k=-\infty}^{\infty}\frac{2k+1}{|2k+v|^s}+v^s(v-1)M_\phi(s+2)\sum_{k=-\infty}^{\infty}\frac{2k(2k+v)}{|2k+v|^{s+2}}\ , \end{aligned} \tag{2.5}$$

where $M_\phi(s):=\int_0^\infty \phi(x)\,x^s\,\frac{dx}{x}$ is the Mellin transform of ϕ. This can be computed explicitly: $M_\phi(s)=\frac{1}{2\sqrt{2\pi}}2^{s/2}\Gamma(s/2)$, but all we need is that $M_\phi(s+2)=sM_\phi(s)$ and $M_\phi(1)=1/2$. To simplify the right-hand side of (2.5), we introduce the following Dirichlet series (cf. [9, eq. 2.4] where a similar notation is used):

$$D^+(s,v):=\sum_{k=-\infty}^{\infty}\frac{1}{|2k+v|^s},\quad D^-(s,v):=\sum_{k=-\infty}^{\infty}\frac{2k+v}{|2k+v|^{s+1}},\quad \Re(s)>1\ . \tag{2.6}$$

The following manipulation of the main term of the right-hand side of (2.5)

$$\frac{2k+1}{|2k+v|^s}=\frac{2k+v}{|2k+v|^s}+\frac{1-v}{|2k+v|^s},\quad \frac{2k(2k+v)}{|2k+v|^{s+2}}=\frac{1}{|2k+v|^s}-\frac{v(2k+v)}{|2k+v|^{s+2}}\ ,$$

allows us to express $M(s)$ in terms of $D^\pm(s,v)$ as follows:

$$\begin{aligned} M(s) &= v^s M_\phi(s)\,D^-(s-1,v)+v^s(v-1)(s-1)M_\phi(s)\,D^+(s,v) \\ &\quad - sv^{s+1}(v-1)M_\phi(s)\,D^-(s+1,v),\quad \Re(s)>2\ . \end{aligned} \tag{2.7}$$

At this point we introduce the Hurwitz zeta function

$$\zeta(s,a):=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\quad \Re(s)>1,\quad a\in(0,1)\ . \tag{2.8}$$

It was discovered by Hurwitz that, as a function of s, $\zeta(s,a)$ can be analytically continued to the entire complex plane, with only a simple pole at $s=1$. Moreover, it is known that [1]:

$$\lim_{s\to1}\left[\zeta(s,a)-\frac{1}{s-1}\right]=-\psi(a),\quad \zeta(0,a)=\frac12-a\ . \tag{2.9}$$

As an immediate consequence, it follows that both

$$D^\pm(s,v)=2^{-s}\left\{\zeta\left(s,\frac{v}{2}\right)\pm\zeta\left(s,1-\frac{v}{2}\right)\right\}$$

have meromorphic continuation to $s\in\mathbb{C}$. Moreover, we deduce from (2.9) that

$$\lim_{s\to1}\left[D^+(s,v)-\frac{1}{s-1}\right]=\log(2)-\frac12\left(\psi(\frac{v}{2})+\psi(1-\frac{v}{2})\right) \tag{2.10}$$

$$D^-(1,v)=\tfrac12\left(\psi(1-\tfrac{v}{2})-\psi(\tfrac{v}{2})\right) \tag{2.11}$$

$$D^+(0,v)=0,\quad D^-(0,v)=1-v \tag{2.12}$$

Step 2. We now let $s \to 1$ in the identity (2.7): we use (2.10) and (2.12) and $M_\phi(1) = 1/2$ to obtain

$$\begin{aligned} M(1) &= \frac{1}{2}v(v-1) + \frac{1}{2}v^2(1-v)D^-(2,v) + \frac{1}{2}vD^-(0,v) \\ &= \frac{1}{2}v^2(1-v)D^-(2,v) \ . \end{aligned} \tag{2.13}$$

Since $s = 2$ is in the domain of convergence of the $D^-(s,v)$,

$$D^-(2,v) = \sum_{k=-\infty}^{\infty} \frac{2k+v}{|2k+v|^3} = \frac{1}{v^2} + \sum_{k=1}^{\infty}\left[\frac{1}{(2k+v)^2} - \frac{1}{(2k-v)^2}\right] \ . \tag{2.14}$$

We conclude that $P(\bar{X} > v) = \int_0^\infty f(x,v)dx = M(1)$, therefore

$$\begin{aligned} P(\bar{X} < v) &= 1 - M(1) = 1 - \frac{1}{2}v^2(1-v)D^-(2,v) \\ &= 1 - \frac{1-v}{2} - \frac{1}{2}v^2(1-v)\sum_{k=1}^{\infty}\left[\frac{1}{(2k+v)^2} - \frac{1}{(2k-v)^2}\right], \end{aligned} \tag{2.15}$$

and this finishes the proof of Theorem 1.1.

2.3 Moments

Let $p_{\bar{X}}(v) = \frac{d}{dv}P(\bar{X} < v)$ be the probability density function of $\bar{X}$. It is clear from the above identity that

$$p_{\bar{X}}(v) = \frac{d}{dv}\left(\frac{1}{2}v^2(v-1)D^-(2,v)\right) \ . \tag{2.16}$$

The numerical calculation of $p_{\bar{X}}(v)$ is given in the Appendix (section 4.2). We can integrate that expression numerically against test functions to obtain (the computations were done in *Matlab*)

$$E[|\bar{X}|] \approx 0.4621, \quad E[\bar{X}^2] \approx 0.2813, \quad E[\bar{X}^4] \approx 0.1418 \ . \tag{2.17}$$

2.4 The Taylor expansion at $v = 0$

By differentiating $(n+1)$ times the identity (1.3), we obtain all the higher order derivatives of $p_{\bar{X}}(v)$ at $v = 0$:

$$p_{\bar{X}}^{(n)}(0) = \begin{cases} 1/2, & n = 0 \\ 0, & n = 1 \\ -(n+1)!\,(n-1)2^{-n}\,\zeta(n), & n \geq 3, \text{ odd} \\ (n+1)!\,n\,2^{-n-1}\,\zeta(n+1), & n \geq 2, \text{ even} \end{cases} \tag{2.18}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. In particular, $p''_{\bar{X}}(0) = \frac{3}{2}\zeta(3) \approx 1.8031$. This indicates that $v = 0$ is a local minimum for $p_{\bar{X}}(v)$, which explains the bimodality illustrated in Fig. 1a. The modes of the distribution of $\bar{X}$ occur near ± 0.554, but it seems difficult to determine them explicitly.

3 Proof of Theorem 1.2

Let $F(y,z) := P(m_1 \leq y, M_1 \leq z)$, the joint distribution function of m_1 and M_1. An explicit expression can be obtained by integrating term-by-term the series (2.1) (cf. [6] and [4, 1.15.4]):

$$F(y,z) = \int_{-y}^{z} w(x,y,z)\,dx = 2\sum_{k=-\infty}^{\infty}(-1)^k\Phi((k+1)y + kz) \ , \tag{3.1}$$

where $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$ is the cumulative distribution function of the standard normal distribution. For a fixed $v \in (0,1)$, $P(Q \le v) = P(\frac{m_1}{m_1+M_1} \le v) = P(m_1 \le \lambda M_1)$, where $\lambda = \frac{v}{1-v}$. Therefore

$$P(Q \le v) = \int_0^\infty dz \int_0^{\lambda z} F''_{yz}(y,z)\,dy = \int_0^\infty F'_z(\lambda z, z)\,dz\ , \tag{3.2}$$

and $F'_z(\lambda z, z)$ can be obtained by differentiating (3.1):

$$F'_z(\lambda z, z) = 2 \sum_{k=-\infty}^{\infty} (-1)^k k\, \phi\big(\frac{k+v}{1-v} z\big)\ . \tag{3.3}$$

A similar analysis applies as in the case of $f(x,v)$: we prove in the Appendix (Proposition 4.2) that $F'_z(\lambda z, z)$ is smooth and rapidly decreasing at both ends of the interval $(0,\infty)$, and we identify $P(Q \le v)$ as a special value of the Mellin transform

$$P(Q \le v) = \int_0^\infty F'_z(\lambda z, z) dz = H(1)\ ,$$

where $H(s) := \int_0^\infty F'_z(\lambda z, z)\, z^s\, \frac{dz}{z}$ is an entire function of s. On the other hand, we can integrate (3.3) term-by-term against z^s, when $\Re(s) > 2$, to obtain

$$H(s) = 2(1-v)^s M_\phi(s) \sum_{k=-\infty}^{\infty} \frac{(-1)^k k}{|k+v|^s}, \quad \Re(s) > 2\ . \tag{3.4}$$

The inner sum can be easily identified as

$$D^-(s-1,v) + D^-(s-1,1-v) + v(D^+(s,1-v) - D^+(s-1,v))\ .$$

Finally, we can use (2.10) and the identity $M_\phi(1) = 1/2$ to compute $H(1)$ and thus arrive at the second identity of (1.4). The equivalence of the two separate expressions of (1.4) follows from the identity $\psi(x) = -\frac{1}{x} - \gamma - \sum_{n=1}^{\infty}(\frac{1}{n+x} - \frac{1}{n})$ (cf. [2, 6.3.16]).

3.1 The behavior near $v = 0$

Let $p_Q(v) = \frac{d}{dv} P(Q \le v)$ the density function of Q. The asymptotic expansion $\psi(z) = -\frac{1}{z} - \gamma + O(z), z \to 0$, where γ is Euler's constant, implies that

$$P(Q \le v) = \frac{1}{2} v\,[-\gamma + \psi(1) - 2\psi(\frac{1}{2})] + O(v^2) = (2\log 2)\, v + O(v^2), \quad v \to 0+$$

hence $p_Q(0) = 2\log(2) \approx 1.3863$. (See [2, 6.3.3, 6.3.14] for the relevant identities.)

3.2 Moments

The symmetry around $1/2$ implies that $E[Q] = 1/2$. The higher moments can be approximated by integrating numerically $p_Q(t)$ against test functions (the computations were done in *Matlab*):

$$E(Q^2) \approx 0.3453, \quad E(Q^3) \approx 0.2679, \quad E(Q^4) \approx 0.2205\ . \tag{3.5}$$

4 Appendix

4.1 Theta functions

The main ingredients of the proof of Proposition 4.2 are two theta functions that are special cases of the classical Jacobi theta functions [8, Chap. 10]. In what follows we define them and derive their main properties. For $v \in (0,1)$, $p \in \{0,1\}$ and $x > 0$, let

$$\vartheta_p(x,v) := \sum_{k=-\infty}^{\infty} (k+v)^p e^{-\pi(k+v)^2 x}, \quad \eta_p(x,v) := \sum_{k=-\infty}^{\infty} k^p e^{2\pi i k v} e^{-\pi k^2 x} . \tag{4.1}$$

By definition, these functions are smooth in $x > 0$, and

$$\vartheta_p(x,v) = O(e^{-cx}), \quad \eta_1(x,v) = O(e^{-cx}), \quad \eta_0(x,v) = 1 + O(e^{-cx}) \ , x \to +\infty \tag{4.2}$$

(for any $c < \pi$). The Poisson summation formula [8, eq. 35.41] applied to the function $t \mapsto (t+v)^p e^{-\pi(t+v)^2 x}$ yields the following functional equations

$$\vartheta_0(x,v) = x^{-1/2}\eta_0(1/x,v), \quad \vartheta_1(x,v) = -ix^{-3/2}\eta_1(1/x,v) . \tag{4.3}$$

As a consequence of these identities we obtain the behavior near 0:

$$\vartheta_0(x,v) = x^{-1/2} + O(e^{-c/x}), \quad \vartheta_1(x,v) = O(e^{-c/x}), \quad x \to 0+ \ . \tag{4.4}$$

We now turn to the analysis of the functions $f(x,v)$ and $F_z'(az,z)$, as defined in (2.3) and (3.3). For simplicity of notation, we define

$$g(x,v) := \sum_{k=-\infty}^{\infty} [(2k+1) - 2\pi x(1-2v)k(k+v)] e^{-\pi(k+v)^2 x} , \tag{4.5}$$

so that $f(x,v) = \frac{1}{\sqrt{2\pi}} g(\frac{2x^2}{\pi v^2}, \frac{v}{2})$. The behavior of f as $x \to 0+$ is deduced from that of g.

Lemma 4.1. *Let $\lambda = \frac{v}{1-v}$ and $x := (\frac{2}{\pi})^{1/2}(\lambda+1)z$. The following two identities hold:*

$$g(x,v) = (1-2v)\,\vartheta_0(x,v) + 2(1-2v)\,x\,\frac{\partial\vartheta_0}{\partial x}(x,v) + 2[1 + \pi\, v(1-2v)\,x]\,\vartheta_1(x,v) . \tag{4.6}$$

$$(\frac{\pi}{8})^{1/2} F_z'(\lambda z, z) = \vartheta_1(x, \frac{v}{2}) - \frac{v}{2}\vartheta_0(x,\frac{v}{2}) + \vartheta_1(x, \frac{1-v}{2}) + \frac{v}{2}\vartheta_0(x, \frac{1-v}{2}) . \tag{4.7}$$

Proof. By definition, $g(x,v)$ is a linear combination of the series $\sum k^j e^{-\pi(k+v)^2 x}$, with $j = 0,1,2$. The series corresponding to $j=0$ and $j=1$ are in the linear span of $\vartheta_0(x,v)$ and $\vartheta_1(x,v)$. As for $j=2$, term by term differentiation of (4.1) yields

$$\frac{\partial\vartheta_0}{\partial x}(x,v) = -\pi \sum_{k=-\infty}^{\infty} (k+v)^2 e^{-\pi(k+v)^2 x},$$

hence the sum corresponding to $j=2$ can be written as

$$\sum_{k=-\infty}^{\infty} k^2 e^{-\pi(k+v)^2 x} = -\frac{1}{\pi}\frac{\partial\vartheta_0}{\partial x}(x,v) - v^2\vartheta_0(x,v) - 2v\,\vartheta_1(x,v) .$$

The exact identity (4.6) results from careful bookkeeping. The proof of (4.7) is similar. □

Proposition 4.2. *As a function of x, $f(x,v)$ is smooth and rapidly decreasing as $x \to 0+$ and $x \to +\infty$. The same holds true for $F_z'(\lambda z, z)$ as a function of $z > 0$.*

Proof. In the case of $f(x,v)$, it is enough to prove the same statement for $g(x,v)$, when $x \to 0+$. We differentiate the first identity from (4.3)

$$\frac{\partial\vartheta_0}{\partial x}(x,v) = -(1/2)\,x^{-3/2}\eta_0(1/x,v) + x^{-5/2}\frac{\partial\eta_0}{\partial x}(1/x,v) ,$$

hence $\frac{\partial\vartheta_0}{\partial x}(x,v) = -(1/2)x^{-3/2} + O(e^{-c/x})$, as $x \to 0+$ (with $c < \pi$). We use this estimate and (4.4) to derive, from (4.6),

$$g(x,v) = (1-2v)[x^{-\frac{1}{2}} + O(e^{-c/x})] + 2(1-2v)\,x\,[-(1/2)x^{-\frac{3}{2}} + O(e^{-c/x})] + O(e^{-c/x}) ,$$

as $x \to 0+$. The leading terms cancel out conveniently and we are left with $O(e^{-c/x})$. Similarly, the proof for $F_z'(\lambda z, z)$ follows from (4.7) and (4.4). □

4.2 The numerical computation of $p_{\bar{X}}(v)$

In this section we obtain an alternative expression for $P(\bar{X} < v)$ that is more convenient for numerical computations than the slowly convergent series (1.3). To do so, we evaluate $D^-(2, v)$ with the aid of the Mellin transform

$$\mathcal{M}(\vartheta_1; s) := \int_0^\infty \vartheta_1(x, v) x^s \, \frac{dx}{x}, \quad \Re(s) > 1 \; . \tag{4.8}$$

On the one hand, we can integrate (4.1) term-by-term against x^s

$$\mathcal{M}(\vartheta_1; s) = 2^{2s-1} \pi^{-s} \Gamma(s) D^-(2s-1, 2v), \quad \Re(s) > 1 \; . \tag{4.9}$$

On the other hand, the functional equation (4.3) allows us to write

$$\begin{aligned} \mathcal{M}(\vartheta_1; s) &= \int_0^1 \vartheta_1(x, s)\, x^s \, \frac{dx}{x} + \int_1^\infty \vartheta_1(x, s)\, x^s \, \frac{dx}{x} \\ &= (-i) \int_1^\infty \eta_1(x, s)\, x^{3/2-s} \, \frac{dx}{x} + \int_1^\infty \vartheta_1(x, s)\, x^s \, \frac{dx}{x} \; , \quad s \in \mathbb{C}, \end{aligned}$$

since both ϑ_1 and η_1 are rapidly decreasing at ∞. The series (4.1) converge uniformly in $x > 1$, so we can integrate the above term-by-term to obtain

$$\begin{aligned} \mathcal{M}(\vartheta_1; s) = \pi^{-s} &\sum_{k=-\infty}^{\infty} \frac{(k+v)\Gamma(s, \pi(k+v)^2)}{|k+v|^{2s}} \\ &+ 2\pi^{s-3/2} \sum_{k=1}^{\infty} k^{2s-2} \sin(2\pi k v) \Gamma(3/2 - s, \pi k^2), \quad s \in \mathbb{C}, \end{aligned} \tag{4.10}$$

where $\Gamma(s, x) := \int_x^\infty e^{-t} t^s \frac{dt}{t}$ is the incomplete gamma function [2, 6.5.3]. Combining (4.9) and (4.10) when $s = 3/2$, we obtain for $v \neq 0$ (after replacing $2v$ by v),

$$D^-(2, v) = \frac{2}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}(k) \Gamma(\frac{3}{2}, \frac{\pi}{4}(2k+v)^2)}{(2k+v)^2} + \pi \sum_{k=1}^{\infty} k \Gamma(0, \pi k^2) \sin(\pi k v) \; . \tag{4.11}$$

(Here $\operatorname{sgn}(k) = 1$ if $k \geq 0$, and $\operatorname{sgn}(k) = -1$ otherwise.) Both series on the right-hand side are rapidly convergent, since $\Gamma(3/2, x)$ and $\Gamma(0, x)$ have exponential decay at ∞. The derivative of $D^-(2, v)$ can also be computed:

$$\begin{aligned} \frac{\partial}{\partial v} D^-(2, v) = &-\frac{4}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \frac{\operatorname{sgn}(k) \Gamma(3/2, \frac{\pi}{4}(2k+v)^2)}{(2k+v)^3} - \frac{\pi}{2} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{4}(2k+v)^2} \\ &+ \pi^2 \sum_{k=1}^{\infty} k^2 \Gamma(0, \pi k^2) \cos(\pi k\, v) \; , \quad v \neq 0 \; . \end{aligned} \tag{4.12}$$

The last two formulas allow us to evaluate numerically $p_{\bar{X}}(v)$ using (2.16), which we re-write as

$$p_{\bar{X}}(v) = v(3v/2 - 1) D^-(2, v) + \frac{1}{2} v^2 (v-1) \frac{\partial}{\partial v} D^-(2, v) \; .$$

In *Matlab* notation, $\Gamma(0, x) = expint(x)$ and $\frac{2}{\sqrt{\pi}} \Gamma(3/2, x) = gammainc(x, 3/2,$ 'upper'$)$. By retaining, on the right-hand side of (4.11), and (4.12), only the terms corresponding to $|k| \leq 2$, in the exponential series, and $k = 1$, in the trigonometric series, we obtain an approximation of $p_{\bar{X}}(t)$ within 10^{-6}, uniformly in the interval $[-1, 1]$.

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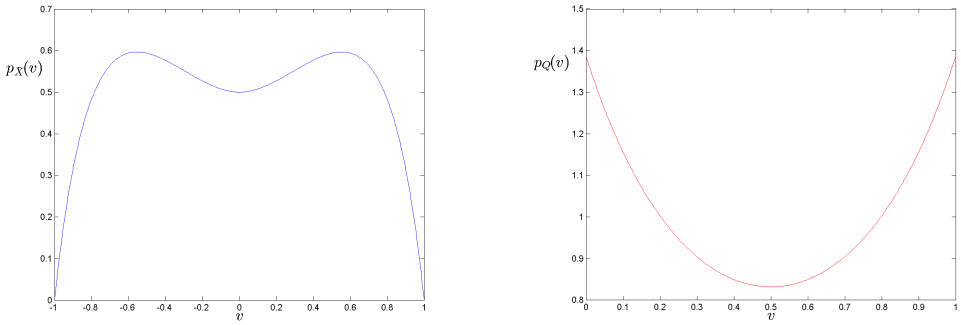

(a) The distribution of $\bar{X}$.

(b) The distribution of Q.

Figure 1

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