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A note on Kesten's Choquet-Deny lemma*

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Abstract

Let $d > 1$ and $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random $d\times d$ matrices with nonnegative entries. This induces a Markov chain $M_n = \mathbf{A}_n M_{n-1}$ on the cone $\mathbb{R}^d_\geq \setminus \{0\} = \mathbb{S}_\geq \times \mathbb{R}_>$. We study harmonic functions of this Markov chain. In particular, it is shown that all bounded harmonic functions in $\mathcal{C}_b(\mathbb{S}_\geq) \otimes \mathcal{C}_b(\mathbb{R}_>)$ are constant. The idea of the proof is originally due to Kesten [*Renewal theory for functionals of a Markov chain with general state space.* Ann. Prob. 2 (1974), 355 – 386], but is considerably shortened here. A similar result for invertible matrices is given as well.

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1 Introduction

Let $d > 1$. Write $\mathbb{R}^d_\geq = [0,\infty)^d$ for the cone of d-vectors with nonnegative entries and $\mathbb{S}_\geq := \{x \in \mathbb{R}^d_\geq \ : \ |x| = 1\}$ for its intersection with the unit sphere $\mathbb{S}$, where $|\cdot|$ is the euclidean norm on $\mathbb{R}^d$. A matrix $\mathbf{a} \in \mathcal{M}_+ := M(d\times d, \mathbb{R}_\geq)$ is called *allowable*, if it has no zero line or column. Any allowable matrix leaves $V := \mathbb{R}^d_\geq \setminus \{0\}$ invariant and one can define its action on $\mathbb{S}_\geq$ by

$$\mathbf{a}\cdot x := \frac{\mathbf{a}x}{|\mathbf{a}x|}, \qquad x \in \mathbb{S}_\geq.$$

If μ is a probability distribution on allowable matrices in $\mathcal{M}_+$ then V is μ-a.s. invariant. Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed (iid) random matrices with law μ, then $M_n = \mathbf{A}_n M_{n-1}$ defines a Markov chain on V.

The aim of this note is to study the bounded harmonic functions of $(M_n)_{n\in\mathbb{N}_0}$ under some additional equicontinuity condition on the functions. Besides being of interest in its own right, the absence of nontrivial bounded harmonic functions appears prominently in the proof of Kesten's renewal theorem [8] and has been recently used in [10] to determine the set of fixed points of the multivariate distributional equation associated with the random matrices $(\mathbf{T}_1,\dots,\mathbf{T}_N) \in \mathcal{M}_+^N$,

$$Y \stackrel{d}{=} \mathbf{T}_1 Y_1 + \dots \mathbf{T}_N Y_N, \tag{1.1}$$

where $N \geq 2$ is fixed, $Y, Y_1, \dots, Y_N$ are iid $\mathbb{R}^d_\geq$-valued random variables and independent of $(\mathbf{T}_1,\dots,\mathbf{T}_N)$.

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The idea of proof is based on the Choquet-Deny lemma used by Kesten in the proof of his renewal theorem [8, Lemma 1]. By restricting to the more specialized setting of Markov chains generated by the action of nonnegative matrices and using recent results on products of random matrices from Buraczewski, Damek and Guivarch [3], this proof can be considerably shortened and, as we hope, thereby made more illuminating.

2 Statement of Results

The Markov chain $(M_n)_{n\in\mathbb{N}_0}$ will be studied in the decomposition $M_n = e^{S_n}X_n$ for $X_n \in \mathbb{S}_\geq$ and $S_n \in \mathbb{R}$. Note that up to an exponential transform, this corresponds to the decomposition $V = \mathbb{S}_\geq \times \mathbb{R}_>$, where $\mathbb{R}_> = (0,\infty)$. One easily deduces that

$$X_n = \mathbf{A}_n \cdot X_{n-1}, \qquad S_n - S_{n-1} = \log|\mathbf{A}_n X_{n-1}|,$$

hence $(X_n, S_n)_{n\in\mathbb{N}_0}$ is a Markov chain on $\mathbb{S}_\geq \times \mathbb{R}$ that carries the additional structure of a Markov random walk.

Writing $\mathrm{int}(A)$ for the topological interior of a set A, recall that by the Perron-Frobenius Theorem, any $\mathbf{a} \in \mathrm{int}(\mathcal{M}_+)$ possesses a unique largest eigenvalue $\lambda_{\mathbf{a}} \in \mathbb{R}_>$ with corresponding normalized eigenvector $w_{\mathbf{a}} \in \mathrm{int}(\mathbb{S}_\geq)$.

Definition 2.1. *A subsemigroup $\Gamma \subset \mathcal{M}_+$ is said to satisfy condition (C), if*

1. *every $\mathbf{a} \in \Gamma$ is allowable*
2. *no subspace $W \subsetneq \mathbb{R}^d$ with $W \cap \mathbb{R}^d_\geq \neq \{0\}$ satisfies $\Gamma W \subset W$ and*
3. *$\Gamma \cap \mathrm{int}(\mathcal{M}_+) \neq \emptyset$.*

Denote by $[\mathrm{supp}\,\mu]$ the smallest closed semigroup of $\mathcal{M}_+$ generated by $\mathrm{supp}\,\mu$ and write $\mathcal{C}_b(E)$ for the set of bounded continuous functions on the space E. Abbreviating $\mathbf{\Pi}_n = \mathbf{A}_n \dots \mathbf{A}_1$, define for each $x \in \mathbb{S}_\geq$ a probability measure $\mathbb{P}_x$ on the path space of $(X_n, S_n)_{n\in\mathbb{N}_0}$ by

$$\mathbb{P}_x\Big((X_0,S_0,\dots,X_n,S_n) \in B\Big) = \mathbb{P}\Big((x,0,\dots,\mathbf{\Pi}_n \cdot x, \log|\mathbf{\Pi}_n x|) \in B\Big)$$

for all $n \in \mathbb{N}$ and measurable B. The corresponding expectation symbol is denoted by $\mathbb{E}_x$.

Theorem 2.2. *Let $[\mathrm{supp}\,\mu]$ satisfy (C). Assume that $L \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ satisfies*

(a) $L(x,s) = \mathbb{E}_x\, L(X_1, s - S_1)$ for all $(x,s) \in \mathbb{S}_\geq \times \mathbb{R}$, and

(b) for all $z \in \mathrm{int}(\mathbb{S}_\geq)$,

$$\lim_{y\to z} \sup_{t\in\mathbb{R}} |L(y,t) - L(z,t)| = 0.$$

Then L is constant.

It is interesting to observe that (b) is an equicontinuity property for the family $(L(\cdot,t))_{t\in\mathbb{R}}$ of functions in $\mathbb{C}_b(\mathbb{S}_\geq)$. In fact, the Arzelà-Ascoli theorem is applicable and yields that for all $t \in \mathbb{R}$

$$\lim_{s\to t} \sup_{y\in\mathbb{R}} |L(y,s) - L(y,t)| = 0.$$

Each pair of functions $f \in \mathcal{C}_b(\mathbb{S}_\geq)$, $h \in \mathcal{C}_b(\mathbb{R})$ defines a composite function $f \otimes h \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ by $(f\otimes h)(u,s) := f(u)h(s)$. Write $\mathcal{C}_b(\mathbb{S}_\geq) \otimes \mathcal{C}_b(\mathbb{R})$ for the set of all finite linear combinations of such functions (tensor product). Then the following corollary is obvious:

Corollary 2.3. *Let $[\operatorname{supp}\mu]$ satisfy condition (C). If $L \in \mathcal{C}_b(\mathbb{S}_\geq) \otimes \mathcal{C}_b(\mathbb{R})$ is harmonic for the Markov chain $(X_n, S_n)_{n\in\mathbb{N}_0}$, then L is constant.*

The further organisation of the paper is as follows. At first, we repeat for the readers convenience important implications of (C), based on [3]. Then we turn to the proof of the main theorem. It will be assumed throughout that $d > 1$ and that $[\operatorname{supp}\mu]$ satisfies (C). In Section 5 we provide some examples for which condition (C) is satisfied and discuss its role in the proof of Kesten's renewal theorem. Finally, we describe briefly how to extend the result to Markov chains on $\mathbb{R}^d$ generated by the action of invertible matrices.

3 Implications of Condition (C)

Under each $\mathbb{P}_x$, $x \in \mathbb{S}_\geq$, $(X_n)_{n\in\mathbb{N}_0}$ constitutes a Markov chain with transition operator $P : \mathcal{C}_b(\mathbb{S}_\geq) \to \mathcal{C}_b(\mathbb{S}_\geq)$ defined by

$$Pf(y) = \int f(\mathbf{a}\cdot y)\,\mu(d\mathbf{a}) = \mathbb{E}f(\mathbf{A}_1 \cdot y), \qquad y \in \mathbb{S}_\geq.$$

Abbreviating $\Gamma = [\operatorname{supp}\mu]$, write

$$W(\Gamma) = \overline{\{w_{\mathbf{a}} \,:\, \mathbf{a} \in \Gamma \cap \operatorname{int}(\mathcal{M}_+)\}}$$

for the closure of the set of normalized Perron-Frobenius eigenvectors, and

$$\Lambda(\Gamma) = \{\log\lambda_{\mathbf{a}} \,:\, \mathbf{a} \in \Gamma \cap \operatorname{int}(\mathcal{M}_+)\}$$

for the logarithms of the corresponding Perron-Frobenius eigenvalues.

Proposition 3.1 ([3, Propositions 3.1 & 3.2])**.** *The set $\Lambda(\Gamma)$ generates a dense subgroup of $\mathbb{R}$. There is a unique P- stationary probability measure ν on $\mathbb{S}_\geq$, and $\operatorname{supp}\nu = W(\Gamma)$.*

Since $\mathbb{S}_\geq$ is compact, the uniqueness of ν implies the following ergodic theorem (see [1])

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(X_k) = \int f(y)\,\nu(dy) \qquad \mathbb{P}_x\text{-a.s.} \tag{3.1}$$

for all $x \in \mathbb{S}_\geq$, $f \in \mathcal{C}_b(\mathbb{S}_\geq)$.

Proposition 3.1 also implies the following "weak" aperiodicity property of $(S_n)_{n\in\mathbb{N}_0}$, which is an adaption of condition $I.3$ in [8]. As usual, $B_\varepsilon(z) := \{y \in E : |y - z| < \varepsilon\}$ for $\varepsilon > 0$, $z \in E$.

Lemma 3.2. *There exists a sequence $(\zeta_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ such that the group generated by $(\zeta_i)_{i\in\mathbb{N}}$ is dense in $\mathbb{R}$ and such that for each ζ_i there exists $z \in \operatorname{int}(\mathbb{S}_\geq)$ with the following properties:*

1. *$\nu\big(B_\varepsilon(z)\big) > 0$ for all $\varepsilon > 0$.*
2. *For all $\delta > 0$ there is $\varepsilon_\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_\delta)$ there are $m \in \mathbb{N}$ and $\eta > 0$, such that for $B := B_\varepsilon(z)$:*

$$\mathbb{P}_x\left(X_m \in B,\ |S_m - \zeta_i| < \delta\right) \geq \eta \quad \textit{for all } x \in B. \tag{3.2}$$

The first property together with (3.1) entails that B is a recurrent set for $(X_n)_{n\in\mathbb{N}}$. By a geometric trials argument (see e.g. [2, Problem 5.10]), it follows that for all $\delta > 0$ and sufficiently small $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\mathbb{P}_x\left(|X_n - z| < \varepsilon, |X_{n+m} - z| < \varepsilon, |S_n - (S_{n+m} - \zeta_i)| < \delta \text{ i.o. }\right) = 1 \tag{3.3}$$

We repeat the short proof of Lemma 3.2 from [3, Prop. 5.5], for it clarifies the importance of Proposition 3.1 and moreover, we want to strengthen the result a bit.

Proof. By Prop. 3.1, the set $\Lambda(\Gamma)$ generates a dense subgroup of $\mathbb{R}$, hence it contains a countable sequence (ζ_i) which still generates a dense subgroup. Fix ζ_i. Then $\zeta_i = \log \lambda_{\mathbf{a}}$ for some $\mathbf{a} \in \Gamma \cap \operatorname{int}(\mathcal{M}_+)$, set

$$z := w_{\mathbf{a}} \in W(\Gamma) \cap \operatorname{int}(\mathbb{S}_\geq).$$

Referring again to Prop. 3.1, $z \in \operatorname{supp} \nu$, thus (1) follows.

Now fix $\delta > 0$. Then for all $\varepsilon > 0$ sufficiently small, since $w_{\mathbf{a}}$ is a Perron-Frobenius eigenvector,

$$\begin{aligned} &\mathbf{a} \cdot B_\varepsilon(w_{\mathbf{a}}) \subset B_{\varepsilon/2}(w_{\mathbf{a}}), \\ &|\log \lambda_{\mathbf{a}} - \log |\mathbf{a}x|| < \delta/2 \qquad \text{for all } x \in B_\varepsilon(w_{\mathbf{a}}). \end{aligned}$$

Since $\mathbf{a} \in [\operatorname{supp} \mu]$, there is $m \in \mathbb{N}$ such that $\mathbf{a} = \mathbf{a}_m \ldots \mathbf{a}_1$, $\mathbf{a}_j \in \operatorname{supp} \mu$, $1 \leq j \leq m$, hence for all $\gamma > 0$,

$$\mathbb{P}\left(\mathbf{A}_n \cdots \mathbf{A}_1 \in B_\gamma(\mathbf{a})\right) = \eta_\gamma > 0.$$

If $\gamma > 0$ is chosen sufficiently small, then for all $\mathbf{a}' \in B_\gamma(\mathbf{a})$,

$$\begin{aligned} &\mathbf{a}' \cdot B_\varepsilon(w_{\mathbf{a}}) \subset B_\varepsilon(w_{\mathbf{a}}), \\ &|\log \lambda_{\mathbf{a}} - \log |\mathbf{a}'x|| < \delta \qquad \text{for all } x \in B_\varepsilon(w_{\mathbf{a}}). \end{aligned}$$

Consequently, for all $x \in B_\varepsilon(w_{\mathbf{a}})$,

$$\mathbb{P}\left(|\mathbf{\Pi}_n \cdot x - w_{\mathbf{a}}| < \varepsilon,\ |\log |\mathbf{\Pi}_n x| - \log \lambda_{\mathbf{a}}| < \delta\right) \geq \eta_\gamma > 0.$$

Recalling the definition of $\mathbb{P}_x$, this gives (3.2). $\square$

4 Proof of the Main Theorem

Let $L \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$. For a compactly supported function $h \in \mathcal{C}_b(\mathbb{R})$ define

$$L_h(x,s) = \int L(x, s+r)\, h(r)\, dr.$$

If for each such h, L_h is constant, then the same holds true for L itself – this can be seen by choosing a sequence h_n of probability densities, such that $h_n(r)\,dr$ converges weakly towards the dirac measure in 0.

Lemma 4.1. *Let $L \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ satisfy properties (a),(b) of Theorem 2.2. Then for any compactly supported $h \in \mathcal{C}_b(\mathbb{R})$, L_h still satisfies (a),(b) and moreover:*

(c) For all $z \in \operatorname{int}(\mathbb{S}_\geq)$,

$$\lim_{y \to z} \lim_{\delta \downarrow 0} \sup_{|t-t'|<\delta} |L_h(z,t) - L_h(y,t')| = 0.$$

Proof. That (a) and (b) persist to hold for L_h is a simple consequence of Fubini's theorem resp. Fatou's lemma.

In order to prove (c), let $|L| \leq C$. Consider

$$\begin{aligned} &\lim_{\delta \to 0} \sup_{y \in \mathbb{S}_\geq} \sup_{|t-t'|<\delta} |L_h(y,t) - L_h(y,t')| \\ =&\lim_{\delta \to 0} \sup_{y \in \mathbb{S}_\geq} \sup_{|t-t'|<\delta} \left| \int L(y, t'+r) h(r-(t-t'))dr - \int L(y,t'+r)h(r)dr \right| \\ \leq&\lim_{\delta \to 0} \sup_{|t-t'|<\delta} C \int |h(r-(t-t')) - h(r)|\, dr = 0, \end{aligned}$$

where the uniform continuity of h was taken into account for the last line. Combine this with (b) to obtain for all $z \in \operatorname{int}(\mathbb{S}_\geq)$,

$$\begin{aligned}
&\lim_{y\to z}\lim_{\delta\downarrow 0}\sup_{|t-t'|<\delta}|L_h(z,t)-L_h(y,t')| \\
&\leq \lim_{y\to z}\lim_{\delta\downarrow 0}\sup_{|t-t'|<\delta}\Big[|L_h(z,t)-L_h(y,t)|+|L_h(y,t)-L_h(y,t')|\Big] \\
&\leq \lim_{y\to z}\sup_{t\in\mathbb{R}}|L_h(z,t)-L_h(y,t)|+\lim_{\delta\to 0}\sup_{y\in\mathbb{S}_\geq}\sup_{|t-t'|<\delta}|L_h(y,t)-L_h(y,t')|=0.
\end{aligned}$$

□

Consequently, in order to proof Theorem 2.2, we may w.l.o.g. assume that L satisfies properties $(a)-(c)$.

Proof of Theorem 2.2. The burden of the proof is to show that for all the ζ_i of Lemma 3.2,

$$L(x,s)=L(x,s+\zeta_i)\quad \text{for all } (x,s)\in\mathbb{S}_\geq\times\mathbb{R}. \tag{4.1}$$

If this holds true, then for any $\sigma=\sum_{i=1}^N c_i\zeta_i$ with $c_i\in\mathbb{N}_0$, $N\in\mathbb{N}$

$$L(x,s)=L(x,s+\sigma)\quad \text{for all } (x,s)\in\mathbb{S}_\geq\times\mathbb{R}.$$

But the set of σ's is dense in $\mathbb{R}$, thus by the continuity of L,

$$L(x,s)=L(x,0)\quad \text{for all } (x,s)\in\mathbb{S}_\geq\times\mathbb{R}.$$

Hence $L(x,s)$ reduces to a function $\tilde{L}$ on $\mathbb{S}_\geq$, which is then bounded harmonic for the ergodic Markov chain $(X_n)_{n\in\mathbb{N}_0}$ (see (3.1)), thus $\tilde{L}$ is constant.

Now we are going to prove (4.1). Considering (a), $L(X_n,s-S_n)_{n\in\mathbb{N}_0}$ constitutes a bounded, hence a.s. convergent martingale under each $\mathbb{P}_x$ with

$$L(x,s)=\mathbb{E}_x\lim_{n\to\infty}L(X_n,s-S_n)\quad \text{for all } (x,s)\in\mathbb{S}_\geq\times\mathbb{R}. \tag{4.2}$$

Fix any ζ_i and the corresponding $z\in\operatorname{int}(\mathbb{S}_\geq)$, defined in Lemma 3.2. Referring to (c), for all $\xi>0$, there are $\delta,\varepsilon>0$ such that

$$\sup_{u,y\in B_\varepsilon(z)}\sup_{|t-t'|<\delta}|L(u,t)-L(y,t')|<\xi.$$

Combining this with (3.3), we infer that for all $s\in\mathbb{R}$,

$$\mathbb{P}_x\left(|L(X_n,s-S_n)-L(X_{n+m},s+\zeta_i-S_{n+m})|<\xi \text{ i.o.}\right)=1.$$

Hence for all $(x,s)\in\mathbb{S}_\geq\times\mathbb{R}$,

$$\lim_{n\to\infty}L(X_n,s-S_n)=\lim_{n\to\infty}L(X_n,s+\zeta_i-S_n)\qquad \mathbb{P}_x\text{-a.s.}$$

and consequently, using (4.2), it follows for all $(x,s)\in\mathbb{S}_\geq\times\mathbb{R}$

$$L(x,s)=\mathbb{E}_x\lim_{n\to\infty}L(X_n,s-S_n)=\mathbb{E}_x\lim_{n\to\infty}L(X_n,s+\zeta_i-S_n)=L(x,s+\zeta_i).$$

□

5 On Conditon (C)

5.1 Comparison with Kesten's Assumptions

Since Kesten's renewal theorem is formulated for Markov chains on a general state space, his Choquet-Deny lemma [8, Lemma 1] holds for a broader class of Markov chains than just those generated by random matrices. Nevertheless, the latter ones provide by far the most important applications, hence condition (C) should be compared to the assumptions of Kesten's renewal theorem for products of random matrices [7, Theorem A].

Firstly, in [7, Theorem A] it is assumed that the matrices in $\operatorname{supp}\mu$ do not have a zero row, instead of no zero row *and* now zero column. But the latter assumption has the advantage of being invariant under taking the transpose. In fact, if (C) holds for Γ, than it holds for $\Gamma^\top$ as well, condition (2) being translated by considering the orthogonal spaces

$$W^\perp = \{y \in \mathbb{R}^d \;:\; \langle x, y\rangle = 0 \;\forall\, x \in W\}.$$

Secondly, Kesten's assumption [7, (1.11)] requests (3) as well, while the "nonlattice" part of [7, (1.11)] is replaced by the more natural assumption (2) that the problem may not be reduced to a lower dimensional one.

5.2 Examples

A convenient way to check the irreducibility assumption (2) is to consider the eigenspaces of matrices generated by μ. In dimension $d = 2$, for example, it is sufficient that there are two matrices the eigenvectors of which are pairwise independent: Any proper subspace W is onedimensional, and if it is invariant for $[\operatorname{supp}\mu]$, then it is in particular invariant for any matrix $\mathbf{a} \in \operatorname{supp}\mu$, i.e. an eigenspace of $\mathbf{a}$.

Hence a simple example of a distribution satisfying (C) is given by the probability law that puts masses $p, 1-q > 0$ on the two matrices

$$\mathbf{a} := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{b} := \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix},$$

the eigenvectors being $(1,1)^\top$, $(-1,1)^\top$ resp. $(1,2)^\top$, $(-1,1)^\top$. Though the second eigenvectors are the same, the corresponding linear spaces do not intersect the positive cone except in $\{0\}$, thus (2) is satisfied, as well as (1) and (3) obviously are.

A second example where (C) holds is when μ has a density with respect to the Lebesgue measure on $\mathcal{M}_+$, seen as a subset of $\mathbb{R}_{\geq}^{d\times d}$. Again conditions (1) and (3) are obviously satisfied. If now W is an invariant subspace, consider a set of independent vectors $v_1, \dots, v_k$ generating the orthogonal space $W^\perp$. W being invariant then implies that for any fixed $x \in W$,

$$\langle \mathbf{a}x, v_1\rangle = \cdots = \langle \mathbf{a}x, v_k\rangle = 0 \qquad \text{for } \mu\text{-a.e. } \mathbf{a}.$$

But the set of matrices satisfying this set of equations has entries from a $k\times d$-dimensional subspace of $\mathbb{R}^{d\times d}$, hence has mass zero under the Lebesgue measure.

Finally, a negative example satisfying the assumptions of [7, Theorem A], but not conditon (C), is given by the law that puts masses $q, 1-q > 0$ on

$$\mathbf{a}' := 1/2 \begin{pmatrix} e & e \\ e & e \end{pmatrix}, \qquad \mathbf{b}' := 1/2 \begin{pmatrix} e^\pi & e^\pi \\ e^\pi & e^\pi \end{pmatrix}.$$

6 Invertible Matrices

Let us finally mention that a result similar to Theorem 2.2 holds for invertible matrices: In the following, let μ be a distribution on $GL(d, \mathbb{R})$.

Definition 6.1. *A subsemigroup* $\Gamma \in GL(d, \mathbb{R})$ *is said to be* irreducible-proximal (i-p)*, if*

1. *no finite union* $W = \bigcup_{i=1}^{n} W_i$ *of proper subspaces* $\{0\} \subsetneq W_i \subsetneq \mathbb{R}^d$ *satisfies* $\Gamma W \subset W$ *(irreducibility) and*
2. *there is* $g \in \Gamma$ *having a algebraically simple dominant eigenvalue* $\lambda_g \in \mathbb{R}$ *such that* $|\lambda_g| = \lim_{n\to\infty} \|g^n\|^{1/n}$ *(proximality).*

This condition has been studied intensively by Guivarc'h and Le Page [4, 5, 6]. Considering condition *i-p*, Proposition 3.1, on which our proof rests, can be replaced by [4, Proposition 2.5] which is the corresponding result for *i-p* matrices. Then following the lines of the proof of Theorem 2.2, one obtains the following:

Theorem 6.2. *Let* $[\operatorname{supp}\mu]$ *satisfy* i-p. *Assume that* $L \in \mathcal{C}_b(\mathbb{S} \times \mathbb{R})$ *satisfies*

(a) $L(x, s) = \mathbb{E}_x\, L(X_1, s - S_1)$ *for all* $(x, s) \in \mathbb{S} \times \mathbb{R}$*, and*

(b) for all $z \in \mathbb{S}$*,*

$$\lim_{y\to z} \sup_{t\in\mathbb{R}} |L(y, t) - L(z, t)| = 0.$$

Then L *is constant.*

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