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From uniform renewal theorem to uniform large and moderate deviations for renewal-reward processes

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Abstract

A uniform key renewal theorem is deduced from the uniform Blackwell's renewal theorem. A uniform LDP (large deviation principle) for renewal-reward processes is obtained, and MDP (moderate deviation principle) is deduced under conditions much weaker than existence of exponential moments.

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1 Introduction

An ordinary renewal-reward process $S(\cdot)$ is a process of the form

$$S(t) = X_1 + \dots + X_n \quad \text{for } \tau_1 + \dots + \tau_n \le t < \tau_1 + \dots + \tau_{n+1} \, ;$$

here $(\tau_1, X_1), (\tau_2, X_2), \dots$ are independent copies of a pair (τ, X) of (generally, correlated) random variables such that $\tau > 0$ a.s. That is,

$$S(t) = \sum_{j=1}^{N(t)} X_j \quad \text{where} \quad N(t) = \min\{n : T_n \le t\} \, , \quad T_n = \sum_{j=1}^{n} \tau_n \, .$$

Large deviation principle (LDP) for $S(t)$ (as $t \to \infty$) is well-known when τ and X have exponential moments. I prove moderate deviation principle (MDP) for $S(t)$ requiring

$$\mathbb{E}\, \tau < \infty \, , \tag{1.1}$$

$$\mathbb{E}\, \exp(\varepsilon X^2 - \tau) < \infty \quad \text{for some } \varepsilon > 0 \, . \tag{1.2}$$

Example 1.1. *Let $X = \pm\sqrt{\tau}$ in the sense that $\mathbb{P}\big(X = -\sqrt{\tau} \,\big|\, \tau \big) = \mathbb{P}\big(X = \sqrt{\tau} \,\big|\, \tau \big) = 0.5$ a.s. In this case* (1.2) *follows from* (1.1)*, thus, MDP is ensured by Theorem 1.2 (below) whenever $\mathbb{E}\,\tau < \infty$. Note that, for instance, large values of X_1 do not contribute to the tail of $S(1)$; indeed, $S(1) = 0$ unless $\tau_1 \le 1$, that is, $|X_1| \le 1$. This is the significance of* (1.2)*: large values of X are "screened out" by large values of τ.*

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Emergence of MDP and/or LDP in some models driven by "bad" random variables is not a new phenomenon. It was observed in the context of renewal processes [4], self-normalization [6, Sect. 3], random walks in random environment [5, Sect. 2.2].

Conditions (1.1), (1.2) imply $\mathbb{E}\, X^2 < \infty$ (see Remark 4.1) and are invariant under linear transformations of X and rescaling of τ (see Remark 4.2); thus we may restrict ourselves to the case

$$\mathbb{E}\, X = 0\,, \quad \mathbb{E}\, X^2 = 1\,, \quad \mathbb{E}\, \tau = 1\,. \tag{1.3}$$

Theorem 1.2. *If* (1.2), (1.3) *are satisfied then*

$$\lim_{x\to+\infty, x/\sqrt{t}\to 0} \frac{1}{x^2} \ln \mathbb{P}\big(S(t) > x\sqrt{t}\big) = -\frac{1}{2}\,.$$

The limit in two variables t, x is taken; that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all t, x satisfying $x > 1/\delta$, $x/\sqrt{t} < \delta$ the function is ε-close to the limit.

Theorem 1.2 (MDP) will be deduced from Theorem 1.4 (LDP uniform up to MDP), and Theorem 1.4 extends Theorem 1.3 (LDP uniform away from MDP) to small λ; due to the uniformity in small λ it covers the limit in two variables $t \to \infty$, $\lambda \to 0$, $\lambda^2 t \to \infty$, which leads readily to MDP. The assumption for Theorem 1.3 is weaker than (1.2) (see Remark 4.2):

$$\forall \lambda \in \mathbb{R}\; \forall \varepsilon > 0 \quad \mathbb{E}\, \exp(\lambda X - \varepsilon\tau) < \infty\,. \tag{1.4}$$

In combination with (1.1) it implies $\mathbb{E}\,|X| < \infty$ but does not imply $\mathbb{E}\, X^2 < \infty$ (see Remark 3.3).

Theorem 1.3. *If* (1.1), (1.4) *hold and* $\mathbb{E}\, X = 0$ *then for every* $\lambda \in \mathbb{R}$, *first,* $\mathbb{E}\, \exp(\lambda S(t)) < \infty$ *for all* $t \ge 0$*; second, there exists one and only one* $\eta_\lambda \in [0, \infty)$ *such that*

$$\mathbb{E}\, \exp(\lambda X - \eta_\lambda \tau) = 1\,; \tag{1.5}$$

and third,

$$\frac{1}{t} \ln \mathbb{E}\, \exp\big(\lambda S(t)\big) = \eta_\lambda + O\Big(\frac{1}{t}\Big) \tag{1.6}$$

as $t \to \infty$*, uniformly in* $\lambda \in [-C, -c] \cup [c, C]$ *whenever* $0 < c < C < \infty$.

Theorem 1.4. *If* (1.2) *and* (1.3) *hold then*

$$\eta_\lambda = \frac{1}{2}\lambda^2 + o(\lambda^2) \quad \text{as } \lambda \to 0\,,$$

and (1.6) *holds uniformly in* $\lambda \in [-C, C]$ *whenever* $0 < C < \infty$.

Theorem 1.2 is used in [7].

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2 Uniform renewal theorems

A uniform version of Blackwell's renewal theorem is available ([8, Th. 1]; see also [1, Th. 2.6(2), 2.7]) and may be formulated as follows.

First, we define the *span* of a probability measure μ on $(0, \infty)$ as

$$\mathrm{Span}(\mu) = \max\big(\big\{\delta > 0 : \mu(\{\delta, 2\delta, 3\delta, \dots\}) = 1\big\} \cup \big\{0\big\}\big)\,;$$

$\mathrm{Span}(\mu) = 0$ if and only if μ is non-arithmetic. A set M of probability measures on $(0, \infty)$ will be called a *set of constant span* δ, if $\mathrm{Span}(\mu) = \delta$ for all $\mu \in M$. Symbolically: $\mathrm{Span}(M) = \delta$. Thus, a set of constant span 0 contains only non-arithmetic measures;

a set of constant span $\delta > 0$ contains only arithmetic measures of span δ (rather than $2\delta, 3\delta, \dots$).

Second, for every probability measure μ on $(0,\infty)$ we introduce the *renewal measure* as the sum of convolutions:

$$U_\mu = \sum_{n=0}^{\infty} \underbrace{\mu * \dots * \mu}_{n} \tag{2.1}$$

(the term for $n = 0$ being the atom at the origin); U_μ is infinite, but locally finite, since $\int \mathrm{e}^{-t} U_\mu(\mathrm{d}t) = \sum_n \bigl(\int \mathrm{e}^{-t}\mu(\mathrm{d}t)\bigr)^n < \infty$.

Theorem 2.1. ([8], [1]) *Assume that a set M of probability measures on $(0,\infty)$ is weakly compact (treated as a set of measures on $\mathbb{R}$), is a set of constant span, and is uniformly integrable, that is,*

$$\lim_{a\to+\infty} \sup_{\mu\in M} \int_{[a,\infty)} t\,\mu(\mathrm{d}t) = 0\,.$$

Then in the non-arithmetic case $(\mathrm{Span}(M) = 0)$, for every $v > 0$,

$$U_\mu\bigl((u, u+v]\bigr) \to \frac{v}{\int t\,\mu(\mathrm{d}t)} \quad \text{as } u \to \infty$$

uniformly in $\mu \in M$; and in the arithmetic case $(\mathrm{Span}(M) = \delta)$,

$$U_\mu(\{n\delta\}) \to \frac{\delta}{\int t\,\mu(\mathrm{d}t)} \quad \text{as } n \to \infty$$

uniformly in $\mu \in M$.

We denote for convenience

$$\lambda_\mu = \frac{1}{\int t\,\mu(\mathrm{d}t)}\,.$$

Remark 2.2. *The uniform integrability of M ensures continuity of the function $\mu \mapsto \lambda_\mu$ on M. By compactness,*

$$0 < \min_{\mu\in M} \lambda_\mu \le \max_{\mu\in M} \lambda_\mu < \infty\,.$$

Remark 2.3. *Under the uniform integrability, M is weakly compact if and only if it is weakly closed (as noted in [8, p. 25]).*

Remark 2.4. *In the non-arithmetic case it follows that $U_\mu(\{u\}) \to 0$ as $u \to \infty$, uniformly in $\mu \in M$. Therefore the interval $(u, u+v]$ in Theorem 2.1 may be replaced with $(u, u+v)$, $[u, u+v]$ or $[u, u+v)$.*

Remark 2.5. *We note well-known inequalities*

$$\begin{aligned} U_\mu([u, u+v]) &\le U_\mu([0, v])\,, \\ U_\mu([u, u+v)) &\le U_\mu([0, v))\,, \\ U_\mu((u, u+v]) &\le U_\mu([0, v)) \end{aligned}$$

for all $u, v \ge 0$. See [2, p. 123] for the third inequality; others follow by approximation.

A uniform version of the key renewal theorem follows. We start with the arithmetic case.

Theorem 2.6. *Let M be a set of probability measures on $(0,\infty)$ satisfying the conditions of Theorem 2.1,* $\mathrm{Span}(M)=\delta>0$*, and H a set of functions $\{0,\delta,2\delta,\dots\}\to\mathbb{R}$ such that*

$$\sup_{h\in H}\sum_{k=0}^{\infty}|h(k\delta)|<\infty \quad \textit{and} \quad \lim_{n\to\infty}\sup_{h\in H}\sum_{k=n}^{\infty}|h(k\delta)|=0\,.$$

Then

$$(U_\mu * h)(n\delta)\to\delta\lambda_\mu\sum_{k=0}^{\infty}h(k\delta) \quad \textit{as } n\to\infty$$

uniformly in $\mu\in M$ and $h\in H$.

Proof. By Remark 2.5, $U_\mu(\{n\delta\})\le U_\mu(\{0\})=1$ for all μ and n. By Theorem 2.1, $U_\mu(\{n\delta\})\to\delta\lambda_\mu$ as $n\to\infty$, uniformly in $\mu\in M$. Lemma 2.7 (below) completes the proof. □

Lemma 2.7. *Let U and H be sets of functions $\{0,1,2,\dots\}\to\mathbb{R}$ such that*

$$\sup_{u\in U}\sup_n|u(n)|<\infty\,,$$

$$\textit{the limit } u(\infty)=\lim_{n\to\infty}u(n) \textit{ exists uniformly in } u\in U\,;$$

$$\sup_{h\in H}\sum_n|h(n)|<\infty\,;$$

$$\sum_{n=N}^{\infty}|h(n)|\to 0 \textit{ as } N\to\infty\,,\ \textit{uniformly in } h\in H\,.$$

Then

$$(u*h)(n)\to u(\infty)\sum_{k=0}^{\infty}h(k) \textit{ as } n\to\infty\,,\ \textit{uniformly in } u\in U \textit{ and } h\in H\,.$$

Proof. Denoting $\|u\|_\infty=\sup_n|u(n)|$, $\|h\|_1=\sum_n|h(n)|$ and $\Sigma(h)=\sum_n h(n)$ we have $\|u*h\|_\infty\le\|u\|_\infty\|h\|_1$, $|u(\infty)|\le\|u\|_\infty$ and $|\Sigma(h)|\le\|h\|_1$. For arbitrary $N\in\{0,1,2,\dots\}$ and $h\in H$ we introduce $h_N,h^N:\{0,1,2,\dots\}\to\mathbb{R}$ by $h_N(n)=h(n)$ for $n\le N$, $h_N(n)=0$ for $n>N$, and $h^N=h-h_N$. We have $\sup_{u\in U}\|u\|_\infty<\infty$, $\sup_{h\in H}\|h\|_1<\infty$, and $\sup_{h\in H}\|h^N\|_1\to 0$ as $N\to\infty$. For arbitrary N and all $n\ge N$,

$$\begin{aligned}
&|(u*h)(n)-u(\infty)\Sigma(h)|\le\\
&\qquad\le|(u*h_N)(n)-u(\infty)\Sigma(h_N)|+|(u*h^N)(n)-u(\infty)\Sigma(h^N)|\le\\
&\qquad\le\Big|\sum_{k=0}^{N}u(n-k)h(k)-u(\infty)\sum_{k=0}^{N}h(k)\Big|+|(u*h^N)(n)|+|u(\infty)\Sigma(h^N)|\le\\
&\qquad\le\sum_{k=0}^{N}|u(n-k)-u(\infty)||h(k)|+\|u*h^N\|_\infty+|u(\infty)||\Sigma(h^N)|\le\\
&\qquad\qquad\le\|h\|_1\sup_{k\ge n-N}|u(k)-u(\infty)|+2\|u\|_\infty\|h^N\|_1\,;
\end{aligned}$$

given $\varepsilon>0$, we choose N such that $\|u\|_\infty\|h^N\|_1\le\varepsilon$ for all $u\in U$ and $h\in H$; then for all n large enough we have $\|h\|_1\sup_{k\ge n-N}|u(k)-u(\infty)|\le\varepsilon$ for all $u\in U$ and $h\in H$, and finally, $|(u*h)(n)-u(\infty)\Sigma(h)|\le 3\varepsilon$. □

The non-arithmetic case needs more effort. Recall that a function $h:[0,\infty)\to\mathbb{R}$ is called *directly Riemann integrable,* if two limits exist and are equal (and finite):

$$\lim_{\delta\to 0+}\delta\sum_{n=0}^{\infty}\inf_{[n\delta,n\delta+\delta)}h(\cdot)=\lim_{\delta\to 0+}\delta\sum_{n=0}^{\infty}\sup_{[n\delta,n\delta+\delta)}h(\cdot)\,.$$

Definition 2.8. *A set H of functions $[0,\infty) \to \mathbb{R}$ is* uniformly directly Riemann integrable, *if*

$$\sup_{h\in H} \sum_{n=0}^{\infty} \sup_{[n,n+1)} |h(\cdot)| < \infty \,,$$

$$\sum_{n=N}^{\infty} \sup_{[n,n+1)} |h(\cdot)| \to 0 \quad \text{as } N \to \infty \,, \text{ uniformly in } h \in H \,;$$

$$\delta \sum_{n=0}^{\infty} \Big(\sup_{[n\delta,n\delta+\delta)} h(\cdot) - \inf_{[n\delta,n\delta+\delta)} h(\cdot) \Big) \to 0 \quad \text{as } \delta \to 0+ \,, \text{ uniformly in } h \in H \,.$$

Remark 2.9. *If $\sup_{h\in H} \sum_{n=0}^{\infty} \sup_{[n\delta,n\delta+\delta)} |h(\cdot)| < \infty$ for some δ then it holds for all δ.*
Proof. *Given $\delta_1, \delta_2 > 0$, we consider $A = \{(n_1,n_2) : [n_1\delta_1, n_1\delta_1+\delta_1) \cap [n_2\delta_2, n_2\delta_2+\delta_2) \ne \emptyset\}$, note that $\#\{n_1 : (n_1,n_2) \in A\} \le \frac{\delta_2}{\delta_1} + 2$, and get*

$$\sum_{n_1=0}^{\infty} \sup_{[n_1\delta_1,n_1\delta_1+\delta_1)} |h(\cdot)| \le \sum_{n_1=0}^{\infty} \max_{n_2:(n_1,n_2)\in A} \sup_{[n_2\delta_2,n_2\delta_2+\delta_2)} |h(\cdot)| \le$$
$$\le \sum_{(n_1,n_2)\in A} \sup_{[n_2\delta_2,n_2\delta_2+\delta_2)} |h(\cdot)| \le \Big(\frac{\delta_2}{\delta_1} + 2 \Big) \sum_{n_2=0}^{\infty} \sup_{[n_2\delta_2,n_2\delta_2+\delta_2)} |h(\cdot)| \,.$$

Remark 2.10. *By Remark 2.9, the first two conditions of Def. 2.8 may be reformulated as*

$$\sup_{h\in H} \sum_{n=0}^{\infty} \sup_{[n\delta,n\delta+\delta)} |h(\cdot)| < \infty \,,$$

$$\sum_{n=N}^{\infty} \sup_{[n\delta,n\delta+\delta)} |h(\cdot)| \to 0 \quad \text{as } N \to \infty \,, \text{ uniformly in } h \in H$$

for some (therefore, all) $\delta > 0$. Similarly,

$$\delta \sum_{n:n\delta>N} \Big(\sup_{[n\delta,n\delta+\delta)} h(\cdot) - \inf_{[n\delta,n\delta+\delta)} h(\cdot) \Big) \le (1+2\delta) \sum_{n=N}^{\infty} \sup_{[n,n+1)} |h(\cdot)| \,.$$

Thus, the third condition of Def. 2.8 may be reformulated as uniform Riemann integrability on bounded intervals: for every N,

$$\delta \sum_{n\ge0:n\delta\le N} \Big(\sup_{[n\delta,n\delta+\delta)} h(\cdot) - \inf_{[n\delta,n\delta+\delta)} h(\cdot) \Big) \to 0 \quad \text{as } \delta \to 0+ \,, \text{ uniformly in } h \in H \,.$$

Remark 2.11. *If each $h \in H$ is a decreasing function $[0,\infty) \to [0,\infty)$ then H is uniformly directly Riemann integrable if and only if*

$$\sup_{h\in H} h(0) < \infty \,, \quad \sup_{h\in H} \int_0^\infty h(s)\,\mathrm{d}s < \infty \,, \quad \text{and}$$
$$\sup_{h\in H} \int_a^\infty h(s)\,\mathrm{d}s \to 0 \quad \text{as } a \to \infty \,.$$

By taking differences, a similar result can be obtained for functions of uniformly bounded variation on $[0,\infty)$ (rather than decreasing).

We turn to the non-arithmetic case (of the uniform version of the key renewal theorem).

Theorem 2.12. *Let M be a set of probability measures on $(0,\infty)$ satisfying the conditions of Theorem 2.1,* $\mathrm{Span}(M)=0$*, and H a uniformly directly Riemann integrable set of functions $[0,\infty)\to\mathbb{R}$. Then*

$$(U_\mu * h)(t) \to \lambda_\mu \int_0^\infty h(s)\,\mathrm{d}s \quad \textit{as } t\to\infty$$

uniformly in $\mu\in M$ and $h\in H$.

Here is a generalization of Lemma 2.7, to be used in the proof of the theorem.

Lemma 2.13. *Let H be as in Lemma 2.7, and V a set of functions $\{0,1,2,\dots\}\times[0,\infty)\to\mathbb{R}$ such that, first, $\sup_{v\in V}\sup_n\sup_t|v_n(t)|<\infty$, and second, the limit $v(\infty)=\lim_{t\to\infty}v_n(t)$ exists uniformly in $v\in V$ for every n, and does not depend on n. Then*

$$\sum_{n=0}^\infty h(n)v_n(t)\to v(\infty)\sum_{n=0}^\infty h(n) \textit{ as } t\to\infty\,, \textit{ uniformly in } v\in V \textit{ and } h\in H\,.$$

Proof. The proof of Lemma 2.7 needs only trivial modifications: $\sum_n h(n)v_n(t)$ instead of $(u*h)(n)$; $\sum_{n=0}^N |v_n(t)-v(\infty)||h(n)|$ instead of $\sum_{k=0}^N|u(n-k)-u(\infty)||h(k)|$; and $\max_{n=0,\dots,N}|v_n(t)-v(\infty)|$ (for large t) instead of $\sup_{k\ge n-N}|u(k)-u(\infty)|$ (for large n). Also, $\|v\|_\infty=\sup_{n,t}|v_n(t)|$. ☐

Here is a special case of Theorem 2.12 for step functions.

Lemma 2.14. *Assume that M and H are as in Theorem 2.12, $\delta>0$, and every $h\in H$ is constant on each $[n\delta,n\delta+\delta)$. Then the conclusion of Theorem 2.12 holds.*

Proof. Lemma 2.13 will be applied to $\tilde H$ and V, where $\tilde H$ consists of all $\tilde h$ of the form $\tilde h(n)=h(n\delta)$ for $h\in H$, and V consists of all v of the form

$$v_n(\cdot)=U_\mu * \mathbb{1}_{[n\delta,n\delta+\delta)}$$

for $\mu\in M$; that is, $v_n(t)=U_\mu\big((t-n\delta-\delta,t-n\delta]\big)$. By Remark 2.5,

$$v_n(t)\le U_\mu\big([0,\delta)\big)\le \mathrm{e}^\delta\int \mathrm{e}^{-s}U_\mu(\mathrm{d}s)=\frac{\mathrm{e}^\delta}{1-\int \mathrm{e}^{-s}\mu(\mathrm{d}s)}\,;$$

by compactness of M,

$$\sup_{v,n,t}|v_n(t)|\le\frac{\mathrm{e}^\delta}{1-\max_\mu\int \mathrm{e}^{-s}\mu(\mathrm{d}s)}<\infty\,.$$

By Theorem 2.1, for every n, $v_n(t)\to\lambda_\mu\delta$ as $t\to\infty$, uniformly in v. Thus, V satisfies the conditions of Lemma 2.13. By Remark 2.10, $\tilde H$ satisfies the conditions (for H) of Lemma 2.13, that is, of Lemma 2.7. It remains to apply Lemma 2.13 and take into account that $v(\infty)=\lambda_\mu\delta$, $\delta\sum_n\tilde h(n)=\int_0^\infty h(s)\,\mathrm{d}s$ and $\sum_n\tilde h(n)v_n(\cdot)=U_\mu*h$ since $\sum_n h(n\delta)\mathbb{1}_{[n\delta,n\delta+\delta)}=h$. ☐

Proof of Theorem 2.12. For arbitrary $\delta>0$ and $h\in H$ we introduce $h_\delta^-,h_\delta^+:[0,\infty)\to\mathbb{R}$ by

$$h_\delta^-(t)=\inf_{[n\delta,n\delta+\delta)}h(\cdot)\,,\quad h_\delta^+(t)=\sup_{[n\delta,n\delta+\delta)}h(\cdot)\quad \text{for } t\in[n\delta,n\delta+\delta)\,,$$

then $h_\delta^-\le h\le h_\delta^+$. The sets $H_\delta^-=\{h_\delta^-:h\in H\}$, $H_\delta^+=\{h_\delta^+:h\in H\}$ are uniformly directly Riemann integrable by the arguments of Remark 2.10. Applying Lemma 2.14 to M and $H_\delta^\pm$ we get

$$(U_\mu * h_\delta^\pm)(t)\to\lambda_\mu\int_0^\infty h_\delta^\pm(s)\,\mathrm{d}s\quad\text{as } t\to\infty$$

uniformly in $\mu \in M$ and $h \in H$.

Given $\varepsilon > 0$, we choose $\delta = \delta_\varepsilon$ such that $\int |h_\delta^\pm(t) - h(t)|\,\mathrm{d}t \le \varepsilon$ for all $h \in H$. Then we choose t_ε such that for all $t \ge t_\varepsilon$, $\mu \in M$ and $h \in H$,

$$\Big|(U_\mu * h_\delta^\pm)(t) - \lambda_\mu \int_0^\infty h_\delta^\pm(s)\,\mathrm{d}s\Big| \le \varepsilon\,.$$

We get

$$\begin{aligned}(U_\mu * h)(t) - \lambda_\mu \int h(s)\,\mathrm{d}s \le \\ \le (U_\mu * h_\delta^+)(t) - \lambda_\mu \int_0^\infty h_\delta^+(s)\,\mathrm{d}s + \lambda_\mu \Big(\int_0^\infty h_\delta^+(s)\,\mathrm{d}s - \int_0^\infty h(s)\,\mathrm{d}s \Big) \le \\ \le \varepsilon + \lambda_\mu \varepsilon\end{aligned}$$

and a similar lower bound; thus (recall Remark 2.2),

$$\Big|(U_\mu * h)(t) - \lambda_\mu \int h(s)\,\mathrm{d}s\Big| \le \varepsilon\Big(1 + \max_{\mu \in M} \lambda_\mu\Big)$$

for all $t \ge t_\varepsilon$, $\mu \in M$ and $h \in H$. □

3 Uniform large deviations

Theorem 1.3 is proved in this section.

Exponential moments of a renewal-reward process boil down to a renewal equation, see [3, Th. 5], and therefore to an auxiliary renewal process, as explained below.

Having η_λ satisfying (1.5) for a given λ (see Lemma 3.5), we introduce a probability distribution ν_λ on $(0,\infty) \times \mathbb{R}$ by its Radon-Nikodym derivative

$$\frac{\mathrm{d}\nu_\lambda}{\mathrm{d}\nu}(t,x) = \mathrm{e}^{\lambda x - \eta_\lambda t}$$

where ν is the joint distribution of τ and X. Our renewal-reward process $S(\cdot)$ is driven by independent pairs (τ_k, X_k) distributed ν. Replacing ν with ν_λ ("change of measure") we may get another renewal-reward process. However, we need only the corresponding renewal measure (the reward being irrelevant). Thus, we introduce the first projection (marginal distribution) μ_λ of ν_λ. That is, if a pair $(\tau_\lambda, X_\lambda)$ is distributed ν_λ then τ_λ is distributed μ_λ. Also, τ is distributed μ. Similarly to (2.1) we define the renewal measure

$$U_\lambda = \sum_{n=0}^\infty \mu_\lambda^{*n}\,, \quad \mu_\lambda^{*n} = \underbrace{\mu_\lambda * \dots * \mu_\lambda}_{n}\,.$$

Lemma 3.1.

$$\mathbb{E}\,\mathrm{e}^{\lambda S(t)} = \mathrm{e}^{\eta_\lambda t}(U_\lambda * h_\lambda)(t)$$

where

$$h_\lambda(t) = \begin{cases} \mathrm{e}^{-\eta_\lambda t}\mathbb{P}\big(\tau > t\big) & \textit{for } t \ge 0, \\ 0 & \textit{for } t < 0. \end{cases} \tag{3.1}$$

Proof. Using the notation $\mathbb{E}\,(Z; A) = \mathbb{E}\,(Z \cdot \mathbb{1}_A)$ we have

$$\mathbb{E}\,\mathrm{e}^{\lambda S(t)} = \sum_{n=0}^\infty \mathbb{E}\,(\mathrm{e}^{\lambda S(t)};\, N(t) = n)\,.$$

It is sufficient to prove that

$$\mathbb{E}\,(\mathrm{e}^{\lambda S(t)};\,N(t)=n)=\mathrm{e}^{\eta_\lambda t}(\mu_\lambda^{*n}*h_\lambda)(t)\quad\text{for } n=0,1,2,\dots$$

We have

$$\begin{gathered}\mathrm{e}^{-\eta_\lambda t}\mathbb{E}\,(\mathrm{e}^{\lambda S(t)};\,N(t)=n)=\mathrm{e}^{-\eta_\lambda t}\mathbb{E}\,(\mathrm{e}^{\lambda(X_1+\dots+X_n)};\,N(t)=n)=\\ \mathbb{E}\,(\mathrm{e}^{\lambda X_1-\eta_\lambda\tau_1}\dots\mathrm{e}^{\lambda X_n-\eta_\lambda\tau_n}\mathrm{e}^{-\eta_\lambda(t-\tau_1-\dots-\tau_n)};\,0\le t-\tau_1-\dots-\tau_n<\tau_{n+1})=\\ \int\nu_\lambda(\mathrm{d}t_1\,\mathrm{d}x_1)\dots\int\nu_\lambda(\mathrm{d}t_n\,\mathrm{d}x_n)\int\nu(\mathrm{d}t_{n+1}\,\mathrm{d}x_{n+1})\mathrm{e}^{-\eta_\lambda(t-t_1-\dots-t_n)}\mathbb{1}_{0\le t-t_1-\dots-t_n<t_{n+1}}=\\ \int\mu_\lambda(\mathrm{d}t_1)\dots\int\mu_\lambda(\mathrm{d}t_n)\underbrace{\int\mu(\mathrm{d}t_{n+1})\mathrm{e}^{-\eta_\lambda(t-t_1-\dots-t_n)}\mathbb{1}_{0\le t-t_1-\dots-t_n<t_{n+1}}}_{h_\lambda(t-t_1-\dots-t_n)}=(\mu_\lambda^{*n}*h_\lambda)(t)\,.\end{gathered}$$

□

Remark 3.2. $\dfrac{\mathrm{d}\mu_\lambda}{\mathrm{d}\mu}(t)=\mathbb{E}\big(\mathrm{e}^{\lambda X-\eta_\lambda\tau}\big|\tau=t\big)$ *for μ-almost all t.*

Proof: $\mu_\lambda(A)=\nu_\lambda(A\times\mathbb{R})=\mathbb{E}\,(\mathrm{e}^{\lambda X-\eta_\lambda\tau};\,\tau\in A)=\mathbb{E}\big(\mathbb{E}\big(\mathrm{e}^{\lambda X-\eta_\lambda\tau}\big|\tau\big);\,\tau\in A\big)=\int_A\mathbb{E}\big(\mathrm{e}^{\lambda X-\eta_\lambda\tau}\big|\tau=t\big)\,\mu(\mathrm{d}t)$ *for all μ-measurable sets $A\subset\mathbb{R}$.*

Recall assumptions (1.1) $\mathbb{E}\,\tau<\infty$ and (1.4) $\forall\lambda\in\mathbb{R}\;\forall\varepsilon>0\quad\mathbb{E}\,\exp(\lambda X-\varepsilon\tau)<\infty$.

Remark 3.3. *Assumptions* (1.1), (1.4) *imply* $\mathbb{E}\,|X|<\infty$ *but do not imply* $\mathbb{E}\,X^2<\infty$.
Proof. *First,* $|X|\le\tau+\mathrm{e}^{|X|-\tau}\le\tau+\mathrm{e}^{-X-\tau}+\mathrm{e}^{X-\tau}$ *is integrable. Second, take* $X=\tau^{2/3}$ *with τ such that* $\mathbb{E}\,\tau<\infty$ *but* $\mathbb{E}\,\tau^{4/3}=\infty$, *then* (1.1), (1.4) *hold but* $\mathbb{E}\,X^2=\infty$.

From now on, till the end of this section, we assume the conditions of Theorem 1.3; that is, (1.1), (1.4), and $\mathbb{E}\,X=0$. We also assume that $\mathbb{P}\big(X=0\big)\ne1$; otherwise Theorem 1.3 is trivial.

Lemma 3.4. *Maps $(\lambda,\eta)\mapsto\exp(\lambda X-\eta\tau)$ and $(\lambda,\eta)\mapsto\tau\exp(\lambda X-\eta\tau)$ are continuous from $\mathbb{R}\times(0,\infty)$ to the space L_1 of integrable random variables.*

Proof. It is sufficient to prove the continuity on $[-C,C]\times[2\varepsilon,\infty)$ for arbitrary $C,\varepsilon>0$. Also, it is sufficient to consider the map $(\lambda,\eta)\mapsto\mathrm{e}^{\varepsilon\tau}\exp(\lambda X-\eta\tau)$, since $\tau\le\frac{1}{\mathrm{e}\varepsilon}\mathrm{e}^{\varepsilon\tau}$ a.s. We apply the dominated convergence theorem, taking into account that $\exp(-CX-\varepsilon\tau)+\exp(CX-\varepsilon\tau)$ is an integrable majorant of $\mathrm{e}^{\varepsilon\tau}\exp(\lambda X-\eta\tau)$ for all $\lambda\in[-C,C]$ and $\eta\in[2\varepsilon,\infty)$. □

Lemma 3.5. *For every λ there is one and only one η_λ satisfying* (1.5) $\mathbb{E}\,\exp(\lambda X-\eta_\lambda\tau)=1$*, and the function $\lambda\mapsto\eta_\lambda$ is continuous on $\mathbb{R}$.*

Proof. The function $\psi:\mathbb{R}\times(0,\infty)\to(0,\infty)$ defined by $\psi(\lambda,\eta)=\mathbb{E}\,\exp(\lambda X-\eta\tau)$ is continuous by Lemma 3.4. For every λ the function $\psi(\lambda,\cdot)$ is strictly decreasing, $\psi(\lambda,+\infty)=0$, and (possibly, infinite) $\psi(\lambda,0+)=\mathbb{E}\,\exp(\lambda X)>\exp(\lambda\mathbb{E}\,X)=1$ provided that $\lambda\ne0$. Thus, for $\lambda\ne0$ we get unique $\eta_\lambda>0$; and trivially, $\eta_0=0$.

It remains to prove continuity of the function $\lambda\mapsto\eta_\lambda$. Given $\lambda_0\ne0$ and $\varepsilon<\eta_{\lambda_0}$ we note that $\psi(\lambda_0,\eta_{\lambda_0}+\varepsilon)<1=\psi(\lambda_0,\eta_{\lambda_0})<\psi(\lambda_0,\eta_{\lambda_0}-\varepsilon)$ and take $\delta>0$ such that $\psi(\lambda,\eta_{\lambda_0}+\varepsilon)<1=\psi(\lambda,\eta_\lambda)<\psi(\lambda,\eta_{\lambda_0}-\varepsilon)$ and therefore $\eta_{\lambda_0}-\varepsilon<\eta_\lambda<\eta_{\lambda_0}+\varepsilon$ for all $\lambda\in(\lambda_0-\delta,\lambda_0+\delta)$. For $\lambda_0=0$ we use a one-sided version of the same argument: given $\varepsilon>0$, we take $\delta>0$ such that $\psi(\lambda,\varepsilon)<1$ and therefore $\eta_\lambda<\varepsilon$ for all $\lambda\in(-\delta,\delta)$. □

Lemma 3.6. *The function $\lambda\mapsto\mu_\lambda$ is continuous from $(-\infty,0)\cup(0,\infty)$ to the space of measures with the (total variation) norm topology.*

Proof. By Remark 3.2, $\mathrm{d}\mu_\lambda/\mathrm{d}\mu = \varphi_\lambda$ where φ_λ is defined by $\varphi_\lambda(\tau) = \mathbb{E}\big(\mathrm{e}^{\lambda X - \eta_\lambda \tau} \,\big|\, \tau\big)$. If $\lambda_n \to \lambda \ne 0$ then, using Lemmas 3.4 and 3.5, $\|\mu_{\lambda_n} - \mu_\lambda\| = \int |\varphi_{\lambda_n} - \varphi_\lambda| \,\mathrm{d}\mu = \mathbb{E}\,|\varphi_{\lambda_n}(\tau) - \varphi_\lambda(\tau)| = \mathbb{E}\,\big|\mathbb{E}\big(\mathrm{e}^{\lambda_n X - \eta_{\lambda_n}\tau} \,\big|\, \tau\big) - \mathbb{E}\big(\mathrm{e}^{\lambda X - \eta_\lambda \tau} \,\big|\, \tau\big)\big| \le \mathbb{E}\,|\mathrm{e}^{\lambda_n X - \eta_{\lambda_n}\tau} - \mathrm{e}^{\lambda X - \eta_\lambda \tau}| \to 0$ as $n \to \infty$. ☐

Lemma 3.7. *The set* $\{\mu_\lambda : \lambda \in [-C,-c] \cup [c,C]\}$ *satisfies the conditions of Theorem 2.1 whenever* $0 < c < C < \infty$.

Proof. Lemma 3.6 ensures compactness (even in a topology stronger than needed). For every λ measures μ_λ and μ are mutually absolutely continuous, therefore $\mathrm{Span}(\mu_\lambda) = \mathrm{Span}(\mu)$. It remains to prove uniform integrability. We have $\int_{[a,\infty)} t\,\mu_\lambda(\mathrm{d}t) = \mathbb{E}\big(\tau \exp(\lambda X - \eta_\lambda \tau) \mathbb{1}_{[a,\infty)}(\tau)\big) \to 0$ as $a \to \infty$ uniformly in $\lambda \in [-C,-c] \cup [c,C]$, since random variables $\tau \exp(\lambda X - \eta_\lambda \tau)$ for these λ are a compact subset of L_1 by Lemmas 3.4, 3.5. ☐

Proof of Theorem 1.3. Existence and uniqueness of η_λ satisfying (1.5) are ensured by Lemma 3.5; finiteness of $\mathbb{E}\,\mathrm{e}^{\lambda S(t)}$ by Lemma 3.1.

We reformulate (1.6) as existence of $T \in (0,\infty)$ such that

$$\sup_{\lambda \in [-C,-c] \cup [c,C],\, t \in [T,\infty)} \Big| -\eta_\lambda t + \ln \mathbb{E}\,\mathrm{e}^{\lambda S(t)} \Big| < \infty\,. \tag{3.2}$$

The set $M = \{\mu_\lambda : \lambda \in [-C,-c] \cup [c,C]\}$ satisfies the conditions of Theorem 2.1 by Lemma 3.7.

By Remark 2.2, $\int t\,\mu_\lambda(\mathrm{d}t)$ is bounded away from 0 and ∞ for $\lambda \in [-C,-c] \cup [c,C]$. The rest of the proof of (3.2) splits in two cases.

Non-arithmetic case: $\mathrm{Span}(\mu) = 0$.

The set $H = \{h_\lambda : \lambda \in [-C,-c] \cup [c,C]\}$ is uniformly directly Riemann integrable by (3.1), (1.1) and Remark 2.11. By Lemma 3.1 and Theorem 2.12,

$$\mathrm{e}^{-\eta_\lambda t}\mathbb{E}\,\mathrm{e}^{\lambda S(t)} \to \frac{\int_0^\infty h_\lambda(s)\,\mathrm{d}s}{\int s\,\mu_\lambda(\mathrm{d}s)} \quad \text{as } t \to \infty$$

uniformly in $\lambda \in [-C,-c] \cup [c,C]$. In order to get (3.2) it remains to check that the ratio of integrals is bounded away from 0 and ∞. For the denominator, see above. The numerator is bounded from above by (3.1), (1.1), and from below, since η_λ is bounded from above for $\lambda \in [-C,-c] \cup [c,C]$ by continuity.

Arithmetic case: $\mathrm{Span}(\mu) = \delta > 0$.

The set H of restrictions to $\{0, \delta, 2\delta, \dots\}$ of the functions h_λ for $\lambda \in [-C,-c] \cup [c,C]$ satisfies the conditions of Theorem 2.6 by (3.1) and (1.1). By Lemma 3.1 and Theorem 2.6,

$$\mathrm{e}^{-\eta_\lambda n\delta}\mathbb{E}\,\mathrm{e}^{\lambda S(n\delta)} \to \frac{\delta \sum_{k=0}^\infty h_\lambda(k\delta)}{\int s\,\mu_\lambda(\mathrm{d}s)} \quad \text{as } n \to \infty$$

uniformly in $\lambda \in [-C,-c] \cup [c,C]$. The numerator is bounded from above by (3.1), (1.1), and from below, since $h_\lambda(0) = 1$. Thus we get (3.2) for $t \in \{0, \delta, 2\delta, \dots\}$, which is sufficient, since $S(\cdot)$ is constant on $[k\delta, k\delta + \delta)$ (and η_λ is bounded). ☐

4 Moderate deviations

Theorems 1.2 and 1.4 are proved in this section.

In order to use small λ we need (1.2): $\mathbb{E}\,\exp(\varepsilon X^2 - \tau) < \infty$ for some $\varepsilon > 0$.

Remark 4.1. *Assumptions* (1.1), (1.2) *imply* $\mathbb{E}\,X^2 < \infty$.
Proof: $\varepsilon X^2 \le \tau + \mathrm{e}^{\varepsilon X^2 - \tau}$ *is integrable.*

Remark 4.2. *Assumption* (1.2) *is invariant under linear transformations of* X*, and rescaling of* τ*; also,* (1.2) *implies* (1.4)*.*

Proof. *Rescaling* X*:* $\mathbb{E}\,\exp\big((c^{-2}\varepsilon)(cX)^2-\tau\big)=\mathbb{E}\,\exp(\varepsilon X^2-\tau)<\infty$.

Shifting X*:* $\mathbb{E}\,\exp\big(\frac{\varepsilon}{2}(X+c)^2-\tau\big)\le\mathbb{E}\,\exp\big(\frac{\varepsilon}{2}(X-c)^2+\frac{\varepsilon}{2}(X+c)^2-\tau\big)=\mathrm{e}^{c^2\varepsilon}\mathbb{E}\,\exp(\varepsilon X^2-\tau)<\infty$.

Rescaling τ*:* $\mathbb{E}\,\exp(c\varepsilon X^2-c\tau)=\mathbb{E}\,\big(\exp(\varepsilon X^2-\tau)\big)^c\le\big(\mathbb{E}\,\exp(\varepsilon X^2-\tau)\big)^c<\infty$ *for* $c\in(0,1)$*, and* $\mathbb{E}\,\exp(\varepsilon X^2-c\tau)\le\mathbb{E}\,\exp(\varepsilon X^2-\tau)<\infty$ *for* $c\in[1,\infty)$*.*

Finally, (1.2) *implies* (1.4) *since* $\mathbb{E}\,\exp(\delta X^2-\tau)<\infty$ *implies* $\mathbb{E}\,\exp(\varepsilon\delta X^2-\varepsilon\tau)<\infty$ *(assuming* $0<\varepsilon<1$*) and therefore* $\mathbb{E}\,\exp(\lambda X-\varepsilon\tau)\le\mathbb{E}\,\exp\big(\frac{\lambda^2}{4\varepsilon\delta}+\varepsilon\delta X^2-\varepsilon\tau\big)<\infty$.

From now on we assume the conditions of Theorem 1.4; that is, (1.2), and (1.3): $\mathbb{E}\,X=0$, $\mathbb{E}\,X^2=1$, $\mathbb{E}\,\tau=1$. Conditions of Theorem 1.3 follow, since (1.2) implies (1.4) by Remark 4.2.

Here is an analytic fact that will give us some integrable majorants.

Lemma 4.3. *For all* $a,\varepsilon,\Lambda\in(0,\infty)$*,*

$$\sup_{t>0,x>0,\lambda\in(0,\Lambda)}\frac{(1+t+x^2)\exp(\lambda x-a\lambda^2t)}{t+\exp(\varepsilon x^2-t)}<\infty\,.$$

Proof. Denoting this supremum by $S(a,\varepsilon,\Lambda)$ we observe that $S(a,\varepsilon,\Lambda)\le\max(1,c^2)$ $S(c^{-2}a,c^2\varepsilon,c\Lambda)$ for arbitrary $c>0$ (by rescaling, $x\mapsto cx$ and $\lambda\mapsto c^{-1}\lambda$). Thus, we restrict ourselves to $\varepsilon=1$.

We note that

$$\max_{\lambda\in\mathbb{R}}(\lambda x-a\lambda^2t)=\frac{x^2}{4at}\,.$$

We choose $\alpha,\beta>0$ such that $\alpha>1$ and $\beta^2<4a(\alpha^2-1)$ (for instance, $\alpha=2$ and $\beta=3\sqrt{a}$) and consider three cases.

Case 1: $x\le\alpha\sqrt{t}$.

We note that $t+\exp(x^2-t)\ge t+\mathrm{e}^{-t}\ge\max(t,1)$, thus,

$$\frac{(1+t+x^2)\exp(\lambda x-a\lambda^2t)}{t+\exp(x^2-t)}\le\frac{(1+t+x^2)\exp\frac{x^2}{4at}}{\max(t,1)}\le$$
$$\le\frac{(1+t+\alpha^2t)\exp\frac{\alpha^2}{4a}}{\max(t,1)}\le(2+\alpha^2)\exp\frac{\alpha^2}{4a}\,.$$

Case 2: $\alpha\sqrt{t}\le x\le\beta t$.

$$\frac{(1+t+x^2)\exp(\lambda x-a\lambda^2t)}{t+\exp(x^2-t)}\le\frac{(1+t+x^2)\exp\frac{x^2}{4at}}{\exp(\alpha^2t-t)}\le$$
$$\le(1+t+\beta^2t^2)\exp\Big(\frac{\beta^2t^2}{4at}-\alpha^2t+t\Big)\le$$
$$\le\sup_{t>0}(1+t+\beta^2t^2)\exp\Big(-\frac{4a(\alpha^2-1)-\beta^2}{4a}t\Big)<\infty\,.$$

Case 3: $x\ge\beta t$.

$$\frac{(1+t+x^2)\exp(\lambda x-a\lambda^2t)}{t+\exp(x^2-t)}\le(1+\beta^{-1}x+x^2)\exp(\lambda x-a\lambda^2t-x^2+t)\le$$
$$\le\sup_x(1+\beta^{-1}x+x^2)\exp(\Lambda x-x^2+\beta^{-1}x)<\infty\,.$$

□

Lemma 4.4. *For all* $a, \varepsilon, \Lambda \in (0, \infty)$*,*

$$\sup_{t>0, x\in\mathbb{R}, \lambda\in(-\Lambda,\Lambda)} \frac{(1+t+x^2)\big(1+\exp(\lambda x - a\lambda^2 t)\big)}{t+\exp(\varepsilon x^2 - t)} < \infty \,.$$

Proof. By Lemma 4.3 applied to $|x|, |\lambda|$,

$$\sup_{t>0, x\in\mathbb{R}, \lambda\in(-\Lambda,\Lambda)} \frac{(1+t+x^2)\exp(|\lambda x| - a\lambda^2 t)}{t+\exp(\varepsilon x^2 - t)} < \infty \,,$$

and $\lambda x \le |\lambda x|$, of course. The new terms are bounded:

$$\frac{1+t}{t+\exp(\varepsilon x^2 - t)} \le \frac{1+t}{t+\mathrm{e}^{-t}} \le \frac{1+t}{\max(1,t)} \le 2 \,;$$
$$\frac{x^2}{t+\exp(\varepsilon x^2 - t)} \le \frac{x^2}{t+\varepsilon x^2 - t} \le \frac{1}{\varepsilon} \,.$$

□

Here is a counterpart of Lemma 3.4. This time, the origin $\lambda = \eta = 0$ is included (but its neighborhood is reduced).

Lemma 4.5. *For every* $a \in (0,\infty)$*, maps* $(\lambda,\eta) \mapsto \exp(\lambda X - \eta\tau)$ *and* $(\lambda,\eta) \mapsto \tau\exp(\lambda X - \eta\tau)$ *are continuous from* $\{(\lambda,\eta) : \lambda \in \mathbb{R}, \eta \in [a\lambda^2, \infty)\}$ *to the space* L_1 *of integrable random variables.*

Proof. We apply the dominated convergence theorem, taking into account that $\tau + \exp(\varepsilon X^2 - \tau)$ is an integrable majorant by Lemma 4.4. □

Lemma 4.6. *For all* $a, \varepsilon, \Lambda \in (0, \infty)$*,*

$$\sup_{t>0, x\in\mathbb{R}, \lambda\in(-\Lambda,0)\cup(0,\Lambda)} \frac{|\exp(\lambda x - a\lambda^2 t) - 1 - (\lambda x - a\lambda^2 t)|}{\lambda^2\big(t+\exp(\varepsilon x^2 - t)\big)} < \infty \,.$$

Proof. Denote $u = \lambda x - a\lambda^2 t$.

Case $|x| \le a|\lambda|t$: we have $|\lambda x| \le a\lambda^2 t$, thus $-2a\lambda^2 t \le u \le 0$ and $|\mathrm{e}^u - 1 - u| = \mathrm{e}^u - 1 - u \le 1 - 1 - u = -u \le 2a\lambda^2 t \le 2a\lambda^2\big(t+\exp(\varepsilon x^2 - t)\big)$.

Case $|x| \ge a|\lambda|t$: we apply the bound $|\mathrm{e}^u - 1 - u| \le \frac{1}{2}u^2\max(1,\mathrm{e}^u)$, note that $u^2/\lambda^2 \le 2x^2 + 2(a\lambda t)^2 \le 4x^2$ and get an upper bound

$$\frac{2x^2\max\big(1,\exp(\lambda x - a\lambda^2 t)\big)}{t+\exp(\varepsilon x^2 - t)} \,,$$

bounded by Lemma 4.4. □

Lemma 4.7. *For all* $a \in (0, \infty)$*,*

$$\frac{\mathbb{E}\,\exp(\lambda X - a\lambda^2\tau) - 1}{\lambda^2} \to \frac{1}{2} - a \quad \text{as } \lambda \to 0 \,.$$

Proof. We have

$$\frac{\exp(\lambda X - a\lambda^2\tau) - 1 - (\lambda X - a\lambda^2\tau)}{\lambda^2} \to \frac{1}{2}X^2 \quad \text{a.s.} \quad \text{as } \lambda \to 0 \,.$$

The left-hand side is dominated by $\tau + \exp(\varepsilon X^2 - \tau)$ by Lemma 4.6, the majorant being integrable (for some ε) by (1.1), (1.2). By the dominated convergence theorem,

$$\frac{\mathbb{E}\,\exp(\lambda X - a\lambda^2\tau) - 1 - \lambda\mathbb{E}\,X + a\lambda^2\mathbb{E}\,\tau}{\lambda^2} \to \frac{1}{2}\mathbb{E}\,X^2 \,;$$

it remains to use (1.3). □

Recall η_λ satisfying (1.5) $\mathbb{E}\exp(\lambda X - \eta_\lambda \tau) = 1$, given by Lemma 3.5; $\eta_0 = 0$, and $\eta_\lambda > 0$ for $\lambda \ne 0$.

Lemma 4.8. *$\eta_\lambda = \frac{1}{2}\lambda^2 + o(\lambda^2)$ as $\lambda \to 0$.*

Proof. If $a > \frac{1}{2}$ then by Lemma 4.7, $\mathbb{E}\exp(\lambda X - a\lambda^2\tau) < 1$ and therefore $\eta_\lambda < a\lambda^2$ for all $\lambda \ne 0$ small enough. Similarly, if $a < \frac{1}{2}$ then $\eta_\lambda > a\lambda^2$ for all $\lambda \ne 0$ small enough. □

Lemma 4.9. *The function $\lambda \mapsto \mu_\lambda$ is continuous from $\mathbb{R}$ to the space of measures with the norm topology.*

Proof. Continuity on $(-\infty, 0) \cup (0, \infty)$ holds by Lemma 3.6. The same proof gives now continuity at 0 due to Lemma 4.5 (and 4.8). □

Lemma 4.10. *The set $\{\mu_\lambda : \lambda \in [-C, C]\}$ satisfies the conditions of Theorem 2.1 whenever $0 < C < \infty$.*

Proof. We repeat the proof of Lemma 3.7 using Lemma 4.9 instead of 3.6, and 4.5 instead of 3.4. □

Proof of Theorem 1.4. The first claim is ensured by Lemma 4.8. For the second claim, the proof of Theorem 1.3 needs only trivial modifications: $[-C, C]$ and related results of Sect. 4 are used instead of $[-C, -c] \cup [c, C]$ and related results of Sect. 3. □

Proof of Theorem 1.2. By the well-known Gärtner(-Ellis) argument it is sufficient to prove that

$$\lim_{t\to\infty,\lambda\to 0,\lambda^2 t\to\infty} \frac{1}{\lambda^2 t} \ln \mathbb{E}\exp \lambda S(t) = \frac{1}{2}\,.$$

By Theorem 1.4,

$$\left| \frac{1}{\lambda^2 t} \ln \mathbb{E}\exp \lambda S(t) - \frac{\eta_\lambda}{\lambda^2} \right| = O\Big(\frac{1}{\lambda^2 t}\Big)$$

as $t \to \infty$, uniformly in $\lambda \in [-C, 0) \cup (0, C]$. By Lemma 4.8,

$$\left| \frac{\eta_\lambda}{\lambda^2} - \frac{1}{2} \right| \to 0$$

as $\lambda \to 0$. □

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