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Edgeworth expansion for the integrated Lévy driven Ornstein-Uhlenbeck process

Hiroki Masuda* Nakahiro Yoshida†

Abstract

We verify the Edgeworth expansion of any order for the integrated ergodic Lévy driven Ornstein-Uhlenbeck process, applying a Malliavin calculus with truncation over the Wiener-Poisson space. Due to the special structure of the model, each coefficient of the expansion can be given in a closed form.

Keywords: Edgeworth expansion; mixing property; Lévy driven Ornstein-Uhlenbeck process.
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1 Introduction

Let $(X,Y) = \{(X_t, Y_t)\}_{t\in\mathbb{R}_+}$ be the bivariate model described by

$$\begin{cases} X_t = X_0 - \lambda \int_0^t X_s ds + Z_t, \\ Y_t = \int_0^t (\gamma + \beta X_s) ds + \rho Z_t, \end{cases} \tag{1.1}$$

where $Z = (Z_t)_{t\in\mathbb{R}_+}$ is a non-trivial Lévy process independent of the initial variable X_0, and the parameter $(\lambda, \gamma, \beta, \rho) \in (0,\infty) \times \mathbb{R} \times (\mathbb{R}\backslash\{0\}) \times \mathbb{R}$ satisfies that

$$\beta + \rho\lambda \neq 0. \tag{1.2}$$

The process X is the exponentially ergodic Lévy driven Ornstein-Uhlenbeck (OU) process; we refer to [4] and the references therein for fundamental facts concerning the OU process. The goal of this note is to provide conditions under which the Edgeworth expansion of the expectation $E[f(T^{-1/2}H_T)]$ as $T \to \infty$ is valid, where

$$H_T := Y_T - E[Y_T] \tag{1.3}$$

and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function of at most polynomial growth. The condition (1.2) will turn out to be necessary for the Gaussian limit of $\mathcal{L}(T^{-1/2}H_T)$ to be non-degenerate: as a matter of fact, the necessity of (1.2) can be seen concisely by the expression

$$T^{-1/2}H_T = (\beta + \rho\lambda)T^{-1/2} \int_0^T (X_t - E[X_t])dt + \rho T^{-1/2} \left\{(X_t - E[X_t]) - (X_0 - E[X_0])\right\},$$

*Institute of Mathematics for Industry, Kyushu University, Japan. E-mail: hiroki@imi.kyushu-u.ac.jp
†Graduate School of Mathematical Sciences, University of Tokyo, Japan.
E-mail: nakahiro@ms.u-tokyo.ac.jp

so that, if $\beta+\rho\lambda=0$ and $(X_t-E[X_t])-(X_0-E[X_0])=O_p(1)$ as $T\to\infty$, then $\mathcal{L}(T^{-1/2}H_T)$ tends in probability to 0 (See Section 2.2).

As is well known, distributional regularity of the underlying model is essential to the validity of the Edgeworth expansion. At first glance, the regularity of the joint distribution $\mathcal{L}(X,H)$, which will play an essential role in derivation of the expansion (see Section 3), does not seem enough since we have only one-dimensional random input Z against the two-dimensional objective (X,H). In particular, for pure-jump Z we have to take distributional regularity over the Poisson space into account, rendering the problem mathematically interesting in its own right. In this case, we will execute the Malliavin calculus under truncation, which enables us to successfully pick out a nice event on which the integration by parts formula can apply to ensure distributional regularity; more specifically, our truncation functional will be constructed through two diffusive jumps, so as to make the Malliavin covariance matrix associated with the flow of (X,H) non-degenerate (As will be mentioned in Section 3.4, a single jump is not enough). The Malliavin calculus conveniently enables us to bypass intractable direct estimate of the characteristic function of $\mathcal{L}(T^{-1/2}H_T)$, and results in fairly simple conditions.

Our result has the following statistical implication. Suppose that we can directly observe $\{X_t : 0\le t\le T\}$, based on which we want to estimate $\theta_0 := E[X_0]$ (the mean of the stationary distribution). A natural estimator is then given by

$$\hat{\theta}_T := \frac{1}{T}\int_0^T X_s ds$$

We easily see that $T^{-1/2}H_T = T^{1/2}(\hat{\theta}_T-\theta_0)$ with $\beta=1$ and $\gamma=\rho=0$, hence the consistency, asymptotic normality, and higher order expansion of $\hat{\theta}_T$ are obtained according to our result.

The distributional property of the integrated OU process $X_T^* := \int_0^T X_t dt$, especially its tail behavior, has been investigated in [2]. There, especially motivated by the OU-based stochastic volatility model, the authors provided several concrete examples of positive OU processes for which the tail behavior of $\mathcal{L}(X_T^*)$ for fixed T resembles that of $\mathcal{L}(X_T)$; it was done by looking at the tail of the Lévy measures. The tail approximation discussed in [2] is typically better for smaller $\lambda>0$. Turning to the present study, our Theorem 2.3 provides the different perspective in the different setting: we here provide a unified way of improving the central-limit effect over long period through $T\to\infty$.

Section 2 presents the main result, followed by the proof in Section 3.

2 Edgeworth expansion

2.1 Statement of result

We are given a stochastic basis $(\Omega,\mathcal{F},\mathbf{F}=(\mathcal{F}_t)_{t\in\mathbb{R}_+},P)$, on which our processes are defined.

Assumption 2.1. *X is strictly stationary with a stationary distribution F admitting moments of any order.*

It is known that X is exponentially β-mixing and ergodic under Assumption 2.1; see [4] for more details.

Denote by (b,C,Π) the generating triplet of Z in the form

$$\varphi(u;Z_t)=\exp\left\{t\left(ibu-\frac{1}{2}Cu^2+\int_{\mathbb{R}}(e^{iuz}-1-iuz)\Pi(dz)\right)\right\},$$

where $b \in \mathbb{R}$, $C \geq 0$, and the Lévy measure Π defined on $\mathbb{R}$ is a σ-finite measure satisfying $\Pi(\{0\}) = 0$ and $\int (z^2 \wedge 1)\Pi(dz) < \infty$. Then the process H of (1.3) satisfies

$$dH_t = \beta(X_t - \kappa_F^{(1)})dt + \rho d\bar{Z}_t, \qquad H_0 = 0,$$

where $\bar{Z}_t := Z_t - E[Z_t] = Z_t - E[Z_1]t$ and

$$\kappa_\xi^{(k)} := i^{-k}\partial_u^k \log E[\exp(iu\xi)]\big|_{u=0},$$

the k-th cumulant of ξ, with ∂_v denoting the (partial) differentiation with respect to a variable v.

Denote by Λ the Poisson random measure associated with jumps of Z. We decompose it as

$$\Lambda(dt, dz) = \mu^\flat(dt, dz) + \mu(dt, dz)$$

for some Poisson random measures $\mu^\flat$ and μ; by the independently scattered property of Λ, such a decomposition is always possible. Correspondingly, we write

$$\Pi(dz) = \nu^\flat(dz) + \nu(dz),$$

where $\nu^\flat$ and ν stand for the Lévy measures on $\mathbb{R}_+$ associated with $\mu^\flat$ and μ, respectively.

Assumption 2.2. *Either one of the following two conditions holds true:*

(i) $C > 0$ *(no condition is imposed on the jump-part characteristic);*

(ii) $C = 0$ *and there exists a non-empty open subset of* $\mathbb{R}\backslash\{0\}$ *on which* ν *admits a positive* $\mathcal{C}^3$*-density, say* g*, with respect to the Lebesgue measure.*

Note that Assumption 2.2 puts no restriction on the structure of $\nu^\flat$.

Let us introduce the notation necessary for the Edgeworth expansion; see [6] for more details. We introduce the r-th cumulant function of $T^{-1/2}H_T$ ($r \in \mathbb{N}$, $r \geq 2$):

$$\chi_{r,T}(u) := \partial_u^r \log E\left[\exp(iuT^{-1/2}H_T)\right].$$

Let $p \geq 3$ be an integer. The $(p-2)$*-th Edgeworth expansion* $\Psi_{p,T}$ (a signed measure) is defined by the Fourier inversion of $u \mapsto \hat{\Psi}_{p,T}(u)$, where

$$\hat{\Psi}_{p,T}(u) := \exp\left(\frac{1}{2}\chi_{T,2}(u)\right) + \sum_{r=1}^{p-2} T^{-r/2}\tilde{P}_{r,T}(u),$$

with $\tilde{P}_{r,T}(u)$ specified via the formal expansion

$$\exp\left(\sum_{r=2}^{\infty}\frac{1}{r!}\chi_{r,T}(u)\right) = \exp\left(\frac{1}{2}\chi_{2,T}(u)\right) + \sum_{r=1}^{\infty} T^{-r/2}\tilde{P}_{r,T}(u).$$

Let $\phi(\cdot;\Sigma)$ stand for the one-dimensional centered normal density having variance $\Sigma > 0$, then the r-th Hermite polynomial associated with $\phi(\cdot;\Sigma)$ is

$$h_r(y;\Sigma) := (-1)^r \phi(y;\Sigma)^{-1}\partial_y^r \phi(y;\Sigma).$$

Let

$$\chi_{r,T} := (-i)^r \chi_{r,T}(0),$$

the r-th cumulant of $T^{-1/2}H_T$; in Section 2.2, we will see that $\chi_{r,T} = O(T^{-(r-2)/2})$ as $T \to \infty$. The density of $\Psi_{p,T}$ with respect to the Lebesgue measure is given by

$$g_p(y; T^{-1/2}H_T) = \left\{ 1 + \sum_{k=1}^{p-2} \sum_{l=1}^{k} \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}: \\ k_1+\cdots+k_l=k}} \frac{\chi_{k_1+2,T} \cdot \cdots \cdot \chi_{k_l+2,T}}{l!(k_1+2)! \cdots (k_l+2)!} h_{k+2l}(y; \Sigma_T) \right\} \phi(y; \Sigma_T),$$

where $\Sigma_T := \chi_{2,T}$; we will approximate $E[f(T^{-1/2}H_T)]$ by

$$\Psi_{p,T}[f] := \int f(y) g_p(y; T^{-1/2}H_T) dy.$$

Let $p_0 := 2[p/2]$ and denote by $\mathcal{E}(M, p_0)$ the set of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x)| \le M(1 + |x|^{p_0})$ for every $x \in \mathbb{R}$.

Now we can state the main result.

Theorem 2.3. *Let X, Y, H be given through (1.1) and (1.3), and suppose that (1.2) and Assumptions 2.1 and 2.2 hold true. Fix any positive number Σ^0 such that*

$$\Sigma^0 > \frac{2}{\lambda}(\beta + \rho\lambda)^2 \kappa_F^{(2)}.$$

Then, for any $M, K > 0$, there exist positive constants M^* and δ^* such that

$$\left| E[f(T^{-1/2}H_T)] - \Psi_{p,T}[f] \right| \le M^* \int_{\mathbb{R}} \sup_{|y| \le T^{-K}} |f(x+y) - f(x)| \phi(x; \Sigma^0) dx + o(T^{-(p-2+\delta^*)/2}) \tag{2.1}$$

for $T \to \infty$ uniformly in $f \in \mathcal{E}(M, p_0)$.

Most often in practice, the first term in the upper bound in (2.1) can be quickly vanishing by taking K large; for example, it is the case when f is an indicator function $f = 1_A$ for various $A \subset \mathbb{R}$, such as $A = (-\infty, a]$, $A = [a, b]$, and so on.

2.2 Explicit coefficients

The approximating density $g_p(\cdot; T^{-1/2}H_T)$ involves the cumulants $\chi_{2,T}, \chi_{3,T}, \dots, \chi_{p,T}$. We here prove the explicit formula for them.

Noticing the explicit solution $X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-s)} dZ_s$, we can apply the stochastic Fubini theorem to obtain the relation

$$\int_0^t X_s ds = \eta(\lambda, t) X_0 + \int_0^t \eta(\lambda, t-s) dZ_s, \tag{2.2}$$

where $\eta(\lambda, u) = \lambda^{-1}(1 - e^{-\lambda u})$; one can consults [2] for a detailed analysis of integrated OU processes, especially in the context of financial econometrics. It follows from (1.1), (2.2), and the special relation $k\lambda\kappa_F^{(k)} = \kappa_{Z_1}^{(k)}$ for $k \in \mathbb{N}$ (see [1, 4]) that we can express H_T as

$$H_T = \beta\eta(\lambda, T)X_0 - T(\beta + \rho\lambda)\kappa_F^{(1)} + \int_0^T \{\rho + \beta\eta(\lambda, T-s)\} dZ_s.$$

Hence, using the independence between X_0 and Z we obtain

$$\begin{aligned} \chi_{r,T} &= (-i)^r \left[\partial_u^r \kappa\left(\beta T^{-1/2}\eta(\lambda, T)u; F\right) + \int_0^T \partial_u^r \kappa\left(\{\rho + \beta\eta(\lambda, T-s)\}T^{-1/2}u; Z_1\right) ds \right] \Bigg|_{u=0} \\ &= \left\{\beta T^{-1/2}\eta(\lambda, T)\right\}^r \kappa_F^{(r)} + \int_0^T \left(\{\rho + \beta\eta(\lambda, v)\}T^{-1/2}\right)^r dv \lambda r \kappa_F^{(r)} \\ &= T^{-(r-2)/2} \left[T^{-1}\{\beta\eta(\lambda, T)\}^r + \lambda r T^{-1} \int_0^T \{\rho + \beta\eta(\lambda, v)\}^r dv \right] \kappa_F^{(r)}, \end{aligned}$$

∎

where $\kappa(\cdot;\xi)$ denotes the cumulant function of ξ. By making use of the differential equation $\partial_s \{\eta(\lambda,s)\}^k = k\{\eta(\lambda,s)\}^{k-1} - \lambda k \{\eta(\lambda,s)\}^k$ with $\eta(\lambda,0)=0$ and then integrating the both sides with respect to s over $[0,T]$, we can proceed as in [5, Section 3] to conclude that

$$\chi_{r,T} = T^{-(r-2)/2}\left[T^{-1}\{\beta\eta(\lambda,T)\}^r + \lambda r \sum_{j=0}^{r}\binom{r}{j}\rho^{r-j}\beta^j \mathcal{M}_{r,T}(j)\right]\kappa_F^{(r)}, \tag{2.3}$$

where $\mathcal{M}_{r,T}(j)$ is given by

$$\mathcal{M}_{r,T}(0) = 1,$$
$$\mathcal{M}_{r,T}(j) = \lambda^{-j} - T^{-1}\lambda^{-(j+1)}\sum_{k=1}^{j} k^{-1}\{\lambda\eta(\lambda,T)\}^k, \quad j \geq 1.$$

Thus we can explicitly write down the coefficients of the Edgeworth expansion $\Psi_{p,T}$ up to any order. It is obvious from (2.3) that $\chi_{r,T} = O(T^{-(r-2)/2})$ for $r \geq 2$;

$$T^{(r-2)/2}\chi_{r,T} \to \lambda r \sum_{j=0}^{r}\binom{r}{j}\rho^{r-j}\beta^j\lambda^{-j}\kappa_F^{(r)}.$$

In particular, we get

$$\Sigma_T = \chi_{2,r} \to 2\lambda^{-1}(\beta+\rho\lambda)^2\kappa_F^{(2)},$$

hence the necessity of the condition (1.2).

3 Proof of Theorem 2.3

We will apply [6, Theorem 1]. In order to ensure distributional regularity necessary for the Edgeworth expansion, we will make use of a Malliavin calculus with an effective truncation functional. The main idea of the proof is in principle similar to that of [5, Section 4] treating the stochastic volatility model, where X expresses the latent positive volatility process. However, the OU process X in the present model can take negative values too, so that the way of constructing a truncation functional is essentially different from that of [5]. To save space, we will sometimes omit the technical details, referring to the pertinent parts of [3, 5].

Let us briefly overview the fundamental device. By means of [6, Theorem 1], in order to deduce Theorem 2.3 it suffices to verify the following conditions:

- $[A1]$ *X is strongly mixing with exponential rate;*
- $[A2]$ $\sup_{t\in[0,T]}\|H_t\|_{L^{p+1}(P)} < \infty$ *for each* $T \in \mathbb{R}_+$;
- $[A3]$ *there exist positive constants t^0, a, a' and B, and a truncation functional $\psi : (\Omega,\mathcal{F}) \to ([0,1],\mathcal{B}([0,1]))$ such that $0<a,a'<1$ and $4a' < (a-1)^2$, and that*

$$E\left[\sup_{|u|\geq B}\left|E[\psi\exp(iuH_{t^0})|X_0,X_{t^0}]\right|\right] < a',$$
$$1 - E[\psi] < a.$$

As was mentioned in Section 2, Assumption 2.1 ensures $[A1]$ and $[A2]$ (see (2.3)), so that it remains to verify $[A3]$, which is a version of *conditional Cramér conditions*. Although it may be difficult in general to verify $[A3]$, we will be able to construct a specific truncation ψ which significantly simplify the task.

We also note that the condition $(\tilde{A}'-4)$ of [3, p. 60 and p.130] (smoothness of the coefficients, and integrability under cut-off through an auxiliary function) is indispensable. We will mention this point in Section 3.2

3.1 Transformation of the Poisson random measure

In order to execute a Malliavin calculus of [3], we introduce a transformation of the absolutely continuous part of the Poisson random measure.

Under Assumption 2.1, Z admits the Lévy-Itô decomposition of the form

$$Z_t = \lambda\kappa_F^{(1)} t + \sqrt{C}\tilde{w}_t + \int_0^t\int_{\mathbb{R}} z\tilde{\mu}^\flat(ds, dz) + \int_0^t\int_{\mathbb{R}} z\tilde{\mu}(ds, dz), \quad t \in \mathbb{R}_+,$$

where $\tilde{w}$ stands for a one-dimensional Wiener process defined on $(\Omega, \mathcal{F}, \mathbf{F}, P)$,

$$\tilde{\mu}^\flat(dt, dz) := \mu^\flat(dt, dz) - \nu^\flat(dz)dt,$$

and $\tilde{\mu}(dt, dz) := \mu(dt, dz) - \nu(dz)dt$.

Assumption 2.2 assures the existence of a bounded domain

$$E_0 = (c_1, c_2) \subset \mathbb{R}\backslash\{0\},$$

for which the Lévy density g of ν satisfies that

$$\inf_{z\in E_0} g(z) > 0.$$

Without loss of generality, we may and do suppose that $0 < c_1 < c_2$: if $\nu(\mathbb{R}_+) \equiv 0$, then take $-Z$ as Z anew. We introduce the change of variables $z^* = z^*(z) = g^+(z)$ through $z^* = z^*(z) = \int_z^{c_2} g(v)dv$ for $z \in E_0$; obviously, g^+ is strictly decreasing on E_0. Let g^- denote the strictly decreasing inverse function of g^+ defined on

$$E = (g^+(c_2), g^+(c_1)).$$

Let μ^* denote the integer-valued random measure defined by

$$\int_0^t\int_{a_1}^{a_2} h(s, z)\mu(ds, dz) = \int_0^t\int_{g^+(a_2)}^{g^+(a_1)} h(s, g^-(z^*))\mu^*(ds, dz^*)$$

for each $t \in \mathbb{R}_+$, $a_1, a_2 \in \mathbb{R}$ such that $a_1 < a_2$, and for any measurable function h on $\mathbb{R}_+ \times \mathbb{R}_+$; in particular,

$$E[\mu^*([0, t], B)] = t\mathrm{Leb}(B).$$

Writing $\tilde{\mu}^*(dt, dz^*) = \mu^*(dt, dz^*) - dtdz^*$, we transform μ (on $[0, t] \times E_0$) into μ^* as follows:

$$\int_0^t\int_{c_1}^{c_2} z\tilde{\mu}(ds, dz) = \int_0^t\int_{g^+(c_2)}^{g^+(c_1)} g^-(z^*)\tilde{\mu}^*(ds, dz^*).$$

The bivariate process (X, H) satisfies the stochastic differential equation

$$\begin{aligned}\begin{pmatrix} dX_t \\ dH_t \end{pmatrix} &= (\kappa_F^{(1)} - X_t)\begin{pmatrix} \lambda \\ -\beta \end{pmatrix}dt + \sqrt{C}\begin{pmatrix} 1 \\ \rho \end{pmatrix}d\tilde{w}_t \\ &\quad + \int_{\mathbb{R}} z\begin{pmatrix} 1 \\ \rho \end{pmatrix}(\tilde{\mu}^\flat + 1_{E_0^c}\tilde{\mu})(dt, dz) + \int_{E\cup[g^+(c_1),\infty)} J(z^*)\begin{pmatrix} 1 \\ \rho \end{pmatrix}\tilde{\mu}^*(dt, dz^*),\end{aligned} \tag{3.1}$$

where $J(z^*) := g^-(z^*)1_E(z^*)$ for $z^* \in E \cup [g^+(c_1), \infty)$. As g^- is strictly decreasing, we have $|\partial J(z^*)| > 0$ for $z^* \in E \cup [g^+(c_1), \infty)$.

3.2 Malliavin covariance matrix

Fix any constant $t^0 > 0$ and define $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$ to be the Wiener-Poisson canonical space (see [5, the last paragraph in page 1178]), on which we are given the flow $(X(\cdot, v), H(\cdot, v))^\top$ associated with (X, H) of (3.1) starting from $v = (x, h)^\top \in \mathbb{R}^2$:

$$\begin{pmatrix} X(t,v) \\ H(t,v) \end{pmatrix} = \begin{pmatrix} x \\ h \end{pmatrix} + \int_0^t (\kappa_F^{(1)} - X(s,v)) \begin{pmatrix} \lambda \\ -\beta \end{pmatrix} ds + \sqrt{C} \begin{pmatrix} 1 \\ \rho \end{pmatrix} \tilde{w}_t$$
$$+ \int_0^t \int_{\mathbb{R}} z \begin{pmatrix} 1 \\ \rho \end{pmatrix} (\tilde{\mu}^\flat + 1_{E_0^c} \tilde{\mu})(ds, dz) + \int_0^t \int_{E \cup [g^+(c_1), \infty)} J(z^*) \begin{pmatrix} 1 \\ \rho \end{pmatrix} \tilde{\mu}^*(ds, dz^*).$$

Under the present assumption, the flow $(X(\cdot, \hat{v}), H(\cdot, \hat{v}))^\top$ clearly satisfies the condition $(\tilde{A}' - 4)$.

Let $\hat{x}$ be a random variable independent of $(\tilde{w}, \mu^\flat + 1_{E_0^c}\mu, \mu^*)$ such that $\mathcal{L}(\hat{x}|\hat{P}) = F$ (the distribution under $\hat{P}$), and $\hat{v} := (\hat{x}, 0)^\top$. We will compute the Malliavin covariance matrix of $(X(t^0, \hat{v}), H(t^0, \hat{v}))^\top$, whose “non-degeneracy” is essential here.

Let $Q \in \mathbb{R}^2 \otimes \mathbb{R}^2$ be given by

$$Q = \begin{pmatrix} -\lambda & 0 \\ \beta & 0 \end{pmatrix}.$$

In view of (3.1), the process $K(t,v) := \partial_v (X(t,v), H(t,v))^\top$ satisfies that, for each v,

$$\frac{d}{dt} K(t,v) = \begin{pmatrix} -\lambda \partial_x X(t,v) & 0 \\ \beta \partial_x X(t,v) & 0 \end{pmatrix} = QK(t,v),$$

so that

$$K(t^0, \hat{v}) = \exp(t^0 Q) = \begin{pmatrix} e^{-\lambda t^0} & 0 \\ \beta \lambda^{-1} (1 - e^{-\lambda t^0}) & 1 \end{pmatrix}.$$

We note that, different from [5, Eq.(25) in page 1180], $K(\cdot, \hat{v})$ is free of $\hat{v}$.

Pick positive constants c_j' and c_j'' ($j = 1, 2$) in such a way that $0 < c_1 < c_1' < c_1'' < c_2'' < c_2' < c_2 < \infty$, and let

$$\check{E} := \left(g^+(c_2''), g^+(c_1'') \right).$$

Then, trivially $\check{E} \Subset E$. Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be any bounded smooth function satisfying the conditions:

(i) $\inf_{z^* \in \check{E}} \eta(z^*) > 0$;
(ii) $\eta(z^*) = 0$ for $z^* \notin (g^+(c_2'), g^+(c_1'))$.

The Malliavin covariance matrix of $(X(t^0, \hat{v}), H(t^0, \hat{v}))^\top$ is then well-defined and given by

$$\begin{aligned} U(t^0, \hat{v}) &:= K(t^0, \hat{v}) S(t^0, \hat{v}) K(t^0, \hat{v})^\top \\ &= \exp(t^0 Q) S(t^0, \hat{v}) \exp(t^0 Q^\top), \end{aligned}$$

where

$$\begin{aligned} S(t, \hat{v}) := C \int_0^t \exp(-sQ) \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} \exp(-sQ^\top) ds \\ + \int_0^t \int_E \exp(-sQ) \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} \exp(-sQ^\top) V(z^*) \mu^*(ds, dz^*), \end{aligned} \tag{3.2}$$

with $V(z^*) := \{\partial J(z^*)\}^2 \eta(z^*)$; see [3, Section 10] for details of (3.2). Thus we arrive at the identity

$$\det U(t^0, \hat{v}) = e^{-2\lambda t^0} \det S(t^0, \hat{v}), \quad \text{a.s.} \tag{3.3}$$

3.3 Completion of the proof under Assumption 2.2 (i)

Suppose that $C > 0$. It follows from (3.2) that, in the matrix sense,

$$
\begin{aligned}
S(t^0, \hat{v}) &\geq C \int_0^{t^0} e^{-sQ} \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} e^{-sQ^\top} ds \\
&= \begin{pmatrix} H_2 & \text{sym.} \\ \chi H_1 - (\beta/\lambda) H_2 & \chi^2 t^0 - 2(\beta/\lambda)\chi H_1 + (\beta/\lambda)^2 H_2 \end{pmatrix}
\end{aligned}
$$

where $H_k := \int_0^{t^0} e^{k\lambda s} ds$ and $\chi := \rho + \beta/\lambda$. The determinant of the rightmost side is

$$
C^2 \lambda^{-4} (\beta + \rho\lambda)^2 \left\{ \frac{\lambda t^0}{2} (e^{2\lambda t^0} - 1) - (e^{\lambda t^0} - 1)^2 \right\},
$$

which is positive as soon as $t^0 \lambda \neq 0$ and $\beta + \rho\lambda \neq 0$. Thus $S(t^0, \hat{v})$ is bounded from below by a positive-definite matrix, hence the non-degeneracy of $U(t^0, \hat{v})$ follows from (3.3) without any non-trivial truncation functional; simply let $\psi \equiv 1$ in $[A3]$. Thus we have obtained the non-degeneracy of the Malliavin covariance matrix (i.e. enough integrability of $\{\mathrm{det} U(t^0, \hat{v})\}^{-1}$), which corresponds to [5, Lemma 6].

We further notice the following.

- The flow $(X(t, \hat{v}), H(t, \hat{v}))_{t \in [0, t^0]}$ satisfies the condition $(\tilde{A}' - 4)$ (as was seen in Section 3.2), hence the analogous assertions as [5, Lemmas 7] holds true.
- Following the same argument as in [5, pp.1184–1185], we see that there exists a random variable $\Phi'_{t^0} \in L^1(\hat{P})$ such that

$$
E\left[\sup_{|u| \geq B} |E[\exp(iuH_{t^0}) | X_0, X_{t^0}]| \right] \leq \frac{1}{B} \hat{E}[|\Phi'_{t^0}|]
$$

  for every $B > 0$.

After all, we have deduced the analogous assertions to [5, Lemmas 6, 7 and 8], completing the proof of Theorem 2.3 under Assumption 2.1 and Assumption 2.2 (i).

3.4 Construction of a truncation functional

It remains to prove Theorem 2.3 under Assumptions 2.1 and 2.2 (ii). Then, in order to verify distributional regularity we have to make an effective use of jumps. We will construct the truncation functional ψ in an explicit way through *two* diffusive jumps.

We continue the argument of Section 3.2. Let $t_1, t_2 \in (0, t^0)$ be constants such that $t_1 < t_2$, and fix $z_0 \in \check{E}$. Let $\epsilon > 0$ be sufficiently small so that:

- $\overline{I_1^\epsilon} \cap \overline{I_2^\epsilon} = \emptyset$ for $I_j^\epsilon := (t_j - \epsilon, t_j + \epsilon)$, $j = 1, 2$;
- $g^+(c_2'') < z_0 - \epsilon < z_0 + \epsilon < g^+(c_1'')$.

Let $E^\epsilon := (z_0 - \epsilon, z_0 + \epsilon)$ and

$$
\mathcal{A}^\epsilon := \{\mu^*(I_j^\epsilon, E^\epsilon) = 1 \text{ for } j = 1, 2.\}. \tag{3.4}
$$

According to the independently scattered property of μ^* and since the Lévy measure associated with μ^* (over E) here is the Lebesgue measure, we have

$$
\hat{P}[\mathcal{A}^\epsilon] = \left\{ \hat{P}\left[\mu^*\left([0, 2\epsilon], [0, 2\epsilon]\right)\right] \right\}^2 = \left\{ 4\epsilon^2 \exp(-4\epsilon^2) \right\}^2 > 0
$$

for each $\epsilon > 0$.

Now, we define the truncation functional $\hat{\psi}_\epsilon$ by $\hat{\psi}_\epsilon = \zeta(\hat{\xi}_\epsilon)$, where $\zeta : \mathbb{R}_+ \to [0,1]$ is a non-increasing smooth function such that $\zeta(x) = 1$ if $0 \le x \le 1/2$ and $\zeta(x) = 0$ if $x \ge 1$, where

$$\hat{\xi}_\epsilon = \frac{2}{1 + 3\mathrm{det}U(t^0, \hat{v})}. \tag{3.5}$$

We will show that the Malliavin covariance matrix $U(t^0, \hat{v})$ is non-degenerate on the event $\mathcal{A}^\epsilon$ for any $\epsilon > 0$ small enough.

Noting that

$$\sup_{s:|s-t_j|\le\epsilon} \left| e^{-sQ} - \begin{pmatrix} e^{\lambda t_j} & 0 \\ \beta\lambda^{-1}(1 - e^{\lambda t_j}) & 1 \end{pmatrix} \right| + \sup_{z:|z-z_0|\le\epsilon} |V(z) - V(z_0)| \to 0$$

as $\epsilon \to 0$ and by virtue of (3.4), we apply Taylor's expansion around z_0 and t_j ($j = 1, 2$) on $\mathcal{A}^\epsilon$ to conclude that

$$\begin{aligned} S(t^0, \hat{v}) &\ge \sum_{j=1}^{2} \int_{I_j^\epsilon} \int_{E^\epsilon} e^{-sQ} \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} e^{-sQ^\top} V(z^*) \mu^*(ds, dz^*) \\ &= \sum_{j=1}^{2} e^{-t_j Q} \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} e^{-t_j Q^\top} V(z_0) + o(1) \\ &= V(z_0) M^\epsilon + o(1) \end{aligned}$$

as $\epsilon \to 0$ (we used the symbol $o(1)$ for matrices too), where

$$M^\epsilon := \begin{pmatrix} J^{(2)} & \text{sym.} \\ (\rho + \beta\lambda^{-1})J^{(1)} - \beta\lambda^{-1}J^{(2)} & 2(\rho + \beta\lambda^{-1})^2 - 2\beta\lambda^{-1}(\rho + \beta\lambda^{-1})J^{(1)} + \beta^2\lambda^{-2}J^{(2)} \end{pmatrix}$$

with $J^{(1)} := e^{\lambda t_1} + e^{\lambda t_2}$ and $J^{(2)} := e^{2\lambda t_1} + e^{2\lambda t_2}$. Therefore

$$\mathrm{det}S(t^0, \hat{v}) \ge V(z_0)^2 \lambda^{-2} (\beta + \lambda\rho)^2 (e^{\lambda t_1} - e^{\lambda t_2})^2 + o(1),$$

which is positive for ϵ sufficiently small whenever $\rho\lambda + \beta \ne 0$ and $t_1 \ne t_2$. [We note that a single jump is not enough: if we instead estimate $S(t^0, \hat{v})$ as

$$S(t^0, \hat{v}) \ge e^{-t_1 Q} \begin{pmatrix} 1 & \rho \\ \rho & \rho^2 \end{pmatrix} e^{-t_1 Q^\top} V(z_0) + o(1),$$

then the determinant of the first term in the right-hand side is identically 0.]

We may set $V(z_0)$ arbitrarily large by choosing the function η suitably. Hence, recalling (3.3) we conclude that $\mathrm{det}U(t^0, \hat{v}) \ge 1$ on $\mathcal{A}^\epsilon$ for some $\epsilon > 0$. The definition (3.5) then leads to the estimate

$$\hat{P}[\hat{\xi}_\epsilon \le 1/2] \ge \hat{P}\left[\left\{ \mathrm{det}U(t^0, \hat{v}) \ge 1 \right\} \cap \mathcal{A}^\epsilon \right] = \hat{P}[\mathcal{A}^\epsilon] > 0,$$

hence the assertion corresponding to [5, Lemma 6] holds true.

We keep using the η and $\epsilon > 0$ chosen in the last paragraph. Clearly, $\hat{\psi}_\epsilon > 0$ implies that $1/3 \le \mathrm{det}U(t^0, \hat{v})$, hence

$$\hat{\psi}_\epsilon \left\{ \mathrm{det}U(t^0, \hat{v}) \right\}^{-1} \in \bigcap_{0<p<\infty} L^p(\hat{P}).$$

This implies that the integration-by-parts formula under the truncation $\hat{\psi}_\epsilon$ is in force. Then, as before, we could deduce the assertions corresponding to [5, Lemmas 7 and 8]:

- The flow $(X(t,\hat{v}),H(t,\hat{v}))_{t\in[0,t^0]}$ satisfies the condition $(\tilde{A}'-4)$ (as was seen in Section 3.2);
- There exists a random variable $\Phi''_{t^0}\in L^1(\hat{P})$ such that

$$E\left[\sup_{|u|\geq B}\left|E[\hat{\psi}_\epsilon\exp(iuH_{t^0})|X_0,X_{t^0}]\right|\right]\leq\frac{1}{B}\hat{E}[|\Phi''_{t^0}|]$$

  for every $B>0$.

The proof of Theorem 2.3 is thus complete.

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