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Extremal Lipschitz functions in the deviation inequalities from the mean*

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Abstract

We obtain an optimal deviation from the mean upper bound

$$D(x) \overset{\text{def}}{=} \sup_{f \in \mathcal{F}} \mu\{f - \mathbf{E}_\mu f \geq x\}, \qquad \text{for } x \in \mathbf{R} \tag{0.1}$$

where $\mathcal{F}$ is the complete class of integrable, Lipschitz functions on probability metric (product) spaces. As corollaries we get exact solutions of (0.1) for Euclidean unit sphere S^{n-1} with a geodesic distance function and a normalized Haar measure, for $\mathbf{R}^n$ equipped with a Gaussian measure and for the multidimensional cube, rectangle, torus or Diamond graph equipped with uniform measure and Hamming distance function. We also prove that in general probability metric spaces the $\sup$ in (0.1) is achieved on a family of negative distance functions.

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1 Introduction

Let us recall a well known result for Lipschitz functions on probability metric spaces, (V, d, μ). Here a probability metric space means that a measure μ is Borel and normalized, $\mu(V) = 1$. Given measurable non-empty set $A \in V$ we denote a distance function by $d(A, u) = \min\{d(u, v), v \in A\}$. We denote by $\mathcal{F} = \mathcal{F}(V)$ the class of integrable, i.e., $f \in L_1(V, d, \mu)$, 1-Lipschitz functions $f : V \to \mathbf{R}$, such that $|f(u) - f(v)| \leq d(u, v)$ for all $u, v \in V$. We will write in short $\{f \in A\}$ instead of $\{u : f(u) \in A\}$, etc. We will say that $\mathcal{F}(V, d, \mu)$ is *complete* if it contains all 1-Lipschitz functions f defined on (V, d, μ). Note that completeness in our sense just means that the distance function $d(x, x_0)$ is μ-integrable (this property does not depend on x_0). Let M_f be a median of the function f, i.e., a number such that $\mu\{f(x) \leq M_f\} \geq \frac{1}{2}$ and $\mu\{x : f(x) \geq M_f\} \geq \frac{1}{2}$. Given probability metric space (V, d, μ), the $\sup$ in

$$\sup_{f \in \mathcal{F}} \mu\{f - M_f \geq x\} \qquad \text{for } x \in \mathbf{R}$$

is achieved on a family, say $\mathcal{F}^*$, of distance functions $f(u) = -d(A, u)$ with measurable $A \subset V$ (for a nice exposition of the results we refer reader to [18, 20]). From this it

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is easy to deduce that this problem is equivalent to the following vertex isoperimetric problem. Given $t \geq 0$ and $h \geq 0$,

$$\text{minimize } \mu(A^h) \text{ over all } A \subset V \text{ with } \mu(A) \geq t, \tag{1.1}$$

where $A^h = \{u \in V : d(u, A) \leq h\}$ is an h-enlargement of A.

Following [5] we say that a space (V, d, μ) is *isoperimetric* if for every $t \geq 0$ there exists a solution, say A_{opt}, of (1.1) which does not depend on h.

However, as was pointed out by Talagrand [25] in practice it is easier to deal with expectation $\mathbf{E}f$ rather than median M_f. In order to get results for the mean instead of median two different techniques were usually used. One way was to evaluate the distance between median and mean, another was to use a martingale technique (see [3, 17, 19, 25] for more detailed exposition of the results). Unfortunately, none of them could lead to tight bounds for the mean.

In this paper we find tight deviation from the mean bounds

$$D(x) \stackrel{\text{def}}{=} \sup_{f \in \mathcal{F}} \mu\{f - \mathbf{E}_\mu f \geq x\}, \qquad \text{for } x \in \mathbf{R} \tag{1.2}$$

for the complete class $\mathcal{F} = \mathcal{F}(V, d, \mu)$. If we change f to $-f$ we get that

$$D(x) = \sup_{f \in \mathcal{F}} \mu\{f - \mathbf{E}_\mu f \leq -x\} \qquad \text{for } x \in \mathbf{R}.$$

Note that the function $D(x)$ depends also on (V, d, μ).

We first state a general result for probability metric spaces.

Theorem 1.1. *If $\mathcal{F}(V, d, \mu)$ is complete, then* $\sup$ *in* (1.2) *is achieved on a family of negative distance functions, i.e.,*

$$\sup_{f \in \mathcal{F}} \mu\{f - \mathbf{E}_\mu f \geq x\} = \sup_{f \in \mathcal{F}^*} \mu\{f - \mathbf{E}_\mu f \geq x\} \qquad x \in \mathbf{R}.$$

Note that $\mathcal{F}^* \subset \mathcal{F}$.

Proof. Fix $x \in \mathbf{R}$. Let $f \in \mathcal{F}$ and $B = \{f - \mathbf{E}_\mu f \geq x\}$. If $B = \emptyset$, then $\mu\{f - \mathbf{E}_\mu f \geq x\} = 0$ and thus $\mu\{f - \mathbf{E}_\mu f \geq x\} \leq \mu\{f^* - \mathbf{E}_\mu f^* \geq x\}$ for any function $f^* \in \mathcal{F}^*$. Let $B \neq \emptyset$. Since $\mathbf{E}_\mu f < \infty$ and f is bounded from below by $\mathbf{E}_\mu f + x$ on B we have that $-\infty < \mathbf{E}_\mu f + x \leq \inf_{u \in B} f(u) < \infty$. Thus without loss of generality we can assume that $\inf_{u \in B} f(u) = 0$. Let g be a function such that $g(u) = 0$ on B and $g(u) = f(u)$ on B^c. It is clear that $g \in \mathcal{F}$ and $f \geq g$ on V and thus $\mathbf{E}_\mu f \geq \mathbf{E}_\mu g$. Next, $x \leq \inf_{u \in B} f(u) - \mathbf{E}_\mu f = g - \mathbf{E}_\mu f \leq g - \mathbf{E}_\mu g$ on B, so $B \subset \{g - \mathbf{E}_\mu g \geq x\}$. Let $f^*(u) = -d(B, u)$. Since g is Lipschitz function, $|g(u)| = |g(u) - g(v)| \leq d(u, v)$ for all $v \in B$, so $|g(u)| \leq d(u, B)$ and thus $g(u) \geq -d(u, B) = f^*(u)$. Again, for all $u \in B$ we have $x \leq g(u) - \mathbf{E}g = f^*(u) - \mathbf{E}g \leq f^*(u) - \mathbf{E}f^*$, so $B \subset \{f^* - \mathbf{E}_\mu f^* \geq x\}$. Thus, $\mu\{f - \mathbf{E}_\mu f \geq x\} \leq \mu\{g - \mathbf{E}_\mu g \geq x\} \leq \mu\{f^* - \mathbf{E}_\mu f^* \geq x\}$. Since f is arbitrary the statement of $Theorem$ 1.1 follows. □

In the special case when $V = \mathbf{R}^n$ and $\mu = \gamma_n$ is a standard Gaussian measure, $Theorem$ 1.1 was proved by Bobkov [9].

Our main result is the following theorem.

Theorem 1.2. *If (V, d, μ) is isoperimetric and $\mathcal{F}$ is complete, then*

$$D(x) = \mu\{f^*_{\text{opt}} - \mathbf{E}_\mu f^*_{\text{opt}} \geq x\} \qquad \text{for } x \in \mathbf{R},$$

where $f^*_{\text{opt}}(u) = -d(A_{\text{opt}}, u)$ is a negative distance function from some extremal set A_{opt}. It turns out that $\mu(A_{\text{opt}}) = D(x)$.

Proof. We have

$$\begin{aligned} \mathbf{E}_\mu d(A, \cdot) &= \int_0^\infty \left(1 - \mu\left\{A^h\right\}\right) \mathrm{dh} \\ &\leq \int_0^\infty \left(1 - \mu\left\{A^h_{\text{opt}}\right\}\right) \mathrm{dh} = \mathbf{E}_\mu d(A_{\text{opt}}, \cdot). \end{aligned} \tag{1.3}$$

Let $f^* \in \mathcal{F}^*$ and $A = \{f^* - \mathbf{E}_\mu f^* \geq x\}$. From (1.3) we get that for all $u \in A$

$$x \leq f^*(u) - \mathbf{E}_\mu f^* \leq f^*(u) - \mathbf{E}_\mu f^*_{\text{opt}},$$

where $f^*_{\text{opt}}(u) = -d(A_{\text{opt}}, u)$. Since $f^*_{\text{opt}}(u) = 0$ for all $u \in A_{\text{opt}}$ we have that $x \leq -\mathbf{E}_\mu f^*_{\text{opt}} = f^*_{\text{opt}}(u) - \mathbf{E}_\mu f^*_{\text{opt}}$ for all $u \in A_{\text{opt}}$ as well. Since f^* (or the set A) is arbitrary and $\mu\{A_{\text{opt}}\} \geq \mu\{A\}$, by Theorem 1.1 we have

$$\sup_{f \in \mathcal{F}} \mu\{f - \mathbf{E}_\mu f \leq -x\} = \sup_{f \in \mathcal{F}^*} \mu\{f - \mathbf{E}_\mu f \geq x\} = \mu\{f^*_{\text{opt}} - \mathbf{E}_\mu f^*_{\text{opt}}\},$$

which completes the proof of $Theorem$ 1.2. □

2 Isoperimetric spaces and corollaries

In this section we provide a short overview of the results on the vertex isoperimetric problem described by (1.1). We also state a number of corollaries following from $Theorem$ 1.1 and $Theorem$ 1.2.

A typical and basic example of isoperimetric spaces is the Euclidean unit sphere $S^{n-1} = \{x \in \mathbf{R}^n : \sum_{i=1}^n |x_i|^2 = 1\}$ with a geodesic distance function ρ and a normalized Haar measure σ_{n-1}. P. Lévy [16] and E. Schmidt [22] showed that if A is a Borel set in S^{n-1} and H is a cap (ball for geodesic distance function ρ) with the same Haar measure $\sigma_{n-1}(H) = \sigma_{n-1}(A)$, then

$$\sigma_{n-1}(A^h) \geq \sigma_{n-1}(H^h) \qquad \text{for all } \ h > 0. \tag{2.1}$$

Thus A_{opt} for the space $(S^{n-1}, \rho, \sigma_{n-1})$ is a cap. We refer readers for a short proof of (2.1) to [2, 11]. The extension to Riemannian manifolds with strictly positive curvature can be found in [12]. Note that if H is a cap, then H^h is also a cap, so we have an immediate corollary.

Corollary 2.1. *For a unit sphere S^{n-1} equipped with normalized Haar measure σ_{n-1} and geodesic distance function we have*

$$D(x) = \sigma_{n-1}\{f^* - \mathbf{E}_\mu f^* \geq x\} \qquad \text{for } x \in \mathbf{R},$$

where $f^*(u) = -d(A_{\text{opt}}, u)$ and A_{opt} is a cap.

Probably the most simple non-trivial isoperimetric space is n-dimensional discrete cube $C_n = \{0,1\}^n$ equipped with uniform measure, say μ, and Hamming distance function. Harper [13] proved that some number of the first elements of C_n in the simplicial order is a solution of (1.1). Bollobas and Leader [6] extended this result to multidimensional rectangle. Karachanjan and Riordan [14, 21] solved the problem (1.1) for multidimensional torus. Bezrukov considered powers of the Diamond graph [4] and powers of cross-sections [5]. We state the results for discrete spaces as corollary.

Corollary 2.2. *For discrete multidimensional cube, rectangle, torus and Diamond graph equipped with uniform measure and Hamming distance function we have*

$$D(x) = \mu\{f^* - \mathbf{E}_\mu f^* \geq x\} \qquad \textit{for } x \in \mathbf{R},$$

where $f^*(u) = -d(A_{opt}, u)$ *and* A_{opt} *are the sets of some first elements in corresponding orders. In particular, for* n*-dimensional discrete cube with Hamming distance function,* A_{opt} *is a set of some first elements of* C_n *in simplicial order.*

There is a vast of papers dedicated to bound $D(x)$ for various discrete spaces. We mention only [7, 15, 24] among others. In [4, 18] a nice overview of isoperimetric spaces and bounds for $D(x)$ are provided.

Another important example of isoperimetric spaces comes from Gaussian isoperimetric problem. Sudakov and Tsirel'son [23] and Borell [10] discovered that if γ_n is a standard Gaussian measure on $\mathbf{R}^n$ with a usual Euclidean distance function d, then $(\mathbf{R}^n, d, \gamma_n)$ is isoperimetric. In [23] and [10] it was shown that among all subsets A of $\mathbf{R}^n$ with $t \geq \gamma_n(A)$, the minimal value of $\gamma_n(A^h)$ is attained for half-spaces of measure t. Thus we have the following corollary of $Theorem\ 1.1$ and $Theorem\ 1.2$.

Corollary 2.3. *For a Gaussian space* $(\mathbf{R}^n, d, \gamma_n)$ *we have*

$$D(x) = \gamma_n\{f^* - \mathbf{E}_{\gamma_n} f^* \geq x\} \qquad \textit{for } x \in \mathbf{R},$$

where $f^*(u) = -d(A_{opt}, u)$ *is a negative distance function from a half-space of space* $\mathbf{R}^n$*.*

The latter result was firstly proved by Bobkov [9]. We also refer for further investigations of extremal sets on $\mathbf{R}^n$ for some classes of measures to [1, 8] among others.

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