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Asymptotic behavior for neutral stochastic partial differential equations with infinite delays*

Jing Cui† Litan Yan‡

Abstract

This paper is concerned with the existence and asymptotic behavior of mild solutions to a class of non-linear neutral stochastic partial differential equations with infinite delays. By applying fixed point principle, we present sufficient conditions to ensure that the mild solutions are exponentially stable in pth-moment ($p \geq 2$) and almost surely exponentially stable. An example is provided to illustrate the effectiveness of the proposed result.

Keywords: Neutral stochastic partial differential equations; exponential stability; infinite delay.
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1 Introduction

In recent years, there has been considerable interest in studying the quantitative and qualitative properties of solutions to stochastic partial differential equations (SPDEs) in a separable Hilbert space such as existence, uniqueness, stability, controllability and asymptotic behavior (see, e.g., [1, 6, 17, 18, 21, 22, 23, 24, 25] and references therein). Moreover, SPDEs with delays have also drawn much attention from many scholars. For example, Liu and Truman [18] introduced some approximating systems with strong solutions, and they presented several criteria about the asymptotic exponential stability for a class of non-autonomous stochastic evolution equations with variable delays; Caraballo and Real [7] studied the stability of the strong solutions of semilinear stochastic evolution equations with delays; Caraballo and Liu [8] considered the exponential stability of mild solutions of SPDEs with delays by utilizing the well-known Gronwall inequality, but the requirement of the monotone decreasing behavior of the delays should be imposed; Liu and Truman [13] established the exponential stability of mild solutions for SPDEs by establishing the corresponding Razuminkhin-type theorem.

However, in some cases, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays (see, e.g., [11] and [12]). Neutral stochastic differential equations with delays are often used to describe such systems. To the best of our knowledge, there is only a little systematic

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†Department of Mathematics, Anhui Normal University, P.R. China.
E-mail: jcui123@126.com
‡Department of Mathematics, College of Science, Donghua University, P.R. China.
E-mail: litanyan@hotmail.com (corresponding author)

investigation on the stability of mild solutions to neutral SPDEs with delays. It is important to note that a number of difficulties exist in the study of the stability of mild solutions to neutral SPDEs with delays due to the presence of the neutral items. For instance, the mild solutions do not have stochastic differentials, and many methods used frequently fail to deal with the stability of mild solutions for neutral SPDEs with delays. Recently, in [2, 3, 4, 5] Burton and his co-authors investigated the stability of deterministic differential equations by fixed point theory, and they pointed out that some of these difficulties can be rectified. Luo [14] applied this valuable method to deal with the stability for stochastic ordinary differential equations. We would also like to mention that some topics similar to the above for stochastic partial functional differential equations have been studied by many authors (see, e.g., [9], [15], [16], [23] and references therein).

Almost all the results on stability in the aforementioned papers are suited to stochastic differential equations with finite delay. However, as mathematical models of phenomena, neutral stochastic differential equations with infinite delays have become an important issue in both the physical and social sciences. For this, see, for example, [10, 19] with the theory development for the description of heat conduction in materials with fading memory. In this paper, motivated by the previous references, we are concerned with the existence and asymptotic behavior of mild solutions to a class of neutral SPDEs with infinite delays. Our approach is based on fixed point theorem and operator theory.

The rest of this paper is organized as follows. In Section 2, we briefly present some basic notations and preliminaries. Section 3 is devoted to the proof of the existence and asymptotic behavior of mild solutions to a class of neutral SPDEs with infinite delays. An example is provided to illustrate the effectiveness of the proposed result.

2 Preliminaries

Let $(H, \langle\cdot,\cdot\rangle_H, |\cdot|_H)$ and $(K, \langle\cdot,\cdot\rangle_K, |\cdot|_K)$ be two real, separable Hilbert spaces. Let $\mathcal{L}(K,H)$ be the space of all linear bounded operators from K to H, equipped with the usual operator norm $\|\cdot\|$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual conditions. We denote by $W = (W_t)_{t\geq 0}$ a K-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with covariance operator Q. That is,

$$E\langle w(t), x\rangle_K \langle w(s), y\rangle_K = (t\wedge s)\langle Qx, y\rangle_K, \quad \forall x, y \in K,$$

where Q is a positive, self-adjoint trace class operator on K. Furthermore, $\mathcal{L}_2^0(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K to H with the norm

$$|\xi|^2_{\mathcal{L}_2^0} := tr(\xi Q \xi^*) < \infty, \quad \xi \in \mathcal{L}(K,H).$$

For the construction of stochastic integral in Hilbert spaces, we refer to Da. Prato [21].

Let A, generally unbounded, be a linear operator from H to H. Assume that $\{S(t), t \geq 0\}$ is an analytic semigroup with its infinitesimal generator A, then it is possible ([20]) under some circumstances to define the fractional power $(-A)^\alpha$ for any $\alpha \in (0,1]$ which is a closed linear operator with domain $D((-A)^\alpha)$; moreover, the subspace $D((-A)^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = |(-A)^\alpha x|_H, \quad x \in D((-A)^\alpha),$$

defines a norm on $D((-A)^\alpha)$.

In this work, we consider the following neutral SPDEs with infinite delays of the form:

$$\begin{cases} d[x(t)+G(t,x(t-\rho(t)))]=[Ax(t)+b(t,\int_{-\infty}^{0}g(\theta,x(t+\theta))d\theta)]dt \\ \qquad\qquad + h(t,\int_{-\infty}^{0}\sigma(\theta,x(t+\theta))d\theta)dW(t), \qquad\qquad t\geq 0, \\ x(t)=\phi(t), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\leq 0, \end{cases} \tag{2.1}$$

where the mappings

$$G,b:[0,+\infty)\times H\to H,\quad h:[0,+\infty)\times H\to\mathcal{L}_2^0(K,H),\quad g,\sigma:(-\infty,0]\times H\to H$$

are all Borel measurable, $\rho(t):[0,+\infty)\to[0,r]$ is continuous, the initial value $\phi:(-\infty,0]\to H$ is an $\mathcal{F}_0$-measurable, continuous function with $E[\sup_{s\leq 0}|\phi(s)|_H^p]<\infty$.

Definition 2.1. *An H-valued stochastic process $\{x(t),t\in(-\infty,T]\}$, $0\leq T<\infty$, is said to be a mild solution to* (2.1) *if*

(a) *$x(t)$ is an $\mathcal{F}_t$-adapted, continuous process with $\int_0^T|x(t)|_H^p dt<\infty$ almost surely;*
(b) *for $t\geq 0$, $x(t)$ satisfies the following integral equation:*

$$\begin{aligned} x(t)=&S(t)[\phi(0)+G(0,\phi(-\rho(0)))]-G(t,x(t-\rho(t)))\\ &-\int_0^t AS(t-s)G(s,x(s-\rho(s)))ds\\ &+\int_0^t S(t-s)b\big(s,\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta\big)ds+\\ &+\int_0^t S(t-s)h\big(s,\int_{-\infty}^0 \sigma(\theta,x(s+\theta))d\theta\big)dW(s), \end{aligned}$$

and $x(t)=\phi(t)$ for $t\leq 0$.

Definition 2.2. *Let $p\geq 2$ be an integer. The mild solution $x(t)$ of* (2.1) *with an initial value $\phi(t)$ is said to be exponentially stable in pth-moment if there exist some constants $M_\alpha\geq 1$, $\eta>0$ such that*

$$E|x(t)|_H^p<M_\alpha E\sup_{\theta\leq 0}|\phi(\theta)|_H^p e^{-\eta t},\quad t\geq 0.$$

In particular, when $p=2$, the mild solution is said to be exponentially stable in mean square.

Definition 2.3. *The mild solution $x(t)$ of* (2.1) *is said to be almost surely exponentially stable if there exists a positive constant $\alpha>0$ such that*

$$\limsup_{t\to\infty}\frac{\ln|x(t)|_H}{t}\leq-\alpha\quad a\cdot s\cdot.$$

In order to obtain our main result, we shall impose the following assumptions:

(A_1) $A:D(A)\to H$ is the infinitesimal generator of a strong continuous semigroup of bounded linear operators $\{S(t),t\geq 0\}$ in H satisfying

$$\|S(t)\|\leq e^{-\gamma t},\quad for\quad some\quad constant\quad \gamma>0.$$

(A_2) For $p\geq 2$, there exist some constants $\alpha\in(\frac{p-1}{p},1]$ and $K_G>0$ such that for any $x,y\in H$, $t\geq 0$, $G(t,x),G(t,y)\in D((-A)^\alpha)$ and

$$|(-A)^\alpha G(t,x)-(-A)^\alpha G(t,y)|_H^p\leq K_G|x-y|_H^p.$$

(A_3) $b : [0, +\infty) \times H \to H$, $h : [0, +\infty) \times H \to \mathcal{L}_2^0(K, H)$ satisfy Lipschitz conditions, i.e., there exist some positive constants K_b, K_h such that for any $x, y \in H$, $t \in \mathbb{R}$,

$$|b(t,x) - b(t,y)|_H \le K_b|x-y|_H, \quad |h(t,x) - h(t,y)|_{\mathcal{L}_2^0} \le K_h|x-y|_H.$$

Moreover, we assume that $|b(t,0)|_H = |h(t,0)|_{\mathcal{L}_2^0} = 0$.

(A_4) There exist some positive constants L_g, L_σ such that for all $t \in \mathbb{R}$, $x, y \in H$,

$$|g(t,x) - g(t,y)|_H \le L_g e^{-\xi|t|}|x-y|_H, \quad 0 < \xi < \gamma;$$
$$|\sigma(t,x) - \sigma(t,y)|_H \le L_\sigma e^{-\xi|t|}|x-y|_H, \quad 0 < \xi < \gamma,$$

we further assume that $|G(t,0)|_H = |\sigma(t,0)|_H = |g(t,0)|_H = 0$.

Lemma 2.1 (Pazy [20]). *Let the assumption (A_1) hold. Then for any $0 < \beta \le 1$, the following equality holds:*

$$S(t)(-A)^\beta x = (-A)^\beta S(t)x, \quad x \in D\big((-A)^\beta\big),$$

and there exists a positive constant M_β such that for any $t > 0$,

$$\|(-A)^\beta S(t)\| \le M_\beta t^{-\beta} e^{-\gamma t}.$$

3 The main theorem

In this section, we establish the existence and asymptotic behavior of the mild solution to (2.1). Our main object is to explain and prove the following theorem.

Theorem 3.1. *Let $p \ge 2$ be an integer. Suppose that $(A_1) - (A_4)$ are satisfied. Suppose also that*

$$\begin{aligned} 4^{p-1}[K_G\|(-A)^{-\alpha}\|^p + M_{1-\alpha}^p K_G \gamma^{-\alpha p}\Gamma^p(\alpha) \\ + K_b^p L_g^p (\xi\gamma)^{-p} + (2\gamma)^{-\frac{p}{2}} K_h^p L_\sigma^p \xi^{-p} c_p] < 1, \end{aligned} \tag{3.1}$$

where $\Gamma(\cdot)$ is the Gamma function, $M_{1-\alpha}$ is the corresponding constant in Lemma 2.1 and $c_p = \big(\frac{p(p-1)}{2}\big)^{\frac{p}{2}}$. If an initial value $\phi(t)$ to (2.1) satisfies

$$E|\phi(t)|_H^p \le M_0 E|\phi(0)|_H^p e^{-\mu t}, \quad t \le 0,$$

for some $M_0 \ge 1$ and $0 < \mu < \xi$, then the mild solution to (2.1) exists uniquely which is exponentially stable in pth-moment.

In order to prove the theorem, we first recall a useful lemma.

Theorem A (Theorem 6.13 in Da Prato and Zabczyk [21]). *Let $r \ge 1$. Then for an arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(t)$,*

$$\sup_{s\in[0,t]} E\left|\int_0^s \Phi(u)dW(u)\right|_H^{2r} \le (r(2r-1))^r \left[\int_0^t (E|\Phi(s)|_{\mathcal{L}_2^0}^{2r})^{\frac{1}{r}} ds\right]^r.$$

Proof of Theorem 3.1. Denote by $\mathcal{S}$ the Banach space of all $\mathcal{F}_t$-adapted continuous processes $x(t)$ endowed with a norm $\|x\|_{\mathcal{S}} := \sup_{t\ge0} E|x(t)|_H^p$ such that there exist some constants $M^\star \ge 1$, $\eta > 0$ satisfying

$$E|x(t)|_H^p < M^\star E \sup_{\theta\le0} |\phi(\theta)|_H^p e^{-\eta t}, \quad t \ge 0$$

and $x(t) = \phi(t)$ for $t \le 0$, where $\phi(t)$ is the initial value of (2.1). Define a mapping $\pi : \mathcal{S} \to \mathcal{S}$ by $\pi(x)(t) = \phi(t)$ for $t \le 0$ and

$$\begin{aligned}\pi(x)(t) =& S(t)\big[\phi(0) + G(0, \phi(-\rho(0)))\big] - G(t, x(t-\rho(t))) \\ &- \int_0^t AS(t-s)G(s, x(s-\rho(s)))ds \\ &+ \int_0^t S(t-s)b\big(s, \int_{-\infty}^0 g(\theta, x(s+\theta))d\theta\big)ds \\ &+ \int_0^t S(t-s)h\big(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta\big)dW(s)\end{aligned} \tag{3.2}$$

for $t \ge 0$.

We begin by verifying the continuity of $(\pi x)(t)$ on $t \ge 0$. To this end, let $x \in \mathcal{S}$, $t_1 \ge 0$ and $|r| > 0$ be sufficiently small. Notice that

$$E|(\pi x)(t_1 + r) - (\pi x)(t_1)|_H^p \le 5^{p-1} \sum_{i=1}^5 E|I_i(t_1 + r) - I_i(t_1)|_H^p.$$

Applying Theorem A together with assumption (A_1), it follows that

$$\begin{aligned}E|I_5(t_1+r) - I_5(t_1)|^p =& E\bigg|\int_0^{t_1+r} S(t_1+r-s)h\big(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta\big)dW(s) \\ &- \int_0^{t_1} S(t_1-s)h\big(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta\big)dW(s)\bigg|_H^p \\ \le& 2^{p-1}c_p\left\{\left[\int_0^{t_1}\Big(E\big|(S(t_1+r-s) - S(t_1-s))h(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta)\big|_{\mathcal{L}_2^0}^p\Big)^{\frac{2}{p}}ds\right]^{\frac{p}{2}}\right. \\ &\left. + \left[\int_{t_1}^{t_1+r}\Big(E\big|S(t_1+r-s)h(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta)\big|_{\mathcal{L}_2^0}^p\Big)^{\frac{2}{p}}ds\right]^{\frac{p}{2}}\right\} \\ \le& 2^{p-1}c_p\left\{\|S(r) - I\|^p\left[\int_0^{t_1}\Big(E|h(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta)|_{\mathcal{L}_2^0}^p\Big)^{\frac{2}{p}}ds\right]^{\frac{p}{2}}\right. \\ &\left. + \left[\int_{t_1}^{t_1+r}\Big(E|h(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta)|_{\mathcal{L}_2^0}^p\Big)^{\frac{2}{p}}ds\right]^{\frac{p}{2}}\right\}.\end{aligned}$$

Noting that for any $s \in [0, T]$, $0 \le T < \infty$, we have

$$\begin{aligned}E&\left|h\big(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta\big)\right|_{\mathcal{L}_2^0}^p \le K_h^p E\left[\int_{-\infty}^0 |\sigma(\theta, x(s+\theta))|_H d\theta\right]^p \\ &\le K_h^p L_\sigma^p \left(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\right)^{p-1} \int_{-\infty}^s e^{\xi(\tau-s)} E|x(\tau)|_H^p d\tau \\ &\le K_h^p L_\sigma^p \xi^{1-p}\left[\int_{-\infty}^0 e^{\xi(\tau-s)} M_0 E|\phi(0)|_H^p e^{-\mu\tau}d\tau\right. \\ &\qquad \left. + \int_0^s e^{\xi(\tau-s)} M^\star E\sup_{\theta\le 0}|\phi(\theta)|_H^p e^{-\eta\tau}d\tau\right] \\ &\le K_h^p L_\sigma^p \xi^{1-p}\left(\frac{M^\star E\sup_{\theta\le 0}|\phi(\theta)|_H^p}{\xi - \eta}e^{-\eta s} + \frac{M_0 E\sup_{\theta\le 0}|\phi(\theta)|_H^p}{\xi - \mu}e^{-\xi s}\right) \\ &\le L,\end{aligned}$$

where L is a positive constant. By the strong continuity of $S(t)$, we get

$$E|I_5(t_1+r)-I_5(t_1)|_H^p \to 0 \quad \text{as} \quad r\to 0.$$

Similarly, we can verify that $E|I_i(t_1+r)-I_i(t_1)|_H^p \to 0$ as $r\to 0$, $i=1,\cdots,4$.

Next, we show that $\pi(\mathcal{S})\subset\mathcal{S}$. Let $x\in\mathcal{S}$. Without loss of generality, we assume that $0<\eta<\xi$. From the definition of π, we have

$$\begin{aligned}
E|(\pi x)(t)|_H^p \le\ & 5^{p-1}E\Big|S(t)\big[\phi(0)+G(0,\phi)\big]\Big|_H^p + 5^{p-1}E|G(t,x(t-\rho(t)))|_H^p \\
&+ 5^{p-1}E\Big|\int_0^t AS(t-s)G(s,x(s-\rho(s)))ds\Big|_H^p \\
&+ 5^{p-1}E\Big|\int_0^t S(t-s)\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta ds\Big|_H^p \\
&+ 5^{p-1}E\Big|\int_0^t S(t-s)\int_{-\infty}^0 \sigma(\theta,x(s+\theta))d\theta dW(s)\Big|_H^p \\
\equiv\ & 5^{p-1}\sum_{i=1}^5 I_i.
\end{aligned} \tag{3.3}$$

By assumption (A_2), we obtain

$$\begin{aligned}
I_2 &= E|G(t,x(t-\rho(t)))|_H^p \\
&\le \|(-A)^{-\alpha}\|^p E|(-A)^\alpha[G(t,x(t-\rho(t)))-G(t,0)]|_H^p \\
&\le K_G\|(-A)^{-\alpha}\|^p E|x(t-\rho(t))|_H^p \\
&\le K_G\|(-A)^{-\alpha}\|^p (M^\star e^{\eta r}E\sup_{\theta\le 0}|\phi(\theta)|_H^p e^{-\eta t}+M_0e^{\mu r}E|\phi(0)|_H^p e^{-\mu t}).
\end{aligned} \tag{3.4}$$

Employing Lemma 2.1 and Hölder's inequality, we get

$$\begin{aligned}
I_3 &= E|\int_0^t AS(t-s)G(s,x(s-\rho(s)))ds|_H^p \\
&\le E[\int_0^t |(-A)^{1-\alpha}S(t-s)(-A)^\alpha G(s,x(s-\rho(s)))|_H ds]^p \\
&\le M_{1-\alpha}^p E\left[\int_0^t (t-s)^{\alpha-1}e^{-\gamma(t-s)}|(-A)^\alpha G(s,x(s-\rho(s)))|_H ds\right]^p \\
&\le M_{1-\alpha}^p \left[\int_0^t (t-s)^{\alpha-1}e^{-\gamma(t-s)}ds\right]^{p-1} \\
&\qquad \cdot\int_0^t (t-s)^{\alpha-1}e^{-\gamma(t-s)}E|(-A)^\alpha G(s,x(s-\rho(s)))|_H^p ds,
\end{aligned}$$

combining this with the assumption (A_2), we further deduce that

$$\begin{aligned}
I_3 &\le M_{1-\alpha}^p K_G\gamma^{-\alpha(p-1)}\Gamma^{p-1}(\alpha)\int_0^t (t-s)^{\alpha-1}e^{-\gamma(t-s)}E|x(s-\rho(s))|_H^p ds \\
&\le M_{1-\alpha}^p K_G\gamma^{-\alpha(p-1)}\Gamma^{p-1}(\alpha)\int_0^t (t-s)^{\alpha-1}e^{-\gamma(t-s)}[M^\star e^{\eta r}E\sup_{\theta\le 0}|\phi(\theta)|_H^p e^{-\eta s} \\
&\quad + M_0e^{\mu r}E|\phi(0)|_H^p e^{-\mu s}]ds \\
&\le M_{1-\alpha}^p K_G\gamma^{-\alpha(p-1)}\Gamma^{p}(\alpha)\left[\frac{M^\star e^{\eta r}E\sup_{\theta\le 0}|\phi(\theta)|_H^p}{(\gamma-\eta)^\alpha}e^{-\eta t}\right. \\
&\quad \left.+\frac{M_0e^{\mu r}E|\phi(0)|_H^p}{(\gamma-\mu)^\alpha}e^{-\mu t}\right].
\end{aligned} \tag{3.5}$$

As for I_4, an application of Hölder's inequality and assumption (A_3), we have

$$
\begin{aligned}
I_4 &\le E\left[\int_0^t \left|S(t-s)b\big(s,\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta\big)\right|_H ds\right]^p \\
&\le K_b^p E\left[\int_0^t e^{-\gamma(t-s)}\Big|\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta\Big|_H ds\right]^p \\
&\le K_b^p\left[\int_0^t e^{-\gamma(t-s)}ds\right]^{p-1} E\int_0^t \left|e^{-\frac{\gamma(t-s)}{p}}\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta\right|_H^p ds \\
&\le K_b^p\gamma^{1-p}E\int_0^t\left[\int_{-\infty}^s L_g e^{-\frac{\gamma(t-s)}{p}}e^{\xi(\tau-s)}|x(\tau)|_H d\tau\right]^p ds \\
&\le K_b^p\gamma^{1-p}L_g^p\int_0^t\left[\left(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\right)^{p-1}\int_{-\infty}^s e^{-\gamma(t-s)}e^{\xi(\tau-s)}E|x(\tau)|_H^p d\tau\right]ds \\
&\le K_b^p\gamma^{1-p}L_g^p\xi^{1-p}\int_0^t\left[\int_{-\infty}^0 e^{-\gamma(t-s)}e^{\xi(\tau-s)}M_0E|\phi(0)|_H^p e^{-\mu\tau}d\tau\right. \qquad (3.6)\\
&\quad\left.+\int_0^s e^{-\gamma(t-s)}e^{\xi(\tau-s)}M^\star E\sup_{\theta\le 0}|\phi(\theta)|_H^p e^{-\eta\tau})d\tau\right]ds \\
&\le K_b^p\gamma^{1-p}L_g^p\xi^{1-p}\left[\frac{M^\star E\sup_{\theta\le 0}|\phi(\theta)|_H^p}{(\gamma-\eta)(\xi-\eta)}e^{-\eta t}+\frac{M_0E|\phi(0)|_H^p}{(\gamma-\xi)(\xi-\mu)}e^{-\xi t}\right].
\end{aligned}
$$

Taking into account Theorem A and assumption (A_3), we obtain that

$$
\begin{aligned}
I_5 &= E\left|\int_0^t S(t-s)h\big(s,\int_{-\infty}^0\sigma(\theta,x(s+\theta))d\theta\big)dW(s)\right|_H^p \\
&\le c_p\left\{\int_0^t\left[E\left|S(t-s)h\big(s,\int_{-\infty}^0\sigma(\theta,x(s+\theta))d\theta\big)\right|_{\mathcal{L}_2^0}^p\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le K_h^p c_p\left\{\int_0^t e^{-2\gamma(t-s)}\left[E\left|\int_{-\infty}^0\sigma(\theta,x(s+\theta))d\theta\right|_H^p\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}},
\end{aligned}
$$

where $c_p=(\frac{p(p-1)}{2})^{\frac{p}{2}}$, by assumption (A_3) and Hölder's inequality, we get

$$
\begin{aligned}
I_5 &\le K_h^p c_p L_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[E(\int_{-\infty}^s e^{\xi(\tau-s)}|x(\tau)|_H d\tau)^p\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le K_h^p c_p L_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[\Big(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\Big)^{p-1}\int_{-\infty}^s e^{\xi(\tau-s)}E|x(\tau)|_H^p d\tau\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le K_h^p c_p L_\sigma^p\xi^{1-p}\left\{\int_0^t e^{-2\gamma(t-s)}\left[\int_{-\infty}^s e^{\xi(\tau-s)}E|x(\tau)|_H^p d\tau\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}}.
\end{aligned}
$$

Noting that $p \geq 2$, we then have

$$
\begin{aligned}
I_5 &\leq K_h^p c_p L_\sigma^p \xi^{1-p} \left\{ \int_0^t e^{-2\gamma(t-s)} \left[\int_0^s e^{\xi(\tau-s)} M^\star E \sup_{\theta\leq 0} |\phi(\theta)|_H^p e^{-\eta\tau}) d\tau \right.\right. \\
&\quad \left.\left. + \int_{-\infty}^0 e^{\xi(\tau-s)} M_0 E|\phi(0)|_H^p e^{-\mu\tau} d\tau \right]^{\frac{2}{p}} ds \right\}^{\frac{p}{2}} \\
&\leq K_h^p c_p L_\sigma^p \xi^{1-p} \left\{ \int_0^t e^{-2\gamma(t-s)} \left[\left(\frac{M^\star E \sup_{\theta\leq 0} |\phi(\theta)|_H^p}{\xi-\eta} e^{-\eta s} \right)^{\frac{2}{p}} \right.\right. \\
&\quad \left.\left. + \left(\frac{M_0 E|\phi(0)|_H^p}{\xi-\mu} e^{-\xi s} \right)^{\frac{2}{p}} \right] ds \right\}^{\frac{p}{2}} \\
&\leq K_h^p c_p L_\sigma^p \xi^{1-p} 2^{\frac{p-2}{2}} \left[\frac{M^\star E \sup_{\theta\leq 0} |\phi(\theta)|_H^p}{\xi-\eta} \left(\frac{p}{2p\gamma-2\eta} \right)^{\frac{p}{2}} e^{-\eta t} \right. \\
&\quad \left. + \frac{M_0 E|\phi(0)|_H^p}{\xi-\mu} \left(\frac{p}{2p\gamma-2\xi} \right)^{\frac{p}{2}} e^{-\xi t} \right].
\end{aligned} \tag{3.7}
$$

Recalling (3.3), from (3.4) to (3.7), we can deduce that there exist $M_1 \geq 1$ such that

$$
E|(\pi x)(t)|_H^p \leq M_1 E \sup_{\theta\leq 0} |\phi(\theta)|_H^p e^{-\eta t}.
$$

Since the $\mathcal{F}_t$-measurability of $(\pi x)(t)$ is easily verified, it follows that π is well defined. Thus, we conclude that $\pi\mathcal{S} \subset \mathcal{S}$.

It remains to show that π has a unique fixed point. For any $x, y \in \mathcal{S}$, we have

$$
E|(\pi x)(t) - (\pi y)(t)|_H^p \leq 4^{p-1} \sum_{i=1}^4 J_i. \tag{3.8}
$$

We now estimate each J_i in (3.8). Noting that $x(s) = y(s) = \phi(s)$ for $s \leq 0$, by assumption (A_2), we have

$$
\begin{aligned}
J_1 &= E|G(t, x(t-\rho(t))) - G(t, y(t-\rho(t)))|_H^p \\
&\leq K_G \|(-A)^{-\alpha}\|^p E|x(t-\rho(t)) - y(t-\rho(t))|_H^p \\
&\leq K_G \|(-A)^{-\alpha}\|^p \sup_{s\geq 0} E|x(s) - y(s)|_H^p.
\end{aligned}
$$

By standard computations involving Lemma 2.1 and Hölder's inequality, we get

$$
\begin{aligned}
J_2 &= E\left| \int_0^t AS(t-s)[G(s, x(s-\rho(s))) - G(s, y(s-\rho(s)))] ds \right|_H^p \\
&\leq E\left[\int_0^t \left| AS(t-s)\big(G(s, x(s-\rho(s))) - G(s, y(s-\rho(s)))\big) \right|_H ds \right]^p \\
&\leq M_{1-\alpha}^p E\left[\int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} |(-A)^{-\alpha}\big(G(s, x(s-\rho(s))) \right. \\
&\quad \left. - G(s, y(s-\rho(s)))\big)|_H ds \right]^p \\
&\leq M_{1-\alpha}^p K_G \left[\int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} ds \right]^{p-1} \times \\
&\quad \int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} E|x(s-\rho(s)) - y(s-\rho(s))|_H^p ds \\
&\leq M_{1-\alpha}^p K_G \gamma^{-\alpha p} \Gamma^p(\alpha) \sup_{s\geq 0} E|x(s) - y(s)|_H^p.
\end{aligned}
$$

A similar argument as before yields

$$
\begin{aligned}
J_3 &= E\left|\int_0^t S(t-s)\left(b(s,\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta)-b(s,g(\theta,y(s+\theta))d\theta\right)ds\right|_H^p \\
&\le K_b^p E\left[\int_0^t e^{-\gamma(t-s)}\left|\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta-\int_{-\infty}^0 g(\theta,y(s+\theta))d\theta\right|ds\right]^p \\
&\le K_b^p\left(\int_0^t e^{-\gamma(t-s)}ds\right)^{p-1}\cdot \\
&\quad \int_0^t\left(\int_{-\infty}^0 |e^{\frac{-\gamma(t-s)}{p}} g(\theta,x(s+\theta))-g(\theta,y(s+\theta))|_H d\theta\right)^p ds \\
&\le K_b^p\gamma^{1-p}L_g^p\int_0^t\left(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\right)^{p-1}\cdot \\
&\quad \left(\int_{-\infty}^s e^{-\gamma(t-s)}e^{\xi(\tau-s)}E|x(\tau)-y(\tau)|_H^p d\tau\right)ds \\
&\le K_b^p\gamma^{-p}L_g^p\xi^{-p}\sup_{s\ge 0}E|x(s)-y(s)|_H^p,
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= E\left|\int_0^t S(t-s)\left[h(s,\int_{-\infty}^0\sigma(\theta,x(s+\theta))d\theta-h(s,\sigma(\theta,y(s+\theta))d\theta\right]dW(s)\right|_H^p \\
&\le c_pK_h^p\left\{\int_0^t e^{-2\gamma(t-s)}\cdot\right. \\
&\quad \left.\left[E\left|\int_{-\infty}^0(\sigma(\theta,x(s+\theta))-\sigma(\theta,y(s+\theta)))d\theta\right|_H^p\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le c_pK_h^pL_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[E\left(\int_{-\infty}^s e^{\xi(\tau-s)}|x(\tau)-y(\tau)|_H d\tau\right)^p\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le c_pK_h^pL_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\cdot\right. \\
&\quad \left.\left[\left(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\right)^{p-1}\int_{-\infty}^s e^{\xi(\tau-s)}E|x(\tau)-y(\tau)|_H d\tau\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le c_pK_h^pL_\sigma^p\xi^{-p}\left[\int_0^t e^{-2\gamma(t-s)}ds\right]^{\frac{p}{2}}\sup_{s\ge 0}E|x(s)-y(s)|_H^p \\
&\le c_pK_h^pL_\sigma^p\xi^{-p}(2\gamma)^{-\frac{p}{2}}\sup_{s\ge 0}E|x(s)-y(s)|_H^p.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\sup_{s\ge 0}E|(\pi x)(t)-(\pi y)(t)|_H^p &\le 4^{p-1}[K_G\|(-A)^{-\alpha}\|^p+M_{1-\alpha}^pK_G\gamma^{-\alpha p}\Gamma^p(\alpha) \\
&\quad +K_b^pL_g^p(\xi\gamma)^{-p}+(2\gamma)^{-\frac{p}{2}}K_h^pL_\sigma^p\xi^{-p}c_p]\sup_{s\ge 0}E|x(s)-y(s)|_H^p,
\end{aligned}
$$

by (3.1) it follows that π is contractive. Thus, Banach fixed point principle implies that there exists a unique $x(t)\in\mathcal{S}$ solves (2.1) with $x(s)=\phi(s)$ on $(-\infty,0]$, moreover, $x(t)$ is exponentially stable in pth-moment. The proof is completed. □

Corollary 3.2. *Under the conditions of Theorem 3.1 with $p = 2$, the mild solution of* (2.1) *exists uniquely which is exponentially stable in mean square.*

Theorem 3.3. *Suppose that all the conditions of Theorem 3.1 hold. Then the mild solution of* (2.1) *is almost surely exponentially stable.*

Proof. The proof is similar to [8], we omit it here. □

At the end of this paper, we give an example to illustrate our results.

Example 3.4. Consider the following neutral stochastic partial differential equation with infinite delays:

$$\begin{cases} d\left[x(t,u) + \dfrac{\alpha_1}{M_{1-\alpha}\|(-A)^\alpha\|} \cdot \dfrac{x(t-\rho(t),u)}{1+|x(t-\rho(t),u)|}\right] = \dfrac{\partial^2}{\partial x^2} x(t,u)dt \\ \qquad\qquad + f_1\left(t, \displaystyle\int_{-\infty}^0 \alpha_2 e^{\xi\theta} x(t+\theta,u)d\theta\right)dt \\ \qquad\qquad + f_2\left(t, \displaystyle\int_{-\infty}^0 \alpha_3 e^{\xi\theta} x(t+\theta,u)d\theta\right)dw(t), \quad \text{t} \geq 0; \\ x(s,u) = \phi(s,u) \in L^2[0,\pi],\ x(t,0) = x(t,\pi) = 0, \quad u \in [0,\pi],\ t \leq 0, \end{cases} \tag{3.9}$$

where $\xi, \alpha_i > 0$, $i = 1,2,3$, $M_{1-\alpha} > 0$, $\alpha \in (\frac{1}{2}, 1]$, $w(t)$ denotes the standard cylindrical Wiener process.

We now rewrite (3.9) into the abstract form of (2.1). Let $H = L^2(0,\pi)$. Define $A : H \to H$ by $-A = \frac{\partial^2}{\partial x^2}$ with domain

$$D(-A) = \left\{x, x^{'} \text{ are absolutely continuous, } x^{''} \in H \text{ and } x(0) = x(\pi) = 0\right\}.$$

Then $(-A)x = \sum\limits_{n=1}^{\infty} n^2\langle x, e_n\rangle e_n$, $x \in D(-A)$, where $e_n(\xi) = \sqrt{\frac{2}{n}}\sin n\xi$, $n = 1,2,\cdots$ is the set of eigenvector of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$, in H and

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n, \quad x \in H,$$

moreover, $\|S(t)\| \leq e^{-t}$, $t \geq 0$.

Let

$$G(t, x(t-\rho(t))) = \frac{\alpha_1}{M_{1-\alpha}\|(-A)^\alpha\|} \cdot \frac{x(t-\rho(t),u)}{1+|x(t-\rho(t),u)|},$$

$$b(t, \int_{-\infty}^0 g(\theta, x(t+\theta))d\theta) = f_1\left(t, \int_{-\infty}^0 \alpha_2 e^{\xi\theta} x(t+\theta,u)d\theta\right),$$

$$h(t, \int_{-\infty}^0 \sigma(\theta, x(t+\theta))d\theta) = f_2\left(t, \int_{-\infty}^0 \alpha_3 e^{\xi\theta} x(t+\theta,u)d\theta\right).$$

We assume that there exist some positive constants K_{f_i}, $i = 1,2$ such that for any $x, y \in H$, $t \geq 0$,

$$|f_1(t,x) - f_1(t,y)|_H \leq K_{f_1}|x-y|_H, |f_2(t,x) - f_2(t,y)|_{\mathcal{L}_0^2} \leq K_{f_2}|x-y|_H,$$

It is obvious that the assumptions $(A_1) - (A_4)$ are satisfied with

$$\gamma = 1, \quad K_G = \frac{\alpha_1^2}{M_{1-\alpha}^2}, K_b = K_{f_1}, K_h = K_{f_2}, \quad L_g = \alpha_2, \quad L_\sigma = \alpha_3.$$

From the definition of $(-A)^{-\alpha}$ (see page 70 of [20]),

$$\|(-A)^{-\alpha}\| \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} \|S(u)\| du \leq \frac{1}{\pi^{2\alpha}}.$$

Thus, by Theorem 3.1 and Theorem 3.3, if $E|\phi(s)|^2 \leq M_0 E|\phi(0)|^2 e^{-\mu s}$ for $s \leq 0$, where $M_0 \geq 1$, $0 < \mu < \xi$, the mild solution of (3.9) exists uniquely and it is exponentially stable in mean square and almost surely exponentially stable provided that

$$4\left[\frac{\alpha_1^2}{(\pi^{2\alpha} M_{1-\alpha})^2} + \alpha_1^2 \Gamma^2(\alpha) K_{f_1}^2 \alpha_2^2 \xi^{-2} + \frac{1}{2} K_{f_2}^2 \alpha_3^2 \xi^{-2}\right] < 1.$$

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