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Scale-free and power law distributions via fixed points and convergence of (thinning and conditioning) transformations

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Abstract

In discrete contexts such as the degree distribution for a graph, *scale-free* has traditionally been *defined* to be *power-law*. We propose a reasonable interpretation of *scale-free*, namely, invariance under the transformation of p-thinning, followed by conditioning on being positive.

For each $\beta \in (1,2)$, we show that there is a unique distribution which is a fixed point of this transformation; the distribution is power-law-β, and different from the usual Yule–Simon power law-β that arises in preferential attachment models.

In addition to characterizing these fixed points, we prove convergence results for iterates of the transformation.

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1 Introduction and statement of results

In the context of of random graphs, many authors define the term *scale-free* to mean that the degree distribution follows a power law – see for example [1, 4]. In this paper, we adopt a different point of view, in which scale-free means that the degree distribution is invariant under a natural transformation on the graph. As we will see, the power law property is then a consequence of this definition.

To motivate our transformation, consider a continuous random variable $X \geq 1$. It appears natural to say that its distribution is scale-free if cX conditioned on $cX \geq 1$ has the same same distribution as X, i.e.,

$$\mathbb{P}(X \geq x) = \mathbb{P}(cX \geq x \mid cX \geq 1).$$

It is not hard to check that the only such distributions are the Pareto distributions

$$\mathbb{P}(X \geq x) = x^{-\alpha}, \quad x \geq 1.$$

See [12, 14] for similar observations. One can also consider convergence to these fixed points, and easily show that

$$\lim_{c \to 0} \mathbb{P}(cX \geq x \mid cX \geq 1) = x^{-\alpha}, \quad x \geq 1$$

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if and only if the tail probabilities $\mathbb{P}(X \geq x)$ are of the form $L(x)x^{-\alpha}$, where L is slowly varying.

We consider now a discrete analogue of this setup. If D is a nonnegative integer valued random variable, cD is no longer integer valued, so we replace multiplication by thinning. A $p-$thinning of D is defined by

$$S_D = \sum_{i=1}^{D} X_i, \tag{1.1}$$

where X_i are i.i.d. Bernoulli (p) random variables that are independent of D. In terms of the probability generating function $G_D(s) = \mathbb{E}\, s^D$, this becomes

$$G_{S_D}(s) = G_D(1-p+ps) = G_D(1-p(1-s)). \tag{1.2}$$

In the graph context, this corresponds to thinning by edges.

We are concerned here with fixed points of the transformation $T = T_p = T_{p,m}$ given by

$$T : D \to (S_D \mid S_D \geq m)$$

where m is an integer ≥ 1, and convergence to these fixed points. (The case $m = 1$ is the most natural.)

There are other contexts in which fixed points and convergence of transformations that are the composition of two operations that change a distribution in opposite directions have been studied. Examples are [2, 7].

Similar questions for other families of transformations acting on discrete distributions have been studied before – see [5, 6, 18] for example. The main feature that distinguishes our setting from these others is the conditioning.

We will use two forms of the power-law-β property:

$$\mathbb{P}(D = k) \sim ck^{-\beta}, \tag{1.3}$$

$$\mathbb{P}(D \geq n) \sim L(n)n^{1-\beta} \tag{1.4}$$

where $\beta > 1$ and L is slowly varying. The latter property is known as regular variation. Our characterization of fixed points is the following. It is proved in Section 3.

Theorem 1.1. *Let m be a positive integer, and let D be a nonnegative integer valued random variable, with $\mathbb{P}(D \geq m) > 0$. The following are equivalent:*

- *The distribution of D is fixed by the transformation $D \mapsto T_{p,m}D$ for all $p \in (0,1)$.*
- *Either $D \equiv m$ is constant, or else D has power-law-β distribution (1.3), with $\beta = \alpha + 1$, $0 < \alpha < m$, $\mathbb{P}(D < m) = 0$, and*

$$\mathbb{P}(D = k+1)/\mathbb{P}(D = k) = (k-\alpha)/(k+1) \text{ for } k \geq m. \tag{1.5}$$

For the convergence results, we consider separately the cases of nontrivial and trivial fixed points. For the motivation for taking $p \downarrow 0$ in these results, see Remark 2.2 in the next section.

Theorem 1.2. *Suppose the distribution of D is power-law-β, as specified by (1.4). Then for every integer $k \geq \beta$*

$$\lim_{p \to 0+} \frac{\mathbb{P}(S_D = k)}{\mathbb{P}(S_D = k-1)} = \frac{k-\beta}{k}. \tag{1.6}$$

Theorem 1.3. *Take $m \geq \beta - 1$, and suppose the distribution of D is such that* (1.6) *holds for $k \geq \beta$. Then the distributions of $(S_D \mid S_D \geq m)$ are tight as $p \downarrow 0$. It follows that these distributions have a limit as $p \downarrow 0$, which is the fixed point described in* (1.5) *in case $\beta < m+1$, or $\mathbb{P}(D = m) = 1$ in case $\beta = m+1$.*

Theorem 1.4. *Suppose $ED^{k-1} < \infty$. Then*

$$\lim_{p\to 0} \frac{\mathbb{P}(S_D \geq k)}{\mathbb{P}(S_D = k-1)} = 0, \tag{1.7}$$

provided that the denominator above is strictly positive. As a consequence, if $ED^m < \infty$ and $\mathbb{P}(D \geq m) > 0$, then

$$\lim_{p\to 0} \mathbb{P}(S_D = m \mid S_D \geq m) = 1.$$

These three results are proved in Sections 4 and 5. In the final section, we prove that the nontrivial fixed points are infinitely divisible.

2 The tranformations $T_{p,m}$ and their fixed points

If D is a nonnegative integer valued random variable and $0 < p < 1$, the p-thinning S_D of D, defined by (1.1), has, using the notation $(z)_k = z(z-1)\cdots(z-k+1)$ for the falling product,

$$\begin{aligned}\mathbb{P}(S_D = n) &= \sum_{l=n}^{\infty} \mathbb{P}(D = l)\binom{l}{n} p^n (1-p)^{l-n} \\ &= \left(\frac{p}{1-p}\right)^n \frac{1}{n!} \sum_{l=n}^{\infty} (l)_n (1-p)^l \mathbb{P}(D = l).\end{aligned} \tag{2.1}$$

Fix an integer $m = 1, 2, \ldots$. For $p \in (0,1)$, the transformations $T \equiv T_p \equiv T_{p,m}$ for which we consider fixed points and convergence of iterates are given by

$$\mathbb{P}(TD = l) = \mathbb{P}(S_D = l \mid S_D \geq m). \tag{2.2}$$

In Section 3, we will prove that the fixed points of the transformation are precisely those described by (2.3) – (2.6) below, and in Section 4 and 5 we will prove results where these fixed points arise as limits of iterates of the transformation.

Remark 2.1. *We are referring here to distributions that are fixed points for all p, not just for some p. It would be interesting to know whether these are the only fixed points for a given p.*

For $m = 1, 2, \ldots$, the distribution with $\mathbb{P}(D = m) = 1$ is a trivial fixed point of $T_{p,m}$. For $m = 1$, all nontrivial fixed points have the form: for some $\alpha \in (0,1)$,

$$G_D(s) := \mathbb{E}\, s^D = 1 - (1-s)^\alpha =: \sum_{k\geq 0} c_k(\alpha) s^k. \tag{2.3}$$

The right hand side of (2.3) *defines* $c_k(\alpha)$ to be the coefficient of s^k in $1 - (1-s)^\alpha$, so for $k \geq 1$, $c_k(\alpha) = (-1)^{k-1}(\alpha)_k/k!$, and for $m = 1$, with the restriction $\alpha \in (0,1)$, $\mathbb{P}(D = k) = c_k(\alpha),\ k = 1, 2, \ldots$.

In general, for $m = 1, 2, \ldots$ and $\alpha \in (0, m)$ there is a nontrivial fixed point for $T_{p,m}$, which is power-law-β for $\beta = 1 + \alpha$, with

$$G_D(s) := \mathbb{E}\, s^D = \frac{1 - (1-s)^\alpha - \sum_{1\leq k<m} c_k(\alpha) s^k}{1 - \sum_{1\leq k<m} c_k(\alpha)}, \tag{2.4}$$

and this gives all nontrivial fixed points of $T_{p,m}$. A unified description of the fixed points (for all p) of $T_{p,m}$, including both the trivial fixed point, obtained by taking $\alpha = m$, is: $1+\alpha = \beta \in (1, m+1]$, $\mathbb{P}(D \in \{m, m+1, m+2, \ldots\}) = 1$, $\mathbb{P}(D = m) > 0$, and

$$\frac{\mathbb{P}(D = k+1)}{\mathbb{P}(D = k)} = \frac{k-\alpha}{k+1}, \quad k \geq m. \tag{2.5}$$

or equivalently, shifting the dummy variable k by 1,

$$\frac{\mathbb{P}(D = k)}{\mathbb{P}(D = k-1)} = \frac{k-\beta}{k}, \quad k > m. \tag{2.6}$$

The Yule–Simon distribution for power-law-β has point probabilities given by $\mathbb{P}(D = k) = (\beta - 1)\,\Gamma(k)\Gamma(\beta)/\Gamma(k+\beta)$, and hence ratios

$$\frac{\mathbb{P}(D = k)}{\mathbb{P}(D = k-1)} = \frac{k-1}{k-1+\beta}. \tag{2.7}$$

In comparison with (2.6), both formulas have denominator minus numerator $= \beta$, for every k, but for non-integer β, (2.6) has the integer in the denominator, while the Yule–Simon ratio (2.7) has the integer in the numerator.

Remark 2.2. *For each $m = 1, 2, \ldots$, it is true that for all $p, q \in (0,1)$ one has $T_q \circ T_p = T_{pq}$; we omit the easy proof. It then follows that the k-fold iterate $(T_q)^k$ of T_q is T_p with $p = q^k$. Theorem 1.3 allows $p \to 0$ with only the restriction $p > 0$, and the special case where p goes to zero along a geometric sequence q^k yields convergence for iterates of the transformation T_q, for one fixed q.*

3 Uniqueness

The goal is to show that, for $m = 1, 2, \ldots$, any distribution D on the nonnengative integers which is unchanged by p-thinning followed by conditioning on being at least m, for all $p \in (0,1)$, is either the constant $D \equiv m$ or else, as specified by (2.6), the law with $1 < \beta < m+1$ and ratios $\mathbb{P}(D = k)/\mathbb{P}(D = k-1) = (k-\beta)/k$ for $k \geq m+1$.

Lemma 3.1. *Suppose A and B are two nonnegative integer valued random variables with probability generating functions G_A, G_B. Let m be a positive integer. Assume $\mathbb{P}(A \geq m) > 0$ and $\mathbb{P}(B \geq m) > 0$. Consider the statements*

(a) $\mathbb{P}(A = k) = \mathbb{P}(B = k)$ *for all $k \geq m$.*

(b) $(A|A \geq m)$ *and* $(B|B \geq m)$ *have the same distribution.*

(c) $G_A^{(m)}(s) = G_B^{(m)}(s)$ *for all $s \in [0,1)$.*

(d) $G_A^{(m)}(s) = c\,G_B^{(m)}(s)$ *for all $s \in [0,1)$, for some constant $c > 0$.*

(Here $G_A^{(m)}(s)$ denotes the mth derivative of $G_A(s)$.) Then (a) if and only if (c), and (b) if and only if (d).

Proof. Let $a_k := \mathbb{P}(A = k)$ and $b_k := \mathbb{P}(B = k)$ so that $G_A(s) = \sum_{k\geq 0} a_k s^k$ and likewise for G_B. These are power series with radius of convergence ≥ 1, hence differentiable term-by-term, with $G_A^{(m)}(s) = \sum_{k\geq m} k_{(m)} a_k s^{k-m}$ for $|s| < 1$, and likewise for G_B. This immediately shows that (a) implies (c); to see that (c) implies (a), given $k \geq m$, differentiate $k-m$ times and evaluate at $s = 0$.

The equivalence of (b) and (d) follows, with $c = \mathbb{P}(B \geq m)/\mathbb{P}(A \geq m)$.

□

We apply this with $A = D$ and $B = S_D$. We are looking for a fixed point of $D \mapsto TD$, where $TD \equiv T_{p,m}D := (S_D | S_D \geq m)$ and S_D is the p-thinning of D. Since $1 = \mathbb{P}(TD \geq m)$, we can have D and TD equal in distribution only if $1 = \mathbb{P}(D \geq m)$. Thus we assume that $1 = \mathbb{P}(D \geq m)$, so that $D = (D | D \geq m)$, and now we have a fixed point of $D \mapsto TD$ if and only if $(D|D \geq m) = (S_D | S_D \geq m)$. Combine Lemma 3.1 with (1.2), so that the two generating functions of interest are $G_A(s) = G(s)$ and $G_B(s) = G(1 - p(1 - s))$.

Write f for the mth derivative of G, so that

$$G_B^{(m)}(s) = (G(1 - p + ps))^{(m)} = p^m\, f(1 - p(1 - s)).$$

Assuming that $1 = \mathbb{P}(D \geq m)$, we have a fixed point of $D \mapsto T_{p,m}D$ if and only if

$$f(s) = c\, p^m \times f(1 - p(1 - s)), \quad \text{for all } s \in [0, 1).$$

Lemma 3.2. *Let f be a continuous function from $[0, 1)$ to $(0, \infty)$, with $f(0) = 1$, and let $p \mapsto c(p)$ be any function on $(0, 1)$. If*

$$\forall p \in (0, 1), \forall s \in [0, 1), \qquad f(1 - p(1 - s)) = c(p) f(s), \tag{3.1}$$

then for some constant d we have $f(s) = (1 - s)^{-d}$.

Proof. First let $s = 1 - t$ so that (3.1) becomes

$$\forall p \in (0, 1), \forall t \in (0, 1], \qquad f(1 - pt) = c(p) f(1 - t),$$

and then consider $g(t) := f(1 - t)$ so that the system to solve becomes

$$\forall p \in (0, 1), \forall t \in (0, 1], \qquad g(pt) = c(p) g(t), \tag{3.2}$$

with $g(1) = 1$. Plugging in $t = 1$ we see that $c(p) = g(p)$, and (3.2) becomes $g(pt) = g(p)g(t)$. It follows that $g(u) = u^{-d}$ for some d. $\square$

Proof of Theorem 1.1. Start by assuming that D is a fixed point. We combine Lemmas 3.1 and 3.2, as in the remarks before Lemma 3.2, so that $G(s) = \mathbb{E}\, s^D$, f is the mth derivative of G, and the conclusion of Lemma 3.2 applied to $f(s)/f(0)$ is that $f(s) = c\,(1 - s)^{-d}$ with $c > 0$. [We have $c = f(0) > 0$ because $\mathbb{P}(D \geq m) > 0$ implies $\mathbb{P}(S_D = m) > 0$, hence $\mathbb{P}(D = m | D \geq m) = \mathbb{P}(S_D = m | S_D \geq m) > 0$, hence $c = m!\mathbb{P}(D = m) > 0$.]

In case $d = 0$, we have f is constant and $D \equiv m$. We cannot have d negative, since then the coefficient of s^1 in f is d, while G has nonnegative coefficients. In case $d > 0$, writing $[s^k]f(s)$ for the coefficient of s^k in f, so that $[s^k]G(s) = \mathbb{P}(D = k)$, we have for $k \geq m$

$$k_{(m)}\mathbb{P}(D = k) = [s^{k-m}]f(s) = [s^{k-m}](c\,(1 - s)^{-d}) = c\,(-1)^{k-m}\, \frac{(-d)_{(k-m)}}{(k - m)!}.$$

Hence for $k \geq m$

$$\mathbb{P}(D = k) = c\,(-1)^{k-m}\, \frac{(-d)_{(k-m)}}{k!}$$

and

$$\frac{\mathbb{P}(D = k + 1)}{\mathbb{P}(D = k)} = \frac{-(-d - (k - m))}{k + 1} = \frac{k - \alpha}{k + 1},$$

with $\alpha = m - d < m$. The requirement $\sum \mathbb{P}(D = k) < \infty$ implies that $\alpha > 0$.

The implication in the opposite direction is easy, again by combining Lemmas 3.1 and 3.2. $\square$

4 Convergence to nontrivial fixed points

Before proving Theorem 1.2, we state part of a Tauberian theorem that can be found on page 447 of [8]. Many other Tauberian theorems can be found in [3].

Theorem 4.1. *Let* $q_l \geq 0$ *and suppose* $Q(s) = \sum_{l=0}^{\infty} q_l s^l$ *converges for* $0 \leq s < 1$. *If* L *is slowly varying,* $\rho > 0$, *and* $q_l \sim l^{\rho-1} L(l)$, *then*

$$Q(s) \sim \frac{\Gamma(\rho)}{(1-s)^{\rho}} L\left(\frac{1}{1-s}\right) \text{ as } s \uparrow 1.$$

Proof of Theorem 1.2. Write $H(k) = \mathbb{P}(D \geq k)$, so that (1.4) gives $H(k) = k^{1-\beta} L(k)$, where L is slowly varying. Sum by parts, make a change of variables in the second sum below, and apply the Tauberian theorem to each of the resulting sums. By (2.1),

$$\begin{aligned}
P(S_D = k) k! \left(\frac{1-p}{p}\right)^k &= \sum_{l=k}^{\infty} (l)_k (1-p)^l \mathbb{P}(D = l) \\
&= \sum_{l=k}^{\infty} (l)_k (1-p)^l [H(l) - H(l+1)] \\
&= \sum_{l=k}^{\infty} (l)_k (1-p)^l H(l) - \sum_{l=k+1}^{\infty} (l-1)_k (1-p)^{l-1} H(l) \\
&= k!(1-p)^k H(k) + \sum_{l=k+1}^{\infty} (l-1)_{k-1} (k - lp)(1-p)^{l-1} H(l) \\
&= k!(1-p)^k H(k) + k \sum_{l=k+1}^{\infty} (l-1)_{k-1} (1-p)^{l-1} H(l) \\
&\qquad - p \sum_{l=k+1}^{\infty} (l)_k (1-p)^{l-1} H(l) \\
&\sim k\Gamma(k - \beta + 1) p^{\beta-k-1} L(p^{-1}) - \Gamma(k-\beta+2) p^{\beta-k-1} L(p^{-1}) \\
&= \Gamma(k - \beta + 1)(\beta - 1) p^{\beta-k-1} L(p^{-1}),
\end{aligned}$$

provided that $k - \beta + 1 > 0$. This gives (1.6) if $k > \beta$. If $k = \beta$, the above computation with k replaced by $k - 1$ gives

$$\sum_{l=k-1}^{\infty} l(l-1)\cdots(l-k+2)(1-p)^l \mathbb{P}(D = l) \sim (k-1)! H(k-1) + (k-1) L^*(p^{-1}),$$

so (1.6) holds in this case as well. □

Convergence of the ratios of probabilities in (1.6) does not immediately imply tightness of the distributions of $(S_D \mid S_D \geq m)$ as $p \downarrow 0$. This tightness is needed to conclude that the iterates of the transformation converge to the appropriate fixed point. We therefore now turn our attention to that issue.

Proof of Theorem 1.3. Tightness of these conditional distributions means that

$$\lim_{k\to\infty} \limsup_{p\to 0+} \frac{\mathbb{P}(S_D \geq k)}{\mathbb{P}(S_D \geq m)} = 0. \tag{4.1}$$

Thus we need to deduce the asymptotics of ratios of tail probabilities from the asymptotics of ratios of point probabilities.

A key identity that allows for this transition is

$$\frac{d}{dp}\mathbb{P}(S_D \geq k) = kp^{-1}\mathbb{P}(S_D = k). \tag{4.2}$$

Students of the theory of percolation will recognize this as a very simple form of Russo's formula – see page 35 of [9], for example. The proof of (4.2) is also simple: Use (2.1) to write

$$\mathbb{P}(S_D \geq k) = \sum_{l=k}^{\infty} \mathbb{P}(D = l)\Big[1 - \sum_{n=0}^{k-1}\binom{l}{n}p^n(1-p)^{l-n}\Big]. \tag{4.3}$$

Differentiating gives

$$\frac{d}{dp}\mathbb{P}(S_D \geq k) = p^{-1}\sum_{l=k}^{\infty}\mathbb{P}(D = l)\sum_{n=0}^{k-1}\binom{l}{n}p^n(1-p)^{l-n-1}(lp-n).$$

To prove (4.2) one needs to check

$$\sum_{n=0}^{k-1}\binom{l}{n}p^n(1-p)^{l-n-1}(lp-n) = k\binom{l}{k}p^k(1-p)^{l-k}. \tag{4.4}$$

The easiest way to check this is to note that the two sides of (4.4) agree for $k = 0$, and differences of the two sides of (4.4) for successive values of k also agree.

By L'Hospital's Rule, whenever (1.6) holds, it follows from (4.2) that

$$\lim_{p\to 0+}\frac{\mathbb{P}(S_D \geq k)}{\mathbb{P}(S_D \geq k-1)} = \frac{k-\beta}{k-1}. \tag{4.5}$$

Using (4.5) repeatedly gives

$$\lim_{p\to 0+}\frac{\mathbb{P}(S_D \geq m+k)}{\mathbb{P}(S_D \geq m)} = \prod_{j=1}^{k}\frac{m+j-\beta}{m+j-1}.$$

Now (4.1) follows from this and the fact that $\sum_j(\beta-1)(m+j-1) = \infty$. □

5 Convergence to trivial fixed points

Next we consider what happens in the less interesting regime $m < \beta - 1$.

Remark 5.1. *If* (1.4) *holds with* $m = \beta - 1$*, then Theorems 1.2 and 1.3 provide the conclusion of Theorem 1.4 even though* ED^m *may be infinite.*

Proof of Theorem 1.4. From (2.1) with $n = k-1$ and the dominated convergence theorem, we see that

$$\mathbb{P}(S_D = k-1) \sim p^{k-1}E\binom{D}{k-1} \quad \text{as } p \downarrow 0. \tag{5.1}$$

We need to show that

$$\lim_{p\to 0}\frac{\mathbb{P}(S_D \geq k)}{p^{k-1}} = 0.$$

This will follow from (4.3) and the dominated convergence theorem provided that

$$1 - \sum_{n=0}^{k-1}\binom{l}{n}p^n(1-p)^{l-n} \leq C(lp)^{k-1} \tag{5.2}$$

for some C depending only on k, and

$$1-\sum_{n=0}^{k-1}\binom{l}{n}p^n(1-p)^{l-n}=o(p^{k-1}) \tag{5.3}$$

as $p\to 0$ for each l. Both (5.2) and (5.3) follow from

$$1-\sum_{n=0}^{k-1}\binom{l}{n}p^n(1-p)^{l-n}\le C(lp)^k \tag{5.4}$$

for some (different) constant C, again depending only on k; (5.4) is a Chernoff bound; see [10, formula (12)]. That (5.3) follows from (5.4) is immediate. To deduce (5.2) from (5.4) write

$$1-\sum_{n=0}^{k-1}\binom{l}{n}p^n(1-p)^{l-n}=1-\sum_{n=0}^{k-2}\binom{l}{n}p^n(1-p)^{l-n}-\binom{l}{k-1}p^{k-1}(1-p)^{l-k+1} \tag{5.5}$$

and apply (5.4) to the first part of (5.5) with k replaced by $k-1$.

The final statement follows from (5.1) with $k=m+1$. □

6 Infinite divisibility

We will show that the distributions in (2.4) are *infinitely divisible*; this is relatively easy, thanks to a result from renewal theory.

Proposition 6.1. *Suppose the sequence* $\{u(n), n\ge 0\}$ *satisfies* $u(0)=1$,

$$u(n)>0, u(n-1)u(n+1)\ge u^2(n) \text{ for } n\ge 1 \text{ and } \lim_n \frac{u(n)}{u(n+1)}>0. \tag{6.1}$$

Let

$$\log\left(\sum_{n=0}^{\infty}u(n)s^n\right)=\sum_{n=1}^{\infty}\lambda(n)s^n. \tag{6.2}$$

Then $\lambda(n)\ge 0$ *for* $n\ge 1$.

Proof. Let $\{f(n), n\ge 1\}$ be the sequence associated to $u(\cdot)$ by the renewal equation:

$$u(n)=\sum_{k=1}^{n}f(k)u(n-k), \tag{6.3}$$

and consider the two generating functions

$$U(s)=\sum_{n=0}^{\infty}u(n)s^n \text{ and } F(s)=\sum_{n=1}^{\infty}f(n)s^n.$$

Multiplying (6.3) by s^n and summing for $n\ge 0$ gives

$$U(s)=1+U(s)F(s), \text{ or equivalently } U(s)=\frac{1}{1-F(s)}.$$

Therefore, (6.2) can be written as

$$\log[U(s)]=-\log(1-F(s))=\sum_{n=1}^{\infty}\frac{[F(s)]^n}{n}.$$

Kaluza ([11]) proved that $f(k)\ge 0$ for all $k\ge 1$. (See [13, Theorem 1] for generalizations of this statement; see also [16].) Therefore the series in (6.2) has nonnegative coefficients. □

The inequality in (6.1) is known as log-convexity of the sequence u. There is a long history of connections between log-convexity and infinite divisibility; see [17] [19] and [15, Thm. 51.3; Notes on p. 426], for example.

Corollary 6.2. *For* $m = 1, 2, \ldots$, *and* $\alpha \in (0, m)$, *the probability distribution for* D *specified by* (2.4) *and* (2.5) *is infinitely divisible.*

Proof. Let $X = D - m$, and define $u(n) = \mathbb{P}(X = n)/\mathbb{P}(X = 0)$ for $n \geq 0$. This yields

$$\sum_{n=0}^{\infty} u(n) s^n = \left[1 - (1-s)^\alpha - \sum_{k=0}^{m-1} (-1)^k \frac{(\alpha)_k}{k!} s^k\right] \Big/ (-1)^m \frac{(\alpha)_m}{m!} s^m,$$

so that $u(0) = 1$, $u(n) > 0$ for all $n > 0$ and $u(n)/u(n+1) = (m+n+1)/(m+n-\alpha)$, which is decreasing in n, so that (6.1) is satisfied. The probability generating function of X is

$$G_X(s) := \mathbb{E}\, s^X = \mathbb{P}(X = 0) \sum_{n=0}^{\infty} u(n) s^n = \mathbb{P}(X = 0) \exp\left(\sum_{n=1}^{\infty} \lambda(n) s^n\right),$$

and Proposition 6.1 shows that $\lambda(n) \geq 0$ for $n = 1, 2, \ldots$. Hence X is equal in distribution to $\sum_{n \geq 1} n Z_n$, where $Z_1, Z_2, \ldots$ are independent, and Z_n is Poisson distributed with parameter $\lambda(n)$. □

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