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Exponential Ergodicity of killed Lévy processes in a finite interval

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Abstract

Following Bertoin who considered the ergodicity and exponential decay of Lévy processes in a finite domain [4], we consider general Lévy processes and their ergodicity and exponential decay in a finite interval. More precisely, given

$$T_a = \inf\{t > 0 : X_t \notin (0, a)\},$$

$a > 0$ and X a Lévy process then we study from spectral-theoretical point of view the killed process $\mathbb{P}(X_t \in ., T_a > t)$. Under general conditions, e.g. absolute continuity of the transition semigroup of the unkilled Lévy process, we prove that the killed semigroup is a compact operator. Thus, we prove stronger results in view of the exponential ergodicity and estimates of the speed of convergence. Our results are presented in a Lévy processes setting but are well applicable for Markov processes in a finite interval once one can establish Lebesgue irreducibility of the killed semigroup and that the killed process is a doubly Feller process. For example, this scheme is applicable to the work of Pistorius [10].

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1 Introduction and results

In this short note we investigate the ergodic properties of general Lévy processes killed upon exiting a finite interval. Exit from such domains is known as the "double-sided exit problem". We stress that this technique is applicable in the far wider context of Markov processes. So far this problem has been previously considered in some generality by Bertoin in [4] for the case when a Lévy process has only negative jumps. There Bertoin uses the so called R-theory developed by Tuominen and Tweedie [13] in order to identify the r-positivity of the process and to identify the r-invariant function and measure. Under similar conditions, i.e. the doubly Feller property of the underlying Lévy process, we derive and discuss the exponential ergodicity of the semigroup of the killed Lévy process in the general case, i.e. when our Lévy process can make both positive and negative jumps. Moreover, we connect this topic to the general theory of semigroups and explicitly demonstrate how the main result can be related to general spectral theory. We achieve this by making use of a result by Schilling and Wang [11] on compactness of Markov semigroups and using the classical theory of compact, positive operators. We strongly believe that this approach is perfectly adapted to studying

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the ergodic properties in the "double-sided exit problem" as it makes use of first and foremost the domain in question namely a compact interval and then of the underlying structure of the one-dimensional Lévy process.

The note is organized as follows: in the first section we introduce the notation and the main results; in the second section we discuss the implications of our results, their conditions and how far they can be extended, also we point out some challenges; in the third section we provide the proof of our results.

2 Notation and Main Result

We denote by $X = (X_t)_{t\geq 0}$ a real-valued Lévy process, i.e. an a.s. right-continuous process with stationary and independent increments. The semigroup of the Lévy process will be denoted by $\mathbb{P}$. We recall that each Lévy process is characterized by its Lévy -Khintchine exponent, i.e.

$$\Psi(z) = \ln \mathbb{E}\left[e^{zX_1}\right] = \frac{\sigma^2}{2}z^2 + \gamma z + \int_{-\infty}^{\infty} \left(e^{zy} - 1 - 1_{\{|y|\leq 1\}}zy\right)\Pi(dy), \tag{2.1}$$

where $\sigma^2 \geq 0$ is the variance of the Brownian component, $\gamma \in \mathbb{R}$ is the linear term and Π is a σ-finite measure which describes the structure of the jumps of X, i.e. their intensity and size.

Fix $a > 0$. Denote the first hitting time to the closed set $\mathbb{R} \setminus (0, a)$ by

$$T_a = \inf\{t > 0 : \, X_t \notin (0, a)\}. \tag{2.2}$$

Then the Lévy process killed upon exiting $(0, a)$ is a Markov process, see [3, IV, Prop. 4, p.46] and its semigroup will be denoted by P. For any $q > 0$, we will denote its resolvent by

$$\Theta^q(x, dy) = \int_0^\infty e^{-qt}\mathbb{P}_x\left(X_t \in dy\right), \text{ for } x \in (0, a) \text{ and } y \in (0, a). \tag{2.3}$$

In the sequel we will call (DF) the assumption that

$$P_t \text{ is a semigroup of a doubly Feller process,}$$

i.e. $P_t f \in C_b(\mathbb{R})$ for any $f \in L^\infty(\mathbb{R})$, where $C_b(\mathbb{R})$ stands for the continuous bounded functions on $\mathbb{R}$ and $L^\infty(\mathbb{R})$ is the set of all bounded measurable functions on $\mathbb{R}$. By Theorem 2.2 in [8], it follows that X is a doubly Feller process when (AC) holds, namely

$$\mathbb{P}\left(X_t \in dx\right) << dx, \text{ for every } t > 0.$$

Call (F) the assumption that

$\sigma > 0$ **or** X is not a negative of subordinator, does not live on a lattice and $\Pi((-a, 0)) > 0$
or X is not a subordinator, does not live on a lattice and $\Pi((0, a)) > 0$.

We note that condition (F) is very general, whereas (DF) seems slightly more restrictive, see Subsection 3.1 for details.

We are now able to state our main result. We will denote by $C_0((0, a))$ the space of continuous functions on $[0, a]$ which vanish at the boundary.

Theorem 2.1. *Let (F) holds and $a > 0$ is fixed. Then P_t is a semigroup of a Lebesgue irreducible Markov process with a state space $(0, a)$. If additionally (DF) holds, for each*

$t > 0$, $P_t : C_0((0,a)) \mapsto C_0((0,a))$ *is a compact operator. Therefore the spectrum of its generator consists of isolated eigenvalues of finite multiplicity, which we ordered according to their real part size in the complex plane, namely* $\Re(\rho_1) \leq \Re(\rho_2) \leq \cdots \leq \Re(\rho_n) \leq \cdots$. *Moreover,*

i) $\rho_1 \in (0,\infty)$, ρ_1 *is of multiplicity* 1 *and* $\rho_1 < \inf_{n\geq 2} \Re(\rho_n)$.

ii) The eigenfunction $W := W^{\rho_1} : [0,a] \mapsto [0,\infty)$, $W \in C_0((0,a))$, $W(x) > 0$ *for* $x \in (0,a)$ *and* $P_t W(x) = e^{-\rho_1 t} W(x)$, *for any* $x \in [0,a]$. *It can be chosen such that*

$$\int_0^a W(x)W(a-x)dx = 1 \tag{2.4}$$

iii) The co-eigenfunction corresponding to ρ_1, $\tilde{W} = \tilde{W}^{\rho_1} : (0,a) \mapsto [0,\infty)$, *i.e. the function such that*

$$\int_0^a P_t f(x)\tilde{W}(x)dx = e^{-\rho_1 t}\int_0^a f(x)\tilde{W}(x)dx \tag{2.5}$$

satisfies in fact the relation $\tilde{W}(x) = W(a-x)$.

iv) The function $W(a-x)$ *defines a measure* $\mathcal{W}(dx) = W(a-x)dx$ *on* $(0,a)$ *such that every* $\epsilon > 0$ *there is a constant* $M_\epsilon > 0$ *such that*

$$\sup_{\{f\in C([0,a]):||f||_\infty\leq 1\}} \left|\left|e^{\rho_1 t}P_t f - W\langle f,\tilde{W}\rangle\right|\right|_\infty \leq M_\epsilon e^{-(\Re(\rho_2)-\rho_1-\varepsilon)t}, \tag{2.6}$$

where $||.||_\infty$ *is the supremum norm on* $[0,a]$, $\langle f,g\rangle = \int_0^a f(x)g(x)dx$. *For any compact set* $C \subset (0,a)$ *and* $A \subset \mathcal{B}(0,a)$, *where* $\mathcal{B}(0,a)$ *is the Borel* σ*-algebra on* $(0,a)$, *we have that for every* $\varepsilon > 0$ *there is some constant* $M'_{\varepsilon,C} > 0$ *that*

$$\sup_{x\in C}\sup_{A\subset(0,a)}\left|e^{\rho_1 t}\mathbb{P}_x\left(X_t\in A|T_a>t\right) - \frac{\mathcal{W}(A)}{\langle 1,\tilde{W}\rangle}\right| \leq M'_{\varepsilon,C}e^{-(\Re(\rho_2)-\rho_1-\varepsilon)t} \tag{2.7}$$

Remark 2.2. *This theorem allows us to conclude that for every measurable* $A \subset (0,a)$

$$\lim_{t\to\infty}\mathbb{P}_x\big(X_t\in A \mid T>t\big) = \frac{\mathcal{W}(A)}{\langle 1,\tilde{W}\rangle}, \tag{2.8}$$

i.e. $\mathcal{W}$ *is the Yaglom limit. Additionally, it provides an exponential speed of convergence to the quasi-stationary distribution represented by the probability measure* $\mathcal{W}(.)/\langle 1,\tilde{W}\rangle$.

Remark 2.3. *Under the conditions in [4, Theorem 2] it is immediately augmented with convergence in total variation and the knowledge of an existing exponential rate of convergence in [4, (v), Th.2]. We note that* $W^{-\rho}$ *in the notation of Bertoin satisfying* $W^{-\rho}(a) = 0$ *is an immediate consequence of the fact that* $P_t f(a) = 0$, *for any* $f \in C([0,a])$ *due to the fact that* X *issued forth from* a *immediately enters* (a,∞) *and* $P_t W^{-\rho}(x) = e^{-\rho t}W^{-\rho}(x)$.

Remark 2.4. *It seems that the* R*-theory with all its might in general state space Markov processes is in this particular instance of "double-exit problem" weaker than the application of spectral theory. We believe this is due to the special case of a certain type of smoothing, i.e. the strong Feller property and the compactness of the closure of the domain* $(0,a)$. *This is due to the fact that those properties imply compactness of the semigroup und thus in particular a gap between the first and the second eigenvalue. The* R*-theory does not directly imply this spectral gap property.*

3 Discussion and Further Remarks

3.1 Condition (F) and (DF)

Condition (DF) is implied by (AC), i.e. the absolute continuity of the transition semigroup of the original Lévy process. Via Fourier inversion it is clear that the transition density p_t exists, equals

$$p_t(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-i\xi x} e^{t\Psi(\xi)} d\xi$$

and is $L^\infty(\mathbb{R})$ provided $\lim_{|\xi|\to\infty} \Re(\Psi(\xi))/\ln(|\xi|) = -\infty$. Thus the class when (AC) holds is enormous. It seems that no necessary and sufficient condition for (DF) in terms of the Lévy triplet is known. Condition (F) is explicit in terms of the Lévy triplet and certainly holds when X is of unbounded variation, i.e.

$$\int_{-1}^{1} |x| \, \Pi(dx) = \infty.$$

Though in many conceivable examples (DF) implies (F) we have not proved this in generality.

3.2 General applicability of our results

Our Theorem 2.1 essentially relies on Proposition 4.2 and the irreducibility of the semigroup P_t and is independent of the fact that X is a Lévy process. Given that any doubly Feller process killed upon hitting an open set is a doubly Feller, see [5], therefore all we need to know to apply our result is that X is a doubly Feller process,

$$T_a = \inf\{t > 0 : X_t \notin [0, a]\}$$

a.s. which implies that the killed process is doubly Feller and that X killed upon exit of $(0, a)$ is a Lebesgue irreducible Markov process. In this vein the results of Pistorius [10] on reflected spectrally one-sided process and its ergodicity can be reduced to the still demanding but yet much shorter task of computation of the resolvent, its properties and the verification of the fact that the reflected killed process satisfies (DF). The R-theory is again superfluous.

It is very difficult however to have information on the eigenvalues. Some trivial estimates for ρ_1 exist but any precise analytical way to computing it is elusive. Furthermore, it seems that numerical schemes will be hard to obtain even for Lévy processes due to the difficulty of computing the resolvents.

3.3 Applicability to Lévy processes

We believe that our results and methodology is very streamlined in view of the classical spectral and Markov theory it relies on. Some results that need a good guess seem to come naturally thanks to the language and notions we use from analysis. Certainly, not all comes for free and for Lévy processes what needs to be computed to have any information on the first eigenfunction and first eigenvalue, namely W and ρ_1, is the resolvent Θ^q. Once this is done as in [4] (see also [12]) then one can have a grasp on these quantities which by no means ensures that they would be known explicitly. Even in [4], where many quantities are tractable we have no clear way to obtain $W^{-\rho}$ in a closed form and even $\rho = \rho_1$. Therefore a new methodology is needed for further progress in this direction.

4 Proof of Theorem 2.1

We prove Theorem 2.1 in several steps. Taking into accound Theorems 4.4, 4.4 and 4.6 we see that the assertions i), ii) and the first part of the assertions iii) and iv) without the specific form of the co-eigenfunction follow in fact from general theory of compact irreducible semigroups. Thus we will first prove irreducibility and then compactness of the killed semigroup.

4.1 Proof of irreducibility

We start with the question of irreducibility as defined in the appendix.

Proposition 4.1. *The killed semigroup $(P_t)_{t\geq 0}$ is irreducible.*

Proof. We need to show, that the resolvent maps a non-trivial and non-negative function to a strictly positve. Fix a generic interval $A = (b, c) \subset (0, a)$, $x \in (0, a)$. It suffices to show that $\Theta^q 1_A(x) > 0$, for each $x \in (0, a)$. Clearly this is the case for $x \in A$ since the Lévy process is a.s. continuous at any deterministic time. Assume that $x \notin A$ and without loss of generality assume that $0 < x \leq b$. Now it is enough to show that with positive probability X enters $(b+\epsilon, c-\epsilon)$ for some very small $\epsilon > 0$ prior to exiting $(0, a)$. If $\Pi(0,\infty) = \infty$ then for a sequence $\epsilon_i \downarrow 0$, $\Pi(\frac{\epsilon_i}{2}, \epsilon_i) > 0$. Decompose $X_t = Y_t + Z_t$, where Y is a Lévy process collecting all jumps of X between $(\frac{\epsilon_i}{2}, \epsilon_i)$ only. Then for all i big enough $\exists\, 0 < S < \infty$ a.s. such that $Y_S \in (b+\epsilon_i, c-\epsilon_i)$. If $\mathbb{P}\left(\sup_{s\leq S} |Z_s| < \epsilon_i/4\right) > 0$, for any such $\epsilon_i > 0$ corresponding to large i, then upon conditioning on $A_i = \{\sup_{s\leq S} |Z_s| < \epsilon_i/4\}$ we obtain that $X_S \in (b + \epsilon_i/2, c - \epsilon_i/2)$ and therefore $\Theta^q 1_A(x) > 0$. So it remains to investigate when $\mathbb{P}(A_i) > 0$, for all i big enough. If X is with infinite variation then by definition Z is as well with infinite variation and from [2, Prop 1.1.] we get that Z has the so-called small deviation property and thus $\mathbb{P}(A_i) > 0$. If X has a bounded variation, i.e. $\int_{-1}^{1} |x|\Pi(dx) < \infty$ put $b = \gamma - \int_{-1}^{1} x\Pi(dx)$ with γ defined in (2.1). If $b \leq 0$ then from [2, Prop 1.1.] we conclude that Z has the small deviation property and thus $\mathbb{P}(A_i) > 0$. However, if $b > 0$ we add a drift to Y, say $Y'_t = 2bt + Y_t$ which also has as a stopping time $\infty > S' > 0$ such that $Y'_{S'} \in (b + \epsilon_i, c - \epsilon_i)$ and clearly $Z'_t = Z_t - 2bt$ is such that $b' < 0$. Applying the same procedure we conclude the statement. If $\Pi(0, \infty) < \infty$ then we decompose $X_t = Y_t + Z_t$ with Y being a compound Poisson process collecting all positive jumps of X. From [4, Prop. 1] Z killed upon exiting $(0, a)$ is Lebesque irreducible if either $\sigma > 0$ or $\Pi(-a, 0) > 0$ holds. Therefore, conditioning upon $\{Y \equiv 0\}$ until Z enters (b, c) we get that $\Theta^q 1_A(x) > 0$. However, when the last condition of (F) is satisfied it may happen that $\Pi(-a, 0) = 0$. Then we put Y to be the compound Poisson process collecting the negative jumps only and use the fact that Z is Lebesgue irreducible from [4, Prop. 1] in the same fashion as above. □

4.2 Compactness of P_t

In the following theorem we demonstrate the compactness of the semigroup by following the ideas of [11]. Similar ideas can be found in the proof of Theorem BIV 2.5 in [1]. In order to make this work self-contained and to make these ideas more widely konwn to the the probabilistic community we provide a complete proof of this very useful result.

Proposition 4.2. *Assume that $(P_t)_{t\geq 0}$ is a semigroup of a doubly Feller process, then for every $t > 0$ the operator $P_t : C([0, a]) \to C([0, a])$ is compact.*

Proof. Choose a continuous function $w > 0$ on $[0, a]$ with $\int_0^a w(x)dx = 1$ and define, for

$t > 0$,

$$\mu_t(\cdot) := \frac{\int_0^a w(x)P_t(x,\cdot)\,dx}{\int_0^a w(x)P_t(x,[0,a])\,dx}$$

If $N \subset [0,a]$ satisfies that $\mu_t(N) = 0$ then $P_t(x,N) = 0$ for Lebesgue all x. Using the strong Feller property we conclude that $x \mapsto P_t\mathbf{1}_N(x) = P_t(x,N)$ is continuous and we thus conclude that $P_t(x,N) = 0$ for all $x \in [0,a]$. Therefore $P_t(x,\cdot)$ is absolutely continuous with respect to μ_t and has a Radon-Nikodym density $p_t(x,y)$. Now, for any $u \in L^\infty(\mu_t)$, define the measurable set $N = \{x \in [0,a] \mid u(x) > ||u||_{L^\infty(\mu_t)}\}$. Then $\mu_t(N) = 0$. Set $\tilde{u} = u \cdot \mathbf{1}_{N^c}$ and note that $\tilde{u}$ is bounded and Borel measurable. We define

$$P_t u(x) := \int_0^a u(y)P_t(x,dy) = \int_0^a u(y)p_t(x,y)\mu_t(dy),\, u \in L^\infty(\mu_t) \tag{4.1}$$

Clearly $P_t u = P_t\tilde{u}$ and P_t is well defined on $C([0,a]) \subset L^\infty(\mu_t)$ and $P_t(L^\infty(\mu_t)) \subset C([0,a])$ due to the strong Feller property. We now need to show that the image of

$$U = \{u \in L^\infty(\mu_t) \mid \|u\|_{L^\infty(\mu_t)} \le 1\}$$

under P_t is sequentially compact in $C([0,a])$. First observe that by the Banach-Alaoglu theorem U is weak*–compact and therefore every sequence $(u_j)_{j\in\mathbb{N}} \subset U$ contains a weak*– subsequence $(u_{j_k})_{k\in\mathbb{N}}$ and for suitable $u \in L^\infty(\mu_t)$ the limit

$$\lim_{k\to\infty} \int_0^a u_{j_k}(y)\varphi(y)\mu_t(dy) = \int_0^a u(y)\varphi(y)\mu_t(dy), \varphi \in L^1(\mu_t) \tag{4.2}$$

exists. Therefore we have for every $x \in [0,a]$

$$\lim_{k\to\infty} \int_0^a u_{j_k}(y)P_t(x,dy) = \lim_{k\to\infty} \int_0^a u_{j_k}(y)p_t(x,y)\mu_t(dy) = \int_0^a u(y)p_t(x,y)\mu_t(dy).$$

Moreover, for every $k,l,m \in \mathbb{N}$ with $k,l \ge m$

$$|P_{2t}u_{j_k} - P_{2t}u_{j_l}| \le P_t\big|P_t u_{j_k} - P_t u_{j_l}\big| \le P_t\big(\sup_{k,l\ge m} |P_t u_{j_k} - P_t u_{j_l}|\big).$$

Note that $h_m := \sup_{k,l\ge m} |P_t u_{j_k} - P_t u_{j_l}|$ decreases to 0 as $m \to \infty$ and so does the sequence $(P_t h_m)_{m\in\mathbb{N}}$ as a consequence of the dominated convergence theorem. According to the strong Feller property the functions $P_t h_m$ are continuous and thus by Dini's theorem we get uniform convergence, which means that $(P_{2t}u_{j_k})_{k\in\mathbb{N}}$ is Cauchy in $C([0,a])$ and the proof is complete as $C([0,a]) \subset L^\infty(\mu_t)$. □

We are now ready to prove compactness for our semigroups. First we show that started from any $x \in (0,a)$

$$T_a \overset{a.s}{=} \tilde{T}_a = \inf\{t > 0 : X_t \notin [0,a]\}.$$

If X is with infinite variation then $T_a \overset{a.s}{=} \tilde{T}_a$ is immediate as the two-half planes are regular for X, i.e. X enters immediately in $\mathbb{R}^+$ and $\mathbb{R}^-$ provided it starts from zero. Let X be of bounded variation and X satisfies (F) and (DF). We next prove that X enters immediately in (a,∞) conditional on $\{X_{T_a} = a\}$ with the other case, i.e. $\{X_{T_a} = 0\}$ studied in the same way. If $\mathbb{P}(X_{T_a} = a) > 0$ then it follows that X creeps up which implies that the ascending ladder height process $H_t^+ = \delta_+ t +$ jumps with $\delta_+ > 0$. This implies that $\mathbb{R}^+$ is regular for X, see [7, Th.22, p.61] and therefore conditioned on $\{X_{T_a} = a\}$, X enters immediately after T_a the set (a,∞). Thus $T_a \overset{a.s}{=} \tilde{T}_a$.

However, since X satisfies (DF) and any doubly Feller process remains doubly Feller upon hitting an open set (recall $T_a \overset{a.s}{=} \tilde{T}_a$), see [5], we can use Proposition 4.2 above to conclude compactness.

4.3 Properties of the first eigenfunction

In this section we establish properties of the eigenfunction and in particular of the co-eigenfunction. In particular we demonstrate the missing assertions stated in iii) and iv) of Theorem 2.1. The compact semigroup $(P_t)_{t\geq 0}$ is irreducible as defined in the appendix and therefore the generalized Perron-Frobenius theorem, i.e. Krein–Rutman theorem, as described in Theorem 4.4 in the appendix below applies and proves the existence of a principal eigenfunction $W(x) \in C_0((0,a))$ with $W(x) > 0$ on $(0,a)$ and a first real eigenvalue $\rho_1 > 0$ of algebraic multiplicity 1. We can choose W such that

$$\int_0^a W(x)W(a-x)dx = 1.$$

Next from duality with respect to the Lebesgue measure of P_t we get the trivial lemma

Lemma 4.3. *If $\tilde{P}_t$ is the semigroup of $-X$ killed upon exit from $(0,a)$ then for any $x \in (0,a)$ and $f \in C_0((0,a))$ we have that*

$$P_t f(x) = \tilde{P}_t \tilde{f}(a-x), \tag{4.3}$$

where $\tilde{f}(x) = f(a-x)$.

This implies that $\tilde{W}(x) = W(a-x) \in C_0((0,a))$ is the eigenfunction for the dual semigroup, i.e. $\tilde{P}_t\tilde{W} = e^{-\rho_1 t}\tilde{W}$. Similarly, this duality via the Hunt's switching identity [3, II, Th.5, p.47] yields (2.5). Therefore the the measure ν appearing in Theorem 4.4 and Theorem 4.6 has the form $\nu(dx) = \tilde{W}(x)\,dx$.

4.4 Ergodicity of the semigroup

The strict positivity of Θ^q and its compactness imply further that the spectral projection $\mathcal{P}$ associated to ρ_1 is a subspace of dimension one generated by $W(x)$ and since P_t is compact and therefore has only point spectrum we deduce by [9, Th 3.1, p.329] or alternatively by Theorem 4.6 in the appendix that

$$P_t = e^{-\rho_1 t}\mathcal{P} + R_t, \tag{4.4}$$

where R_t is a one-parameter family of bounded operators satisfying

$$\lim_{t\to\infty} e^{(\Re\rho_2 - \epsilon)t}||R_t|| = 0$$

for any $\epsilon > 0$ and where the action $\mathcal{P}$ is given by

$$\mathcal{P}(f) = \int_0^a f(y)\tilde{W}(y)\,dy\,W(x).$$

Therefore the proof of the theorem is complete.

Appendix

In this appendix we collect some essential results connected to the spectral theory of positive semigroups on the Banach space E of continuous functions defined on a compact set K. For a rather complete account we refer to the book [1]. Let us denote by $(\mathcal{T}_t)_{t\geq 0}$ a Sub-Markov semigroup on the Banach space $E = C_0(X)$ of continuous functions on a locally compact set X which vanish at infinity. According to Definition

BIII 3.1 in [1] irreducibility of the semigroup $(\mathcal{T}_t)_{t\geq 0}$ is defined by the requirement that for every given $0 < f \in E$ and $\phi \in E'$ there is some $t_0 > 0$ such that

$$\left(\mathcal{T}_{t_0} f, \phi\right) > 0$$

or equivalently by the property that there is some $\lambda > 0$ such that for every $0 < f \in E$ the continuous function

$$g := \int_0^\infty e^{-\lambda t} \mathcal{T}_t f \, dt$$

is strictly positive.

Irreducible positive semigroups have some fundamental spectral properties, which are usually referred to results of Perron-Frobenius or Krein-Rutman type. We denote by $\sigma(B)$ the spectrum of B, by $r(B)$ the spectral radius and by $s(B)$ the spectral bound of an operator B, i.e.

$$s(B) := \sup\{\Re\lambda \mid \lambda \in \sigma(B)\}.$$

Theorem 4.4 (Proposition BIV 3.5 in [1]). *Suppose that A is the generator of an irreducible positive semigroup $(\mathcal{T}_t)_{t\geq 0}$ on the Banach space E of continuous functions on some locally compact space which vanish at infinity. Then the following assertions are true:*

1) *The spectrum $\sigma(A)$ of A is not empty.*
2) *every positive eigenfunction of A is strictly positive*
3) *if $ker(s(A) - A)$ is contains a positive element then $\dim ker(s(A) - A) \leq 1$.*
4) *if $s(A)$ is a pole of the resolvent then it is algebraically simple. The residue has the form $P = \phi \otimes u$ where $\phi \in E'$ and $u \in E$ are strictly positive eigenelements of A' and A, respectively, satisfying $(\phi, u) = 1$.*

The influence of the generator A upon the spectral properties of the semigroups is content of the following result, where we denote by $\sigma(B)$ the spectrum of an operator B.

Theorem 4.5 (compare Corollary AIII 6.7 in [1]). *The sepctral mapping theorem*

$$\sigma(\mathcal{T}_t) \setminus \{0\} = e^{-\sigma(A)}, \, t \geq 0$$

holds true for every compact semigroup $(\mathcal{T}_t)_{t\geq 0}$.

The previous results in combination with compactness of the semigroup the following asymptotic result is true:

Theorem 4.6 (compare Theorem BIV 2.1 and Corollary BIV 2.1 in [1]). *Let $(\mathcal{T}_t)_{t\geq 0}$ be a compact irreducible Sub-Markov-semigroup on the Banach space $E = C_0(X)$ for some locally compact space X with generator A. Then the spectrum A is discrete*

$$\sigma(A) = \{-\rho_1, -\rho_2, \dots\}$$

with $\Re\rho_{n+1} \geq \Re\rho_n$ for $n > 1$ and $\mathbb{R} \in \rho_1 < \Re\rho_2$ and there exists a strictly positive continuous function h and a strictly positive bounded measure ν on K such that for every $\delta \in (0, \rho_1 - \Re\rho_2)$ and some $M_\delta \geq 1$ and all $t \geq 0$

$$\left\| e^{\omega(T)t} \mathcal{T}_t - \nu \otimes h \right\| \leq M_\delta e^{-\delta t},$$

where

$$\omega(\mathcal{T}) := \inf\{w \mid \exists M_w \, \forall t \geq 0 : \|\mathcal{T}_t\| \leq M_w e^{-wt}\}$$

is the growth bound of the semigroup and under the above conditions the growth bound coincides with the spectral radius of the semigroup and the spectral radius of the generator.

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