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The travel time in a finite box in supercritical Bernoulli percolation

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Abstract

We consider the standard site percolation model on the three dimensional cubic lattice. Starting solely with the hypothesis that $\theta(p) > 0$, we prove that, for any $\alpha > 0$, there exists $\kappa > 0$ such that, with probability larger than $1 - 1/n^\alpha$, every pair of sites inside the box $\Lambda(n)$ are joined by a path having at most $\kappa(\ln n)^2$ closed sites.

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1 Introduction

We consider the site percolation model on $\mathbb{Z}^3$. Each site is declared open with probability p and closed with probability $1-p$, and the states of the sites are independent. One of the most important problems in percolation is to prove that, in three dimensions, there is no infinite cluster at the critical point. The most promising strategy so far seems to perform a renormalization argument [1]. The missing ingredient is a suitable construction helping to define a "good block", starting solely with the hypothesis that $\theta(p) > 0$. Our main result here is an estimate on the travel time in a finite box under the hypothesis that $\theta(p) > 0$. For $n \in \mathbb{N}$, we denote by $\Lambda(n)$ the cubic box $\Lambda(n) = [-n, n]^3$.

Theorem 1.1. *Let p be such that $\theta(p) > 0$ and let $\alpha > 0$. There exists a constant κ, depending on α and p, such that*

$$\forall n \geq 2 \qquad P\begin{pmatrix} \text{every pair of sites of the box } \Lambda(n) \\ \text{are joined by a path in } \Lambda(n) \text{ having} \\ \text{at most } \kappa(\ln n)^2 \text{ closed sites} \end{pmatrix} \geq 1 - \frac{1}{n^\alpha}\,.$$

This result can be recast in the language of first passage percolation. If we declare that the travel time is null through an open site and one through a closed site, and if we denote by $T_{\Lambda(n)}(x,y)$ the travel time between two sites x, y in $\Lambda(n)$, that is, the infimum of the travel time over all the paths joining x and y in $\Lambda(n)$, then the above estimate can be rewritten as

$$\forall n \geq 2 \qquad P\big(\forall x, y \in \Lambda(n) \quad T_{\Lambda(n)}(x,y) \,\leq\, \kappa(\ln n)^2\,\big) \,\geq\, 1 - \frac{1}{n^\alpha}\,.$$

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The bound $\kappa(\ln n)^2$ is probably not optimal. If we start with the hypothesis that $p > p_c$, then we get a bound of order $\kappa \ln n$ with the help of the slab technology. The goal of the game here is to see what we can get starting only with the hypothesis that $\theta(p) > 0$. We focus here on the dimension three. The two dimensional case is very special, and it is known that there is no percolation at the critical point in dimensions two. The geometric argument presented here is best visualized in dimensions three. The proof relies essentially on the BK and the FKG inequalities for the probabilistic part (see [1]), and on a tiling of the sphere into 48 spherical triangles for the geometric part. The vertices of these triangles are the vertices of a Catalan solid called the disdyakis dodecahedron or the hexakis octahedron (see [2, 3]). We could write a proof using only cubes, however it would require more geometric computations. With the help of the tiling of the sphere into spherical triangles, we can build in a straightforward way a path converging at geometric speed to a prescribed target. The main point is that the spherical triangles have a diameter strictly less than one.

2 Basic notation

The usual Euclidean norm in $\mathbb{R}^3$ is denoted by $|\cdot|$. Two sites x, y of the lattice $\mathbb{Z}^3$ are said to be connected if they are nearest neighbours, i.e., if $|x-y|=1$. For $x \in \mathbb{Z}^3$, we denote by $C(x)$ the open cluster containing x, i.e., the connected component of the set of the open sites containing x. If x is closed, then $C(x)=\varnothing$. Let A be a subset of $\mathbb{Z}^3$. We define its internal boundary $\partial^{\,in} A$ and its external boundary $\partial^{\,out} A$ by

$$\begin{aligned}\partial^{\,in} A \,&=\, \big\{\, x \in A : \exists y \notin A \quad |x-y|=1 \,\big\}\,,\\ \partial^{\,out} A \,&=\, \big\{\, x \notin A : \exists y \in A \quad |x-y|=1 \,\big\}\,.\end{aligned}$$

For x a point in $\mathbb{Z}^3$, the distance $d(x,A)$ between x and A is defined as

$$d(x,A) \,=\, \inf_{y \in A} \,|x-y|\,.$$

Recall that a path $z_0, \dots, z_r$ is a sequence of sites such that each site is a neighbour of its predecessor:

$$\forall i \in \{\, 0, \dots, r-1 \,\} \qquad |z_{i+1} - z_i| = 1\,.$$

Let A be a subset of $\mathbb{Z}^3$. For x, y in A, we define the travel time $T_A(x,y)$ between x and y in A by

$$T_A(x,y) \,=\, \inf \,\Big\{\, \sum_{i=0}^{r} 1_{z_i \text{ closed}} : z_0, \dots, z_r \text{ path in } A \text{ from } z_0 = x \text{ to } z_r = y \,\Big\}\,.$$

For x in A and E a subset of A, we define the travel time between x and E in A by

$$T_A(x,E) \,=\, \inf_{y \in E} \, T_A(x,y)\,.$$

3 An application of the BK inequality

An application of the BK inequality gives a control on the travel time until the infinite cluster, and this yields a control on the travel time to exit a finite domain.

Lemma 3.1. *Let A be a finite subset of $\mathbb{Z}^3$ and let $x \in A$. We have*

$$\forall k \geq 1 \qquad P\big(\, T_A(x, \partial^{\,in} A) \,\geq\, k \,\big) \,\leq\, (1-\theta(p))^k\,.$$

Proof. Let A be a finite subset of $\mathbb{Z}^3$ and let $x \in A$. The event $\{ x \longleftrightarrow \infty \}$ is included in the event $\{ x \longleftrightarrow \partial^{in} A \}$, thus

$$P\big(T_A(x, \partial^{in} A) = 0\big) \,\geq\, P\big(x \longleftrightarrow \infty\big) \,=\, \theta(p)\,,$$

or, passing to the complementary event,

$$P\big(T_A(x, \partial^{in} A) \geq 1\big) \,\leq\, 1 - \theta(p)\,.$$

In fact, if $T_A(x, \partial^{in} A) \geq 1$, then $\partial^{out} C(x)$, the outer boundary of the open cluster of x, contains a set of closed sites which separates x from ∞. We iterate next this argument. We set $C_0(x) = C(x)$ and we define successively, for $k \geq 0$,

$$C_{k+1}(x) \,=\, C_k(x) \,\cup\, \partial^{out} C_k(x) \,\cup\, \big\{ y \in \mathbb{Z}^3 : y \longleftrightarrow \partial^{out}\big(\partial^{out} C_k(x)\big) \big\}\,.$$

If $C_0(x) = \varnothing$, we define $\partial^{out} C_0(x) = \{ x \}$. It follows directly from this construction that

$$\forall k \geq 0 \qquad C_k(x) \,=\, \big\{ y \in \mathbb{Z}^3 : T_{\mathbb{Z}^3}(x, y) \,\leq\, k \big\}\,.$$

Therefore,

$$T_A(x, \partial^{in} A) \,\geq\, k \qquad \Longrightarrow \qquad C_{k-1}(x) \,\subset\, A \setminus \partial^{in} A\,, \quad \partial^{out} C_{k-1}(x) \,\subset\, A\,.$$

The set $C_k(x)$ becomes infinite when it meets the infinite open cluster. Whenever $C_k(x)$ is finite, its outer boundary $\partial^{out} C_k(x)$ contains a set of closed sites which separates x from ∞. This set realizes the event $\big\{ x \not\longleftrightarrow \infty \big\}$. Moreover, by construction, the sets $\partial^{out} C_k(x)$, $k \geq 0$, are pairwise disjoint. Thus we have

$$\big\{ T_A(x, \partial^{in} A) \,\geq\, k \big\} \,\subset\, \big\{ x \not\longleftrightarrow \infty \text{ occurs disjointly } k \text{ times} \big\}\,.$$

Applying the BK inequality (see for instance [1]), we conclude that

$$P\big(T_A(x, \partial^{in} A) \,\geq\, k \big) \,\leq\, P\big(x \not\longleftrightarrow \infty \big)^k \,\leq\, (1 - \theta(p))^k$$

as required. □

4 Cubic boxes

We consider here the case of a cubic box Λ centered at the origin. Let F_i, $1 \leq i \leq 6$, be the faces of Λ. Each face F_i is a square, which is itself the union of four squares F_i^j, $1 \leq j \leq 4$. Each of these squares shares a vertex with a vertex of Λ and admits the center of F_i as another vertex. We have

$$T_\Lambda(0, \partial^{in} \Lambda) \,=\, \min_{1 \leq i \leq 6} \min_{1 \leq j \leq 4} T_\Lambda(0, F_i^j)$$

and, by the FKG inequality,

$$\begin{aligned} P\big(T_\Lambda(0, \partial^{in} \Lambda) \geq k\big) \,&=\, P\big(\forall i \in \{ 1, \dots, 6 \} \quad \forall j \in \{ 1, \dots, 4 \} \quad T_\Lambda(0, F_i^j) \geq k\big) \\ &\geq \prod_{1 \leq i \leq 6} \prod_{1 \leq j \leq 4} P\big(T_\Lambda(0, F_i^j) \geq k\big) \,=\, P\big(T_\Lambda(0, F_1^1) \geq k\big)^{24}\,. \end{aligned}$$

The last inequality is a consequence of the symmetry of the model, indeed the random variables $T_\Lambda(0, F_i^j)$, $1 \leq i \leq 6$, $1 \leq j \leq 4$, are identically distributed. It follows then from lemma 3.1 that

$$P\big(T_\Lambda(0, F_1^1) \geq k\big) \,\leq\, (1 - \theta(p))^{k/24}\,.$$

We deal next with translated boxes. If Γ is a cubic box, we denote its faces by $F_i(\Gamma)$, $1 \leq i \leq 6$, and we denote by $F_i^j(\Gamma)$, $1 \leq i \leq 6$, $1 \leq j \leq 4$, the tiling of the faces of Γ into four squares. We consider the event $\mathcal{E}(\Lambda, k)$ defined as follows: for every cubic box Γ included in Λ, whose center is a site of $\mathbb{Z}^3$, whose sidelength is an integer, the center of Γ can be joined to each of the 24 squares on the faces of Γ with a path having at most k closed sites. More precisely,

$$\mathcal{E}(\Lambda, k) \,=\, \big\{\, \forall \Gamma = x + \Lambda(m) \subset \Lambda \quad T_\Lambda(x, F_i^j(\Gamma)) \leq k \text{ for } 1 \leq i \leq 6,\, 1 \leq j \leq 4 \,\big\}\,.$$

Proposition 4.1. *Let p be such that $\theta(p) > 0$ and let $\alpha > 0$. There exists a constant c, depending on α and p, such that*

$$\forall n \geq 2 \qquad P\big(\mathcal{E}(\Lambda(n), c \ln n)\big) \,\geq\, 1 - \frac{c}{n^\alpha}\,.$$

Proof. Let us estimate the probability of the complement of the event $\mathcal{E}(\Lambda, k)$:

$$\begin{aligned} P\big(\mathcal{E}(\Lambda, k)^c\big) \,&=\, P\big(\exists\, \Gamma = x + \Lambda(m) \subset \Lambda \quad \exists\, i, j \quad T_\Lambda(x, F_i^j(\Gamma)) > k\big) \\ &\leq \sum_{x \in \Lambda} \,\sum_{m : x + \Lambda(m) \subset \Lambda} \,\sum_{i,j} \, P\big(T_\Lambda(x, F_i^j(x + \Lambda(m))) > k\big)\,. \end{aligned}$$

By translation invariance and symmetry, the probability inside the sum depends neither on x nor on i, j. The number of subboxes Γ included in Λ is bounded by $|\Lambda| \times \text{diameter}(\Lambda)$, so we conclude with the help of the previous estimate that

$$P\big(\mathcal{E}(\Lambda, k)^c\big) \,\leq\, |\Lambda| \times \text{diameter}(\Lambda) \times 24 \times (1 - \theta(p))^{k/24}\,.$$

We take now $\Lambda = \Lambda(n)$ and $k = c \ln n$. For any $\alpha > 0$, we can choose the constant c sufficiently large so that the righthand side is smaller than $c n^{-\alpha}$ for any $n \geq 1$. ☐

Proposition 4.2. *Let $n \geq 1$ and suppose that the event $\mathcal{E}(\Lambda(n), c \ln n)$ occurs. There exists a constant c' such that*

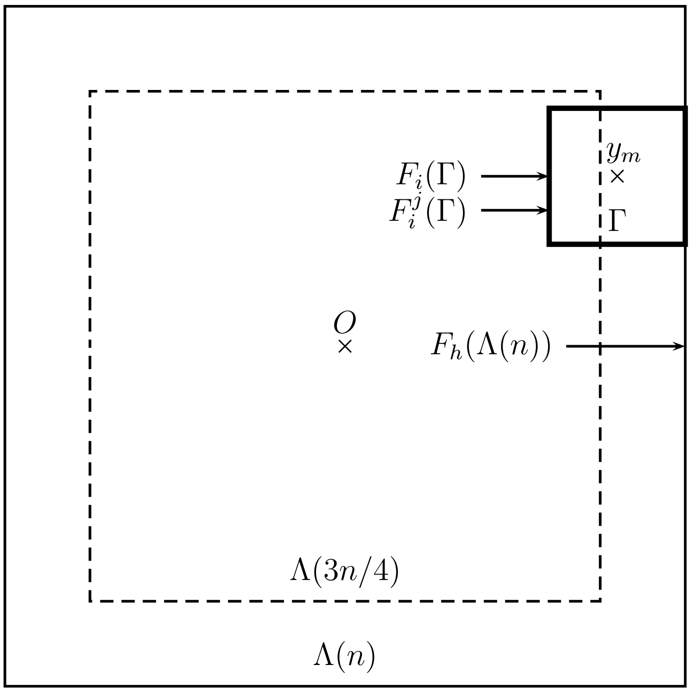
$$\forall x \in \Lambda(n) \setminus \Lambda(n/4) \quad \exists\, y \in \Lambda(n/4) \qquad T_{\Lambda(n)}(x, y) \,\leq\, c' (\ln n)^2\,.$$

Proof. Let x be a site in $\Lambda(n) \setminus \Lambda(n/4)$. We consider the largest cubic box centered at x included in $\Lambda(n)$. We tile the boundary of this box into 24 squares, and we consider the square which "goes" towards the center of $\Lambda(n)$. We look for a point on this square which has the smallest travel time starting from x. We restart this construction starting from this point, and we iterate until we reach the box $\Lambda(n/4)$. More precisely, we build iteratively a sequence travelling from x to the box $\Lambda(3n/4)$. We start from $y_0 = x$. If x belongs to $\partial^{\,in} \Lambda(n)$, we choose for y_1 a site in $\Lambda(n) \setminus \partial^{\,in} \Lambda(n)$ such that $|x - y_1| \leq 2$. If x belongs to $\Lambda(n) \setminus \partial^{\,in} \Lambda(n)$, we set $y_1 = x$. Suppose that $y_0, \dots, y_m$ have been built in such a way that the following four conditions are satisfied for any $l \in \{\, 1, \dots, m-1 \,\}$:

- $y_l \in \Lambda(n) \setminus \Lambda(3n/4)$.
- $T_{\Lambda(n)}(y_l, y_{l+1}) \,\leq\, c \ln n$.
- $\forall i \in \{\, 1, \dots, 6 \,\} \qquad d(y_{l+1}, F_i(\Lambda(n))) \,\geq\, \min\big(d(y_l, F_i(\Lambda(n))), n/4\big)$.
- If h is the smallest index such that $d(y_l, \partial^{\,in} \Lambda(n)) \,=\, d(y_l, F_h(\Lambda(n)))$, then

$$d(y_{l+1}, F_h(\Lambda(n))) \,\geq\, 2 d(y_l, F_h(\Lambda(n)))\,.$$

If y_m belongs to $\Lambda(3n/4)$, the construction terminates. Suppose that y_m does not belong to $\Lambda(3n/4)$. We will next find a site y_{m+1} so that the sequence $y_0, \dots, y_{m+1}$ satisfies the



The box Γ and the face $F_i(\Gamma)$

four conditions above. Let Γ be the largest cubic box centered at y_m included in $\Lambda(n)$. Let h be the smallest index such that

$$d(y_m, \partial^{in}\Lambda(n)) \,=\, d(y_m, F_h(\Lambda(n)))\,.$$

One face of Γ is included in $F_h(\Lambda(n))$. Let $i \in \{\,1,\dots,6\,\}$ be the index such that $F_i(\Gamma)$ is the face of Γ opposite to this face. We have then

$$d(F_i(\Gamma), F_h(\Lambda(n))) \,\geq\, 2d(y_m, F_h(\Lambda(n)))\,.$$

We choose next the index j so that $F_i^j(\Gamma)$ is the square included in $F_i(\Gamma)$ which is among the deepest inside the box $\Lambda(n)$. More precisely, we choose j in $\{\,1,\dots,4\,\}$ such that

$$\forall l \in \{\,1,\dots,6\,\} \qquad d(F_i^j(\Gamma), F_l(\Lambda(n))) \,\geq\, \min\big(d(y_m, F_l(\Lambda(n))), n/4\big)\,.$$

Since the event $\mathcal{E}(\Lambda(n), c\ln n)$ occurs, there exists y_{m+1} in $F_i^j(\Gamma)$ such that

$$T_{\Lambda(n)}(y_m, y_{m+1}) \,\leq\, c\ln n\,.$$

With this choice of y_{m+1}, the sequence $y_0,\dots,y_{m+1}$ satisfies the four conditions above. We prove next that the construction stops, i.e., that the sequence enters the box $\Lambda(3n/4)$ after a finite number of steps. In fact, as long as the sequence is outside of $\Lambda(3n/4)$, the distance to the face of $\Lambda(n)$ which is closest is multiplied by 2, while the distances to the other faces increase or stay larger than $n/4$. For $l \in \{\,0,\dots,m\,\}$, let $h(l)$ be the smallest index such that

$$d(y_l, \partial^{in}\Lambda(n)) \,=\, d(y_l, F_{h(l)}(\Lambda(n)))\,.$$

Until the entrance into $\Lambda(3n/4)$, the index $h(l)$ varies between three values corresponding to the three faces of $\Lambda(n)$ which are closest to the starting point y_0. Thus, every three steps, one of these values is repeated and the distance to the boundary of $\Lambda(n)$ is doubling:

$$\forall l \in \{\,1,\dots,m-3\,\} \qquad d(y_{l+3},\partial^{\,in}\Lambda(n))\,\geq\,2d(y_l,\partial^{\,in}\Lambda(n))\,.$$

It follows that

$$d(y_m,\partial^{\,in}\Lambda(n))\,\geq\,2^{\left\lfloor\frac{m-1}{3}\right\rfloor}\,.$$

If the construction has not stopped after m steps, then y_{m-1} is still outside of $\Lambda(3n/4)$, thus

$$d(y_{m-1},\partial^{\,in}\Lambda(n))\,\leq\,\frac{n}{4}+1\,.$$

These two inequalities imply that

$$2^{\left\lfloor\frac{m-2}{3}\right\rfloor}\,\leq\,\frac{n}{4}+1\,,$$

thus the construction stops at some step m^* satisfying $m^*\leq c'\ln n$, where c' is a positive constant. Now the point y_{m^*} is inside the box $\Lambda(3n/4)$ and we have

$$T_{\Lambda(n)}(x,y_{m^*})\,\leq\,T_{\Lambda(n)}(x,y_1)\,+\sum_{0<m<m^*}T_{\Lambda(n)}(y_m,y_{m+1})\,\leq\,3+cc'(\ln n)^2\,.$$

The site y_{m^*} belongs to the box $\Lambda(3n/4)$ and its distance to the boundary of $\Lambda(n)$ is larger or equal than $n/4$. By using a few more cubic boxes of side $n/4$ (nine boxes are certainly enough), we can join y_{m^*} to the box $\Lambda(n/4)$ with a path having at most $9c\ln n$ closed sites. More precisely, we consider the box $y_{m^*}+\Lambda(n/4)$, we tile the boundary of this box into 24 squares and we consider the square which is closest to the box $\Lambda(n/4)$ (or one of them if there are several). We choose a point on this square minimizing the travel time starting from y_m^* and we reiterate this construction starting from this point. At each step, as long as we are outside of $\Lambda(n/4)$, we gain a distance $n/4$ along one coordinate axis. Thus we will enter the box $\Lambda(n/4)$ in at most nine steps. We concatenate the two paths in order to obtain the desired estimate. □

5 The tiling of the sphere

We denote by S the two dimensional sphere of $\mathbb{R}^3$. We consider the hyperplanes of equations:

$$x=0\,,\quad y=0\,,\quad z=0\,,$$
$$x=y\,,\quad x=-y\,,\quad x=z\,,\quad x=-z\,,\quad y=z\,,\quad y=-z\,.$$

Let $\mathcal{S}$ be the set of the orthogonal symmetries with respect to these hyperplanes. These hyperplanes induce a tiling of the sphere S into 48 spherical triangles. We denote by $\mathcal{T}$ the collection of these triangles. The vertices of the triangles of $\mathcal{T}$ are the vertices of a convex polyhedron which is a Catalan solid, it is called the disdyakis dodecahedron or the hexakis octahedron [2, 3]. The group of the isometries generated by $\mathcal{S}$ acts transitively on the collection $\mathcal{T}$ of spherical triangles. Let us consider one of these triangles, for instance the triangle having for vertices

$$(1,0,0)\,,\quad\Big(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\Big)\,,\quad\Big(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\Big)\,.$$

The longest arc of this triangle is the arc joining the vertices $(1,0,0)$ and $(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ and its length is $\arccos\big(1/\sqrt{3}\big)<0.96$. Let $r>0$. We define

$$B(r)\,=\,\big\{\,(x,y,z)\in\mathbb{Z}^3:x^2+y^2+z^2\leq r^2\,\big\}\,.$$

Let T belong to $\mathcal{T}$. We define

$$T_r \,=\, \big\{\, x \in B(r) : d(x, rT) \leq 3 \,\big\}\,.$$

We have

$$\forall y,z \in T_r \qquad |y-z| \,\leq\, 6 + r\,\text{diameter}(T) \,\leq\, 6 + 0.96r\,.$$

Therefore

$$\forall r \geq 600 \quad \forall y,z \in T_r \qquad |y-z| \,\leq\, 0.97r\,.$$

For any symmetry s in $\mathcal{S}$, we have $s\big(B(r)\big) = B(r)$ and $s\big(T_r\big) = s(T)_r$. The percolation model is invariant under the action of $\mathcal{S}$, therefore

$$P\big(0 \longleftrightarrow T_r \text{ in } B(r)\big) \,=\, P\big(0 \longleftrightarrow s(T)_r \text{ in } B(r)\big)$$

and the above probability is the same for any triangle T in $\mathcal{T}$. Moreover

$$\partial^{\,in} B(r) \,\subset\, \bigcup_{T \in \mathcal{T}} T_r\,.$$

Proceeding as in the case of the cube, we have, for any $r > 0$,

$$T_{B(r)}(0, \partial^{\,in} B(r)) \,\geq\, \min_{T \in \mathcal{T}} T_{B(r)}(0, T_r)$$

and, by the FKG inequality,

$$\begin{aligned} P\big(T_{B(r)}(0, \partial^{\,in} B(r)) \geq k\big) \,&\geq\, P\big(\forall T \in \mathcal{T} \quad T_{B(r)}(0, T_r) \geq k\big) \\ &\geq\, \prod_{T \in \mathcal{T}} P\big(T_{B(r)}(0, T_r) \geq k\big)\,. \end{aligned}$$

By symmetry of the model, all the probabilities appearing in the product are equal. It follows then from lemma 3.1 that

$$\forall T \in \mathcal{T} \quad \forall r > 0 \qquad P\big(T_{B(r)}(0, T_r) \geq k\big) \,\leq\, (1 - \theta(p))^{k/48}\,.$$

We deal next with translates of S. We consider the event $\mathcal{F}(\Lambda, k)$ defined as follows: for any $x \in \Lambda \cap \mathbb{Z}^3$ and any $r > 0$ such that $x + B(r) \subset \Lambda$ and such that the boundary of $x + B(r)$ intersects the lattice $\mathbb{Z}^3$, the site x can be joined to each of the 48 sets $x + T_r$, $T \in \mathcal{T}$, with a path having at most k closed sites. More precisely,

$$\begin{aligned} \mathcal{F}(\Lambda, k) \,=\, \Big\{ &\forall x \in \Lambda \cap \mathbb{Z}^3 \quad \forall r > 0 \\ &x + B(r) \subset \Lambda,\, (x + \partial B(r)) \cap \mathbb{Z}^3 \neq \varnothing \quad \Longrightarrow \quad \forall T \in \mathcal{T} \quad T_{B(r)}(0, T_r) \leq k \Big\}\,. \end{aligned}$$

Proposition 5.1. *Let p be such that $\theta(p) > 0$ and let $\alpha > 0$. There exists a constant c, depending on α and p, such that*

$$\forall n \geq 2 \qquad P\big(\mathcal{F}(\Lambda(n), c \ln n)\big) \,\geq\, 1 - \frac{c}{n^\alpha}\,.$$

Proof. The important point is to notice that the number of choices for the site x and the radius r is bounded by $|\Lambda(n)|^2$. The rest of the proof is the same as proposition 4.1. □

Proposition 5.2. *Let $n \geq 1$ and suppose that the event $\mathcal{F}(\Lambda(n), c \ln n)$ occurs. There exists a constant c' such that*

$$\forall x,y \in \Lambda(n/4) \qquad T_{\Lambda(n)}(x,y) \,\leq\, c'(\ln n)^2\,.$$

Proof. We build a sequence starting at x and which converges at geometric speed towards y, and which stops when it is at distance less than 600 from y. We start from $y_0 = x$. Suppose that $y_0, \dots, y_m$ have been built in such a way that for any $l \in \{1, \dots, m\}$:

- $y_l \in \Lambda(n/4)$.
- $|y_l - y| \leq 0.97\,|y_{l-1} - y|$.
- $T_{\Lambda(n)}(y_{l-1}, y_l) \leq c \ln n$.

We build now y_{m+1}. Let $r > 0$ be such that y is on the boundary of $y_m + B(r)$. Since y and y_m are in $\Lambda(n/4)$, then $y_m + B(r)$ is included in $\Lambda(n)$. If $r < 600$, then $|y_m - y| < 600$ and the construction is finished. Suppose that $r \geq 600$. There exists $T \in \mathcal{T}$ such that y is in $y_m + T_r$. Since the event $\mathcal{F}(\Lambda(n), c \ln n)$ occurs, then there exists $y_{m+1} \in y_m + T_r$ such that

$$T_{y_m + B(r)}(y_m, y_{m+1}) \leq c \ln n\,.$$

Since $r \geq 600$, then

$$|y_{m+1} - y| \leq 0.97 |y_m - y|\,,$$

and the sequence $y_0, \dots, y_{m+1}$ satisfies the required constraints. Since the sequence converges at geometric speed towards y, after at most $c' \ln n$ steps, where c' is a constant, it is at distance less than 600 from y and the construction terminates at some index $m^* \leq c' \ln n$. Now we have

$$T_{\Lambda(n)}(x, y) \leq \sum_{0 \leq m < m^*} T_{\Lambda(n)}(y_m, y_{m+1}) + T_{\Lambda(n)}(y_{m^*}, y) \leq cc'(\ln n)^2 + 1800\,.$$

By enlarging the constants, we obtain the statement of the proposition. □

In the previous argument, it was convenient to use the tiling of the sphere instead of the cube, because the diameter of a tile is less than the radius multiplied by 0.97. This would not have been the case with a cube.

6 Completion of the proof of theorem 1.1

We need only to prove the statement for n large enough. Indeed, if it holds for $n \geq N$, we simply enlarge the constant κ so that $\kappa(\ln 2)^2 \geq 3(2N+1)$. We have then

$$\forall n \in \{2, \dots, N\} \quad \forall x, y \in \Lambda(n) \quad T_{\Lambda(n)}(x, y) \leq \kappa(\ln 2)^2 \leq \kappa(\ln n)^2\,.$$

Let $\alpha > 0$. By propositions 4.1 and 5.1, there exists a constant $c > 0$ such that

$$\forall n \geq 2 \qquad P\big(\mathcal{E}(\Lambda(n), c \ln n) \cap \mathcal{F}(\Lambda(n), c \ln n)\big) \geq 1 - \frac{2c}{n^{\alpha+1}}\,.$$

For n large enough, we have $2c/n^{\alpha+1} < 1/n^{\alpha}$. Suppose now that the events $\mathcal{E}(\Lambda(n), c \ln n)$ and $\mathcal{F}(\Lambda(n), c \ln n)$ occur simultaneously. Let $x, y \in \Lambda(n)$. By proposition 4.2, there exist x^*, y^* in $\Lambda(n/4)$ such that

$$T_{\Lambda(n)}(x, x^*) \leq c'(\ln n)^2\,, \qquad T_{\Lambda(n)}(y, y^*) \leq c'(\ln n)^2\,.$$

By proposition 5.2, since x^*, y^* are in $\Lambda(n/4)$, then

$$T_{\Lambda(n)}(x^*, y^*) \leq c'(\ln n)^2\,.$$

We conclude that

$$T_{\Lambda(n)}(x, y) \leq T_{\Lambda(n)}(x, x^*) + T_{\Lambda(n)}(x^*, y^*) + T_{\Lambda(n)}(y^*, y) \leq 3c'(\ln n)^2\,.$$

This holds for any x, y in $\Lambda(n)$, so we are done.

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