Asymptotics of the probability distributions of the first hitting times of Bessel processes[*]

Yuji Hamana[†]   Hiroyuki Matsumoto[‡]

Abstract

The asymptotic behavior of the tail probabilities for the first hitting times of the Bessel process with arbitrary index is shown without using the explicit expressions for the distribution function obtained in the authors' previous works.

Keywords: Bessel process; hitting time; tail probability.
AMS MSC 2010: 60G40.
Submitted to ECP on December 20, 2013, final version accepted on January 29, 2014.

1 Introduction and main results

Let $\mathbb{P}_a^{(\nu)}$ be the probability law on the path space $W = C([0,\infty); \mathbf{R})$ of a Bessel process with index $\nu \in \mathbf{R}$ or dimension $\delta = 2(\nu + 1)$ starting from $a > 0$. For $w \in W$ we denote the first hitting time to $b > 0$ by $\tau_b = \tau_b(w)$:

$$\tau_b = \inf\{t > 0; w(t) = b\}.$$

In recent works [4, 5] the authors have shown explicit forms of the distribution function and the density of the distribution of τ_b under $\mathbb{P}_a^{(\nu)}$ in the case where $0 < b < a$. The other case, which is easier since we do not need to consider a natural boundary, has been known by Kent [6]. See also Getoor-Sharpe [2].

When $\nu > 0$ and $b < a$, it is shown in [5] that there exists a positive constant $C(\nu)$ such that

$$\mathbb{P}_a^{(\nu)}(\tau_b > t) = 1 - \left(\frac{b}{a}\right)^{2\nu} + C(\nu)t^{-\nu} + o(t^{-\nu}).$$

The constant $C(\nu)$ may be expressed explicitly and we could treat all the cases. However, we need to consider separately the case where δ is an odd integer and the expression for $C(\nu)$ is different from the othere cases. This is because the expression for the distribution function itself is different.

The aim of this note is to show that the constant $C(\nu)$ has the same simple expression also when δ is an odd integer by considering the asymptotics without using the explicit expressions for the distribution functions obtained in [5].

When $a < b$ and $\nu > 0$, the explicit expressions for $\mathbb{P}_a^{(\nu)}(\tau_b > t)$ have been shown in [6] and, from his result, it is easily shown that the tail probability decays exponentially. Hence we concentrate on the case of $b < a$.

[†]Kumamoto University, Japan. E-mail: hamana@kumamoto-u.ac.jp
[‡]Aoyama Gakuin University, Japan. E-mail: matsu@gem.aoyama.ac.jp

Theorem 1.1. *Let $\nu > 0$ and $0 < b < a$. Then, as $t \to \infty$, it holds that*

$$\mathbb{P}_a^{(\nu)}(t < \tau_b < \infty) = b^{2\nu}\Big\{1 - \Big(\frac{b}{a}\Big)^{2\nu}\Big\}\frac{1}{\Gamma(1+\nu)(2t)^\nu} + O(t^{-\nu-\varepsilon})$$

for any $\varepsilon \in (0, \frac{\nu}{1+\nu})$, where Γ denotes the usual Gamma function.

Theorem 1.2. *Let $\nu < 0$ and $0 < b < a$. Then, as $t \to \infty$, it holds that*

$$\mathbb{P}_a^{(\nu)}(\tau_b > t) = a^{2|\nu|}\Big\{1 - \Big(\frac{b}{a}\Big)^{2|\nu|}\Big\}\frac{1}{\Gamma(1+|\nu|)(2t)^{|\nu|}} + O(t^{-|\nu|-\varepsilon})$$

for any $\varepsilon \in (0, \frac{|\nu|}{1+|\nu|})$.

When $\nu = 0$, it is known that

$$\mathbb{P}_a^{(0)}(\tau_b > t) = \frac{2\log(a/b)}{\log t} + o((\log t)^{-1}).$$

This identity has been discussed in [5] and we omit the details.

2　Proof of Theorem 1.1

We assume $\nu > 0$ in this section. At first we give some lemmas. The first one is shown by Byczkowski and Ryznar [1].

Lemma 2.1. *There exists a constant C, independent of t, such that*

$$\mathbb{P}_a^{(\nu)}(t < \tau_b < \infty) \leqq Ct^{-\nu}.$$

Lemma 2.2. *If $0 < b < a$, one has*

$$\mathbb{P}_a^{(\nu)}(\tau_b > t) = 1 - \Big(\frac{b}{a}\Big)^{2\nu} + \mathbb{E}_a^{(\nu)}\Big[\Big(\frac{b}{R_t}\Big)^{2\nu}\mathbf{1}_{\{\inf_{0\leqq s\leqq t} R_s > b\}}\Big] \qquad (2.1)$$

for any $t > 0$, where $\mathbb{E}_a^{(\nu)}$ is the excpectation with respect to $\mathbb{P}_a^{(\nu)}$ and $\{R_s\}_{s\geqq 0}$ denotes the coordinate process.

Proof. It is well known that

$$\mathbb{P}_a^{(\nu)}(\tau_b = \infty) = \mathbb{P}_a^{(\nu)}(\inf_{s\geqq 0} R_s > b) = 1 - \Big(\frac{b}{a}\Big)^{2\nu}.$$

By the Markov property of Bessel processes, we have for $t > 0$

$$\begin{aligned}
\mathbb{P}_a^{(\nu)}(\tau_b = \infty) &= \mathbb{P}_a^{(\nu)}(\inf_{0\leqq s\leqq t} R_s > b \text{ and } \inf_{s\geqq t} R_s > b)\\
&= \mathbb{E}_a^{(\nu)}[\mathbb{P}_{R_t}^{(\nu)}(\tau_b = \infty)\mathbf{1}_{\{\inf_{0\leqq s\leqq t} R_s > b\}}]\\
&= \mathbb{E}_a^{(\nu)}\Big[\Big\{1 - \Big(\frac{b}{R_t}\Big)^{2\nu}\Big\}\mathbf{1}_{\{\inf_{0\leqq s\leqq t} R_s > b\}}\Big],
\end{aligned}$$

which implies (2.1). $\qquad\square$

Lemma 2.3. *For any $a > 0$ and p with $0 < p < 1 + \nu$, it holds that*

$$\frac{\Gamma(1+\nu-p)}{\Gamma(1+\nu)}\frac{1}{(2t)^p}e^{-\frac{a^2}{2t}} \leqq \mathbb{E}_a^{(\nu)}[(R_t)^{-2p}] \leqq \frac{\Gamma(1+\nu-p)}{\Gamma(1+\nu)}\frac{1}{(2t)^p} + Ct^{-1-p} \qquad (2.2)$$

for $t \geqq 1$, where C is a positive constant independent of t.

Proof. By the explicit expression for the transition density of the Bessel process, we have

$$\mathbb{E}_a^{(\nu)}[(R_t)^{-2p}] = \int_0^\infty y^{-2p} \frac{1}{t} \left(\frac{y}{a}\right)^\nu y e^{-\frac{a^2+y^2}{2t}} I_\nu\left(\frac{ay}{t}\right) dy,$$

where I_ν is the modified Bessel function of the first kind with index ν (cf [8]) given by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{(z/2)^{2n}}{\Gamma(n+1)\Gamma(1+\nu+n)} \quad (z \in \mathbf{R} \setminus (-\infty, 0)).$$

Hence, it is easy to get

$$\mathbb{E}_a^{(\nu)}[(R_t)^{-2p}] = \frac{1}{(2t)^p} e^{-a^2/2t} \sum_{n=0}^\infty \frac{a^{2n}\Gamma(n+\nu+1-p)}{\Gamma(n+1)\Gamma(1+n+\nu)(2t)^n}$$

and the assertion of the lemma. $\qquad\square$

Remark 2.4. *The moments of R_t for fixed t have explicit expressions by means of the Whittaker functions (cf. [3], p.709), but it does not seem useful.*

We are now in a position to give a complete proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.2 we have

$$\mathbb{P}_a^{(\nu)}(t < \tau_b < \infty) = b^{2\nu}\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}] - b^{2\nu}\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}\mathbf{1}_{\{\tau_b \leqq t\}}].$$

For the first term we have shown in Lemma 2.3

$$\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}] = \frac{1}{\Gamma(1+\nu)(2t)^\nu}(1 + O(t^{-1})).$$

Hence, if we could show

$$\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}\mathbf{1}_{\{\tau_b \leqq t\}}] = \frac{1}{\Gamma(\nu+1)(2t)^\nu}\left(\frac{b}{a}\right)^{2\nu} + O\left(\frac{1}{t^{\nu+\varepsilon}}\right) \tag{2.3}$$

for any $\varepsilon \in (0, \frac{\nu}{\nu+1})$, we obtain the assertion of the theorem.

For this purpose, we let $\alpha \in (0, \frac{1}{\nu+1})$, choose p satisfying

$$\frac{1}{1-\alpha} < p < \frac{1+\nu}{\nu}$$

and let q be such that $p^{-1} + q^{-1} = 1$. We devide the expectation on the right hand side of (2.3) into the sum of

$$I_1 = \mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}\mathbf{1}_{\{\tau_b \leqq t^{\alpha q}\}}] \quad \text{and} \quad I_2 = \mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}\mathbf{1}_{\{t^{\alpha q} < \tau_b \leqq t\}}]$$

We simply apply the Hölder inequality to I_2. Then we get

$$\begin{aligned}
I_2 &\leqq \mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}\mathbf{1}_{\{t^{\alpha q} < \tau_b < \infty\}}] \\
&\leqq \left\{\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu p}]\right\}^{1/p} \left\{\mathbb{P}_a^{(\nu)}(t^{\alpha q} < \tau_b < \infty)\right\}^{1/q}.
\end{aligned}$$

and, by Lemmas 2.1 and 2.3, we see that there exists a constant C_1 such that

$$I_2 \leqq C_1 t^{-\nu-\alpha\nu}.$$

In the following we mean by C_i's some constants independent of t.

For I_1, the strong Markov property of Bessel processes implies

$$I_1 = \int_0^{t^{\alpha q}} \mathbb{E}_b^{(\nu)}[(R_{t-s})^{-2\nu}]\mathbb{P}_a^{(\nu)}(\tau_b \in ds) = I_{11} + I_{12},$$

where

$$I_{11} = \int_0^{t^{\alpha q}} \frac{1}{2^\nu \Gamma(\nu+1)} \frac{1}{(t-s)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b \in ds).$$

Since $\alpha q < 1$, Lemma 2.1 imples

$$|I_{12}| = |I_1 - I_{11}| \leqq \int_0^{t^{\alpha q}} \frac{C_2}{(t-s)^{\nu+1}} \mathbb{P}_a^{(\nu)}(\tau_b \in ds) \leqq \frac{C_3}{t^{\nu+1}}.$$

We devide I_{11} into the sum of

$$J_1 = \int_{t^\alpha}^{t^{\alpha q}} \frac{1}{2^\nu \Gamma(\nu+1)} \frac{1}{(t-s)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b \in ds)$$

and

$$J_2 = \int_0^{t^\alpha} \frac{1}{2^\nu \Gamma(\nu+1)} \frac{1}{(t-s)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b \in ds).$$

For J_1 we have by Lemma 2.1

$$0 \leqq J_1 \leqq \frac{C_4}{(t-t^{\alpha q})^\nu} \mathbb{P}_a^{(\nu)}(t^\alpha < \tau_b < \infty) \leqq \frac{C_5}{t^{\nu+\alpha\nu}}.$$

For J_2 we have

$$J_2 \leqq \frac{1}{2^\nu \Gamma(\nu+1)(t-t^\alpha)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b \leqq t^\alpha)$$

$$\leqq \frac{1}{\Gamma(\nu+1)(2t)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b < \infty) \frac{1}{(1-t^{-(1-\alpha)})^\nu}$$

$$\leqq \frac{1}{\Gamma(\nu+1)(2t)^\nu} \left(\frac{b}{a}\right)^{2\nu} \left(1 + \frac{C_6}{t^{1-\alpha}}\right).$$

On the other hand we have by Lemma 2.1

$$J_2 \geqq \frac{1}{\Gamma(\nu+1)(2t)^\nu} \mathbb{P}_a^{(\nu)}(\tau_b \leqq t^\alpha)$$

$$= \frac{1}{\Gamma(\nu+1)(2t)^\nu} \left\{\mathbb{P}_a^{(\nu)}(\tau_b < \infty) - \mathbb{P}_a^{(\nu)}(t^\alpha \leqq \tau_b < \infty)\right\}$$

$$\geqq \frac{1}{\Gamma(\nu+1)(2t)^\nu} \left(\frac{b}{a}\right)^{2\nu} - \frac{C_7}{t^\nu t^{\alpha\nu}}.$$

Combining the above estimates, we obtain

$$\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}\mathbf{1}_{\{\tau_b \leqq t\}}] = \frac{1}{\Gamma(\nu+1)(2t)^\nu} \left(\frac{b}{a}\right)^{2\nu} + \frac{1}{t^\nu} O\left(\frac{1}{t^{\alpha\nu}} + \frac{1}{t} + \frac{1}{t^{1-\alpha}}\right).$$

Since

$$0 < \alpha\nu < \frac{\nu}{\nu+1} < 1-\alpha < 1$$

and we can choose arbitrary α satifying this condition,

$$\mathbb{E}_a^{(\nu)}[(R_t)^{-2\nu}\mathbf{1}_{\{\tau_b \leqq t\}}] = \frac{1}{\Gamma(\nu+1)(2t)^\nu} \left(\frac{b}{a}\right)^{2\nu} + O\left(\frac{1}{t^{\nu+\varepsilon}}\right)$$

holds for any $\varepsilon \in (0, \frac{\nu}{\nu+1})$.

Now we have shown (2.3) and the assertion of Theorem 1.1. $\qquad\square$

3 Proof of Theorem 1.2

Theorem 1.2 is easily obtained from Theorem 1.1.

We recall explcit expressions for the Laplace transforms of the distributions of τ_b: for $\nu \in \mathbf{R}$, it is known ([2, 5]) that

$$\mathbb{E}_a^{(\nu)}[e^{-\lambda \tau_b}] = \left(\frac{b}{a}\right)^\nu \frac{K_\nu(a\sqrt{2\lambda})}{K_\nu(b\sqrt{2\lambda})}, \qquad \lambda > 0,$$

where K_ν is the modified Bessel function of the second kind. From this identity we easily obtain for $\nu > 0$

$$\mathbb{P}_a^{(-\nu)}(\tau_b \in dt) = \left(\frac{a}{b}\right)^{2\nu} \mathbb{P}_a^{(\nu)}(\tau_b \in dt).$$

Hence we get from Theorem 1.1

$$\mathbb{P}_a^{(-\nu)}(\tau_b > t) = \left(\frac{a}{b}\right)^{2\nu} \mathbb{P}_a^{(\nu)}(t < \tau_b < \infty)$$
$$= a^{2\nu}\left\{1 - \left(\frac{b}{a}\right)^{2\nu}\right\}\frac{1}{\Gamma(1+\nu)(2t)^\nu}(1 + o(1)).$$

Acknowledgement. We thank Professor Yuu Hariya for valuable discussions.

References

[1] Byczkowski, T. and Ryznar, T.: Hitting distribution of geometric Brownian motion. *Studia Math.* **173**, (2006), 19–38. MR-2204460

[2] Getoor, R. K. and Sharpe, M. J.: Excursions of Brownian motion and Bessel processes. *Z. Wahr. Ver. Gebiete* **47**, (1979), 83–106. MR-0521534

[3] Gradshteyn, I. S. and Ryzhik, I. M.: Table of Integrals, Series, and Products, 7th ed. *Academic Press*, Amsterdam, 2007. xlviii+1171 pp. MR-2360010

[4] Hamana, Y. and Matsumoto, H.: The probability densities of the first hitting times of Bessel processes. *J. Math-for-Industry* **4**, (2012), 91–95. MR-3072321

[5] Hamana, Y. and Matsumoto, H.: The probability distributions of the first hitting times of Bessel processes. *Trans. AMS* **365**, (2013), 5237–5257. MR-3074372

[6] Kent, J.T.: Eigenvalue expansions for diffusion hitting times. *Z. Wahr. Ver. Gebiete* **52**, (1980), 309–319. MR-0576891

[7] Revuz, D. and Yor, M.: Continuous Martingales and Brownian Motion, 3rd ed. *Springer-Verlag*, Berlin, 1999. xiv+602 pp. MR-1725357

[8] Watson, G. N.: A Treatise on the Theory of Bessel Functions. Reprinted of 2nd ed. *Cambridge University Press*, Cambridge, 1995. viii+804 pp. MR-1349110

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards

- Free for authors, free for readers

- Quick publication (no backlog)

Economical model of EJP-ECP

- Low cost, based on free software (OJS[1])

- Non profit, sponsored by IMS[2], BS[3], PKP[4]

- Purely electronic and secure (LOCKSS[5])

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)

- Submit your best articles to EJP-ECP

- Choose EJP-ECP over for-profit journals

---

[1]OJS: Open Journal Systems http://pkp.sfu.ca/ojs/
[2]IMS: Institute of Mathematical Statistics http://www.imstat.org/
[3]BS: Bernoulli Society http://www.bernoulli-society.org/
[4]PK: Public Knowledge Project http://pkp.sfu.ca/
[5]LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[6]IMS Open Access Fund: http://www.imstat.org/publications/open.htm