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Law of large numbers for critical first-passage percolation on the triangular lattice

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Abstract

We study the site version of (independent) first-passage percolation on the triangular lattice $\mathbb{T}$. Denote the passage time of the site v in $\mathbb{T}$ by $t(v)$, and assume that $P(t(v) = 0) = P(t(v) = 1) = 1/2$. Denote by $a_{0,n}$ the passage time from **0** to $(n, 0)$, and by $b_{0,n}$ the passage time from **0** to the halfplane $\{(x, y) : x \geq n\}$. We prove that there exists a constant $0 < \mu < \infty$ such that as $n \to \infty$, $a_{0,n}/\log n \to \mu$ in probability and $b_{0,n}/\log n \to \mu/2$ almost surely. This result confirms a prediction of Kesten and Zhang. The proof relies on the existence of the full scaling limit of critical site percolation on $\mathbb{T}$, established by Camia and Newman.

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1 Introduction

Standard first-passage percolation (FPP) was introduced by Hammersley and Welsh [8] in 1965 as a model of fluid flow through a random medium. See [11] for the basic theory and Section 2 in [7] for a summary of recent progress. The usual setup is that of FPP on lattice $\mathbb{Z}^d$, where i.i.d. non-negative random variables are assigned to nearest-neighbor edges in $\mathbb{Z}^d$. We call this setting the bond version of FPP on $\mathbb{Z}^d$. However, unless otherwise stated, we will focus on the site version of FPP on the triangular lattice $\mathbb{T}$, which is defined precisely in the following, and the reason will be explained later. The classic results of FPP are mainly stated for the bond version of FPP on $\mathbb{Z}^d$, but most of them also hold for the site version of FPP on $\mathbb{T}$. Unless otherwise stated, we just state them directly for the latter in this paper.

Let $\mathbb{T} = (\mathbb{V}, \mathbb{E})$ denote the triangular lattice , where $\mathbb{V}$ is the set of sites, and $\mathbb{E}$ is the set of bonds, connecting adjacent sites. Let $\{t(v) : v \in \mathbb{V}\}$ be an i.i.d. family of non-negative random variables with common distribution function F. A path is a sequence of distinct sites connected by nearest neighbor bonds. A circuit is a path which starts and ends at the same site and does not visit the same site twice, except for the starting site. Sometimes we see the circuit as a simple closed curve consisting of bonds of $\mathbb{E}$. Given a path γ, we define its **passage time** as

$$T(\gamma) := \sum_{v\in\gamma} t(v).$$

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The passage time between two site sets A, B is defined as

$$T(A, B) := \inf\{T(\gamma) : \gamma \text{ is a path connecting some site of } A \text{ with some site of } B\},$$

and a time minimizing path between A, B is called a **geodesic**. Denote the origin by **0**. Define

$$a_{0,n} := \inf\{T(\gamma) : \gamma \text{ is a path from } \mathbf{0} \text{ to } (n, 0)\},$$
$$b_{0,n} := \inf\{T(\gamma) : \gamma \text{ is a path from } \mathbf{0} \text{ to } \{(x, y) : x \geq n\}\}.$$

These are called the point to point and point to line passage times respectively. It is well known (Kingman [13], Wierman and Reh [23]) that if $Et(v) < \infty$, there is a nonrandom constant $\mu = \mu(F) < \infty$ such that

$$\lim_{n\to\infty} \frac{a_{0,n}}{n} = \lim_{n\to\infty} \frac{b_{0,n}}{n} = \mu \ \ a.s. \text{ and in } L^1, \tag{1.1}$$

where μ is called the **time constant**. Kesten [11] showed that

$$\mu = 0 \ \text{ iff } F(0) \geq p_c(\mathbb{T}, \text{site}) = \frac{1}{2}, \tag{1.2}$$

where $p_c(\mathbb{T}, \text{site})$ is the critical probability for site percolation on $\mathbb{T}$. Since there is a transition of the time constant at $F(0) = p_c$, Kesten and Zhang [12] call this “critical” FPP.

In this paper, we shall restrict ourselves to a special critical FPP, that is, we assume that

$$P(t(v) = 0) = P(t(v) = 1) = \frac{1}{2}. \tag{1.3}$$

Note that we can view this model as the critical site percolation on $\mathbb{T}$. Recall that it can be obtained by coloring the faces of the honeycomb lattice randomly, each cell being open (black) or closed (white) with probability 1/2 independently of the others. For this critical FPP, from (1.2) it is natural to ask whether or not the sequences in (1.1) converge to positive limits after properly normalizing. We give a historical note related to this problem here. Let θ stands for a or b. In a survey paper [10] (see the paragraph right below (3.16P) in [10]), Kesten pointed out that the results proved in [2] that $E\theta_{0,n}$ lies between two positive multiples of $\log n$ would imply that $\{\theta_{0,n}/\log n\}$ is a tight family, furthermore, using RSW and FKG, one may show that $P(\theta_{0,n} \leq \varepsilon \log n)$ is small for small ε, which implies that any limit distribution of $\theta_{0,n}/\log n$ has no mass at zero. Later, Kesten and Zhang [12] indicated that the estimates they developed in their paper can be used to prove a strong law of large numbers (SLLN) for $b_{0,n}$: $b_{0,n}/Eb_{0,n} \to 1 \ \ a.s.$ Further, they expected that $Eb_{0,n}/\log n$ and $Ea_{0,n}/\log n$ converge to finite, strictly positive limits as $n \to \infty$. In this paper, we continue the study from [12], the following is our main theorem:

Theorem 1.1. *For the critical FPP satisfying (1.3) on* $\mathbb{T}$*, there exists a constant* $0 < \mu < \infty$*, such that*

$$\lim_{n\to\infty} \frac{a_{0,n}}{\log n} = \mu \ \ \text{in probability}, \tag{1.4}$$

$$\lim_{n\to\infty} \frac{b_{0,n}}{\log n} = \frac{\mu}{2} \ \ a.s. \tag{1.5}$$

Furthermore, the convergence in (1.4) does not occur almost surely.

Remark 1.2. *In fact, using our method one can easily generalize Theorem 1.1 to point to point and point to line passage times along any given direction, and the limits will coincide with the theorem, since Camia and Newman's full scaling limit (see Section 2 below) is invariant under rotations. Furthermore, it is expected that the theorem holds for the classic bond version of FPP on $\mathbb{Z}^2$. Once the existence of full scaling limit of critical bond percolation on $\mathbb{Z}^2$ is established, one may derive the theorem by our strategy. Also, the limits will be the same as Theorem 1.1 because of the conjectural universality of critical percolation.*

Remark 1.3. *One may consider more general critical FPP on $\mathbb{T}$, for example, the distribution function F satisfying the conditions (1.4)–(1.6) in [12]. That is,*

$$P(t(v)=0)=\frac{1}{2}, E[t^{\delta}(v)]<\infty \text{ for some } \delta>4, P(0<t(v)<C_0) \text{ for some } C_0>0.$$

It is expected that Theorem 1.1 still holds for the F above (with $\mu(F)$ as a function of F).

For each $r>0$, let $\mathbb{D}_r$ denote the Euclidean disc of radius r centered at $\mathbf{0}$ and $\partial\mathbb{D}_r$ denote its boundary. Let $\mathbb{D}$ denote the unit disk for short. For $v\in\mathbb{V}$, let $B(v,r)$ denote the discrete ball of radius r centered at v in the triangular lattice:

$$B(v,r):=\mathbb{V}\cap\{v+\overline{\mathbb{D}}_r\}.$$

We denote by $\partial B(v,r)$ its boundary, which is the set of sites in $B(v,r)$ that have at least one neighbor outside $B(v,r)$. For short, we let $B(r):=B(\mathbf{0},r)$.

Remark 1.4. *We can express $a_{0,n}$ and $b_{0,n}$ in terms of circuits. For example, it is easy to see that $a_{0,n}$ and the maximum number of disjoint closed circuits which separate $\mathbf{0}$ and $(n,0)$ differ by at most 2. Note that with probability 1 there is no infinite cluster for the critical site percolation on $\mathbb{T}$, therefore the cluster boundaries form loops. Now we introduce two quantities for this model, which are similar as $a_{0,n}$ and $b_{0,n}$ respectively:*

$$a'_{0,n}:=\text{the number of loops which separate } \mathbf{0} \text{ and } (n,0),$$
$$b'_{0,n}:=\text{the number of loops which separate } \mathbf{0} \text{ and } \{(x,y):x\ge n\}.$$

Note that $a'_{0,n}$ is essentially introduced in [4]. Using the strategy in the present paper and the result of [18], one may get the following result, which is analogous to Theorem 1.1 but with explicit limit values:

$$\lim_{n\to\infty}\frac{a'_{0,n}}{\log n}=\frac{1}{\sqrt{3}\pi} \text{ in probability,} \tag{1.6}$$
$$\lim_{n\to\infty}\frac{b'_{0,n}}{\log n}=\frac{1}{2\sqrt{3}\pi} \quad a.s.$$

Furthermore, the convergence in (1.6) does not occur almost surely.

The explicit limits above mainly relies on the work of [18]. However, it seems very hard to give the explicit value of μ in Theorem 1.1. Nevertheless, it need not much work to deduce that $\mu>1/(2\sqrt{3}\pi)$ from above. We just give a sketch of the proof here. First, let us introduce some notations from [18]. Camia and Newman defined the conformal loop ensemble CLE$_6$ in $\mathbb{D}$ (see Section 3.2 in [3]), which is almost surely a countably infinite collection of (oriented) continuum nonsimple loops and is the scaling limit of the cluster boundaries of critical site percolation on $\eta\mathbb{T}\cap\mathbb{D}$ with monochromatic boundary conditions. We inductively define L_k to be the outermost loop surrounding $\mathbf{0}$ in $\mathbb{D}$ when the loops $L_1,\ldots,L_{k-1}$ are removed. Note that the loops L_k exist for all $k\ge 1$

with probability 1. Define $A_0 = \mathbb{D}$ and let A_k be the component of $\mathbb{D}\backslash L_k$ that contains $\mathbf{0}$. If D is a simply connected planar domain and $\mathbf{0} \in D$, the conformal radius of D viewed from $\mathbf{0}$ is defined to be $CR(D) := |g'(\mathbf{0})|^{-1}$, where g is any conformal map from D to $\mathbb{D}$ that sends $\mathbf{0}$ to $\mathbf{0}$. For $k \geq 1$, define

$$B_k := \log CR(A_{k-1}) - \log CR(A_k).$$

From Proposition 1 in [18], we know that $B_k, k \geq 1$ are i.i.d random variables. Furthermore, it is shown (see (2) in [18]) that

$$E[B_k] = 2\sqrt{3}\pi.$$

A well known consequence of the Schwarz Lemma and the Koebe $1/4$ Theorem (see e.g., Lemma 2.1 and Theorem 3.17 in [14], see also (2.1) in [15] for a similar application) is that

$$\frac{CR(A_n)}{4} \leq dist(\mathbf{0}, L_n) \leq CR(A_n).$$

Let $N(\varepsilon)$ be the number of loops surrounding $\mathbf{0}$ in $\mathbb{D}\backslash\mathbb{D}_\varepsilon$. From the definition it is clear that $\log CR(A_n) = -\sum_{k=1}^n B_k$. Combining above issues, one may conclude

$$\lim_{\varepsilon\to 0} -\frac{N(\varepsilon)}{\log\varepsilon} = \lim_{\varepsilon\to 0} -\frac{EN(\varepsilon)}{\log\varepsilon} = \frac{1}{2\sqrt{3}\pi} \quad a.s. \tag{1.7}$$

(1.7) is an analog of Lemma 2.7 below. One can get analogs of Lemma 2.5 and Lemma 2.8 following our method. Combining these issues, the result would be proved.

Remark 1.5. *Consider the oriented (directed) FPP on $\mathbb{Z}^2$ (see e.g., Section 12.8 in [6] for background). We assign independently to each bond e i.i.d. passage time $t(e)$. Let $\vec{p}_c$ denote the critical probability for oriented bond percolation on $\mathbb{Z}^2$. Assume*

$$P(t(e) = 0) = \vec{p}_c, \quad P(t(e) = 1) = 1 - \vec{p}_c.$$

We denote by $\vec{T}(\mathbf{0}, (r, \theta))$ the passage time from $\mathbf{0}$ to $(\lfloor r\sin\theta\rfloor, \lfloor r\cos\theta\rfloor)$ by a northeast path for $(r, \theta) \in \mathbb{R}^+ \times [0, \pi/2]$. Based on Conjecture 4 in [25], we conjecture that there is a constant $0 < \vec{\mu} < \infty$, such that as $r \to \infty$,

$$\frac{\vec{T}(\mathbf{0}, (r, \pi/4))}{\log r} \to \vec{\mu} \ \text{ in probability.}$$

Remark 1.6. *Camia and Newman's full scaling limit plays a central role in the proof of Theorem 1.1. We want to note that the scaling limit of a critical system may help to show laws of large numbers for many different variables. For example, consider the largest winding angle (the interested reader is referred to [24] for a more general discussion and references of winding angles) $\theta_{max,n}$ of the paths from $\mathbf{0}$ to $\partial B(n)$ in Kesten's incipient infinite cluster (IIC) [9]. Heuristically, once the existence of an appropriate scaling limit of IIC on $\mathbb{T}$ is established, one may derive a SLLN for $\theta_{max,n}$ by our strategy.*

Idea of the Proof. We show a SLLN for $c_n := T(\mathbf{0}, \partial B(n))$, that is, $c_n/\log n \to \mu/2 \ \ a.s.$, then Theorem 1.1 follows from this easily. First, using the estimates developed by Kesten and Zhang [12], we prove that $c_n/Ec_n \to 1 \ \ a.s.$ Next, we want to show $Ec_n/\log n \to \mu/2$, which implies the required SLLN immediately. For this, we divide the discrete ball $B(n)$ into long annuli, which have the same shape. The summation of the passage times of these annuli approximates c_n. Inspired by Beffara and Nolin [1], we express the passage time of an annulus in terms of the collection of cluster interfaces (see Fig. 2). When the annulus is very large, this quantity can be approximated well by the passage time defined analogously for the corresponding annulus with respect to Camia and Newman's full scaling limit [3] (see Fig. 1). For this scaling limit, by the subadditive ergodic theorem we get a SLLN for the passage times of annuli, which can be used to approximate the passage times of the large and long annuli for the discrete model.

Throughout this paper, $C, C_1, C_2, \ldots$ denote positive finite constants that may change from line to line or page to page according to the context.

2 Preliminary results

We shall use some estimates developed in [12]. Let us give some notations from [12]. For $0 < m < n$, define the annulus

$$A(m,n) := B(n)\backslash B(m).$$

Let $A(p) := A(2^p, 2^{p+1}),\ \ p \geq 0$. Next define for $p \geq 0$

$$\begin{aligned} m(p) &:= \inf\{t \in \{p, p+1, \ldots\} : A(t) \text{ contains an open circuit surrounding } \mathbf{0}\},\\ \mathcal{C}_p &:= \text{innermost open circuit surrounding } \mathbf{0} \text{ in } A(m(p)),\ \text{and set } \mathcal{C}_{-1} := \emptyset. \end{aligned}$$

For $p \geq 0$, define

$$\overline{\mathcal{C}}_p := \mathcal{C}_p \cup \text{interior sites of } \mathcal{C}_p, \quad \mathcal{F}_p := \{\sigma\text{-field generated by}\{t(v) : v \in \overline{\mathcal{C}}_p\}\},$$

and let $\mathcal{F}_{-1}$ be the trivial σ-field. Let $\Delta_p = \Delta_{p,q} := E[T(\mathbf{0}, \mathcal{C}_{m(q)})|\mathcal{F}_p] - E[T(\mathbf{0}, \mathcal{C}_{m(q)})|\mathcal{F}_{p-1}]$. Then we can write

$$T(\mathbf{0}, \mathcal{C}_{m(q)}) - ET(\mathbf{0}, \mathcal{C}_{m(q)}) = \sum_{p=0}^{q} \Delta_p. \tag{2.1}$$

Essentially the same as the proof of (2.28), (2.29) in [12], one may get the following lemma, we omit the proof here. Note that (2.3) is stronger than (2.29) in [12], since the distribution of the passage time in [12] is more general than ours. One needs no new technique here.

Lemma 2.1. *There exist constants $C_1, C_2, C_3 > 0$ such that for $q \geq 1$*

$$P(m(p) - p \geq t) \leq \exp(-C_1 t), \quad t, p \geq 0, \tag{2.2}$$

$$P(|\Delta_p| \geq x) \leq C_2 \exp(-C_3 x), \quad x \geq 0, 0 \leq p \leq q. \tag{2.3}$$

Define $R(m,n) := \{v \in \mathbb{V} : |\arg(v)| < \frac{\pi}{10}\} \cap A(m,n)$. We say a path $\gamma \subset R(m,n)$ is a crossing path in $R(m,n)$ if the endpoints of γ lie adjacent (Euclidean distance smaller than 1) to the rays of argument $\pm\frac{\pi}{10}$ respectively. By step 3 of the proof of Theorem 5 in [1], we obtain the following lemma.

Lemma 2.2. *There exist constants $C_1, C_2, K_0 > 0$, such that for all $k > K_0$ and $n > m > 0$,*

$$\begin{aligned} &P(\text{there exist } k\log(n/m) \text{ disjoint closed crossing paths in } R(m,n))\\ &\quad \leq C_1 \exp(-C_2 k \log(n/m)). \end{aligned}$$

Observe that $T(\partial B(m), \partial B(n))$ equals the maximal number of disjoint closed circuits which surround $\mathbf{0}$ in $A(m-1,n)$ (see (2.39) and the Appendix in [12]), from Lemma 2.2 we immediately get:

Corollary 2.3. *There exist constants $C_1, C_2, K_0 > 0$, such that for all $k > K_0$ and $n > m > 0$,*

$$P(T(\partial B(m), \partial B(n)) \geq k \log(n/m)) \leq C_1 \exp(-C_2 k \log(n/m)).$$

Note that (2.48) in [12] also implies this corollary. Recall $c_n = T(\mathbf{0}, \partial B(n))$. By RSW, FKG and Corollary 2.3, one easily obtains the following well-known result, which is (3.23) in [2].

Corollary 2.4. *There exist constants $C_1, C_2 > 0$, such that for all $n \geq 1$,*

$$C_1 n \leq E c_n \leq C_2 n.$$

Lemma 2.5.

$$\lim_{n\to\infty} \frac{c_n}{E c_n} = 1 \;\; a.s.$$

Proof. As we have discussed before Theorem 1.1, Kesten and Zhang got similar result for $b_{0,n}$ in [12] (see (1.15) in [12]), but they did not give the proof. Now let us use the estimates in [12] to prove Lemma 2.5. First we claim that for each $\varepsilon \in (0, \frac{1}{4})$, there exist constants $\delta_1 > 0, C_6 > 0$ such that

$$P(|T(\mathbf{0}, \mathcal{C}_{m(q)}) - ET(\mathbf{0}, \mathcal{C}_{m(q)})| \geq q^{1-\varepsilon}) \leq C_6 q^{-(1+\delta_1)}. \tag{2.4}$$

Let us prove this. First we define

$$\widetilde{\Delta}_{p,q} := \Delta_{p,q} I[m(p) - p \leq C_4 \log q],$$

where $C_4 > 3/C_1$ and C_1 is from Lemma 2.1. By (2.1), we write

$$\begin{aligned} P(|T(\mathbf{0}, \mathcal{C}_{m(q)}) - ET(\mathbf{0}, \mathcal{C}_{m(q)})| \geq q^{1-\varepsilon}) &\leq P(\Delta_{p,q} \neq \widetilde{\Delta}_{p,q} \text{ for some } p \leq q) \\ &\quad + P(|\sum_{p=0}^{q} \widetilde{\Delta}_{p,q}| \geq q^{1-\varepsilon}). \end{aligned}$$

Let us now estimate each term separately. For the first term,

$$\begin{aligned} P(\Delta_{p,q} \neq \widetilde{\Delta}_{p,q} \text{ for some } p \leq q) &\leq \sum_{p=0}^{q} P(m(p) - p \geq C_4 \log q) \\ &\leq (q+1) \exp(-C_4 C_1 \log q) \quad \text{ by (2.2)}. \end{aligned}$$

Now we estimate the second term. Similar as the second half of Lemma 1 in [12], we have: For any $0 \leq p, r \leq q$, if $|p - r| \geq C_4 \log q + 2$, then $\widetilde{\Delta}_{p,q}$ and $\widetilde{\Delta}_{r,q}$ are independent. The proof is omitted here. Therefore, by Chebyshev's inequality and (2.3), there exists a constant $C_5 > 0$ such that,

$$\begin{aligned} P\left(|\sum_{p=0}^{q} \widetilde{\Delta}_{p,q}| \geq q^{1-\varepsilon}\right) &\leq \frac{E[\sum_{p=0}^{q} \widetilde{\Delta}_{p,q}]^4}{q^{4(1-\varepsilon)}} \\ &\leq \frac{24 E[\sum_{0 \leq p_1 \leq \ldots \leq p_4 \leq q, |p_1 - p_4| \leq 3C_4 \log q + 6} \prod_{i=1}^{4} \widetilde{\Delta}_{p_i,q}]}{q^{4(1-\varepsilon)}} \\ &\leq \frac{C_5 q^2}{q^{4(1-\varepsilon)}} = C_5 q^{-2+4\varepsilon} \end{aligned} \tag{2.5}$$

By the bounds of the two terms given above, (2.4) is proved. (2.74) in [12] says that there exist constants $\delta_2 > 2, C_7 > 0$ such that

$$P(|c_{2^q} - T(\mathbf{0}, \mathcal{C}_{m(q)})| \geq x) \leq C_7 x^{-\delta_2}. \tag{2.6}$$

By (2.4) and (2.6), for all large q we have

$$\begin{aligned}P(|c_{2^q}-Ec_{2^q}|\geq 3q^{1-\varepsilon}) &\leq P(|c_{2^q}-T(\mathbf{0},\mathcal{C}_{m(q)})|\geq q^{1-\varepsilon})\\ &\quad+P(|T(\mathbf{0},\mathcal{C}_{m(q)})-ET(\mathbf{0},\mathcal{C}_{m(q)})|\geq q^{1-\varepsilon})\\ &\quad+P(|ET(\mathbf{0},\mathcal{C}_{m(q)})-Ec_{2^q}|\geq q^{1-\varepsilon})\\ &\leq C_7q^{-(2-2\varepsilon)}+C_6q^{-(1+\delta_1)}. \qquad (2.7)\end{aligned}$$

By RSW, FKG and Corollary 2.3, there exist constants $C_8, C_9, C_{10} > 0$ such that for all $x \geq 1$,

$$\begin{aligned}&P(c_{2^q}-c_{2^{q-1}}\geq x)\\ &\leq P(\text{there exists no open circuits surrounding } \mathbf{0} \text{ in } B(2^q)\backslash B(2^{q-\lfloor\sqrt{x}\rfloor}))\\ &\quad+P(T(\partial B(2^{q-\lfloor\sqrt{x}\rfloor}),\partial B(2^q))\geq x)\\ &\leq \exp(-C_8\sqrt{x})+C_9\exp(-C_{10}x), \qquad (2.8)\end{aligned}$$

where $B(2^{q-\lfloor\sqrt{x}\rfloor}) := B(1)$ for $x > q^2$. Then for all $q \geq 1$ we get

$$Ec_{2^q}-Ec_{2^{q-1}}\leq C_{11}, \qquad (2.9)$$

where C_{11} is a universal constant. Define event

$$\mathcal{A}_q := \{|c_{2^q}-Ec_{2^q}|\geq 3q^{1-\varepsilon}\}\cup\{c_{2^q}-c_{2^{q-1}}\geq q^{1-\varepsilon}\}.$$

It follows from (2.7) and (2.8) that $\sum_{q=1}^{\infty}P(\mathcal{A}_q)<\infty$. Then the Borel-Cantelli lemma implies that, a.s. $\mathcal{A}_q$ happens only finitely many times as $q\to\infty$. For large $2^{q-1}\leq n\leq 2^q$, from $c_{2^{q-1}}\leq c_n\leq c_{2^q}$ and (2.9) we have

$$\begin{aligned}&\{|c_n-Ec_n|\geq 9q^{1-\varepsilon}\}\\ &\subset\{|c_n-c_{2^q}|\geq 3q^{1-\varepsilon}\}\cup\{|c_{2^q}-Ec_{2^q}|\geq 3q^{1-\varepsilon}\}\cup\{|Ec_{2^q}-Ec_n|\geq 3q^{1-\varepsilon}\}\\ &\subset\mathcal{A}_q.\end{aligned}$$

Then we know for each $\varepsilon\in(0,\frac{1}{4})$, as $n\to\infty$,

$$|c_n-Ec_n|\leq 9q^{1-\varepsilon}\ \ a.s., \qquad (2.10)$$

where $2^{q-1}\leq n\leq 2^q$. By Corollary 2.4, Ec_n lies between two positive multiples of q, then Lemma 2.5 follows from (2.10). □

As it is well discussed in [19], there are several different ways to describe the scaling limit of critical planar percolation. In the present paper, we focus on the **full scaling limit** constructed by Camia and Newman in [3], described in detail below.

First, we compactify $\mathbb{R}^2$ as usual into $\dot{\mathbb{R}}^2 := \mathbb{R}^2\cup\{\infty\}\simeq\mathbb{S}^2$. Let $d_{\mathbb{S}^2}$ be the induced metric on $\dot{\mathbb{R}}^2$. We call a continuous map from the circle to $\mathbb{R}^2$ a loop, and the loops are identified up to reparametrization by homeomorphisms of the circle with positive winding. We equip the space L of loops with the following metric:

$$d_L(\ell_1,\ell_2):=\inf_{\phi}\sup_{t\in\mathbb{R}/\mathbb{Z}}d_{\mathbb{S}^2}(\ell_1(t),\ell_2(\phi(t))),$$

where the infimum is taken over all homeomorphisms of the circle which have positive winding. Let $\mathcal{L}$ be the space of countable collections of loops in L. Consider the Hausdorff topology on $\mathcal{L}$ induced by d_L. That is, for $c_1, c_2\in\mathcal{L}$, let

$$d_{\mathcal{L}}:=\inf\{\varepsilon:\forall\ell_1\in c_1,\exists\ell_2\in c_2\text{ such that }d_L(\ell_1,\ell_2)\leq\varepsilon\text{ and vice versa}\}.$$

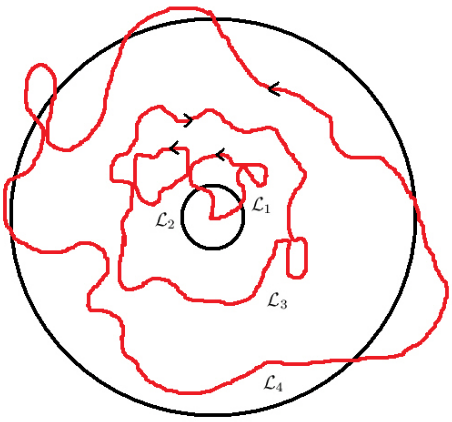

Figure 1: A chain $\mathcal{C} = \ell_1\ell_2\ell_3\ell_4$ connecting two circles. It is easy to see that $T(\mathcal{C}) = 2$

For the critical site percolation on $\mathbb{T}$, with probability 1 there is no infinite cluster, therefore the cluster boundaries form loops. We orient a loop counterclockwise if it has open sites on its inner boundary and closed sites on its outer boundary, otherwise we orient it clockwise.

The following celebrated theorem is shown in [3]:

Theorem 2.6. *As $\eta \to 0$, the collection of all cluster boundary loops of critical site percolation on $\eta\mathbb{T}$ converges in law, under the topology induced by $d_{\mathcal{L}}$, to a probability distribution on $\mathcal{L}$, which is a continuum nonsimple loop process.*

The continuum nonsimple loop process in Theorem 2.6 is just the full scaling limit introduced by Camia and Newman in [3]. Since it is also called the **conformal loop ensemble** CLE_6 in [20] (for the general $\text{CLE}_\kappa, 8/3 \le \kappa \le 8$, see [20, 18, 21]), we just call it CLE_6 in the present paper. Although extracting geometric information is far from being straightforward from CLE_6 (according to [19]), it was used to show the uniqueness of the quad-crossing percolation limit in Subsection 2.3 in [5] and the existence of the monochromatic arm exponents in Section 4 in [1]. In fact, the key idea of the proof of (2.11) is stimulated by the latter.

Several properties of CLE_6 are established. For example, if two loops touch each other and have the same orientation, then almost surely one loop cannot lie inside the other one. Conversely, if two loops of different orientations touch each other, then one has to be inside the other one. See [3] for more details. For CLE_6, we want to define the passage time between two circles. First, we call a sequence of loops $\mathcal{C} = \ell_1 \dots \ell_l$ a **chain** which connects $\partial\mathbb{D}_m$ and $\partial\mathbb{D}_n$, if $\mathcal{C}$ satisfies the following conditions:

- $\ell_1 \cap \mathbb{D}_m \neq \emptyset, \ell_l \cap \{\mathbb{R}^2 \backslash \mathbb{D}_n\} \neq \emptyset$, $\ell_i \subset \mathbb{D}_n \backslash \mathbb{D}_m, 1 < i < l$.
- For $1 \le i \le l-1$, if ℓ_i is counterclockwise, then ℓ_{i+1} touches ℓ_i.
- For $1 \le i \le l-1$, if ℓ_i is clockwise, then ℓ_{i+1} is the minimal counterclockwise loop surrounding ℓ_i.

See Fig. 1. Define the **passage time** of chain $\mathcal{C}$ as

$$T(\mathcal{C}) := \text{the number of occurrences that } \ell_{i+1} \text{ touches counterclockwise loop } \ell_i.$$

The passage time between $\partial\mathbb{D}_m$ and $\partial\mathbb{D}_n$ is defined as

$$T(\partial\mathbb{D}_m, \partial\mathbb{D}_n) := \inf\{T(\mathcal{C}) : \mathcal{C} \text{ is chain connecting } \partial\mathbb{D}_m \text{ and } \partial\mathbb{D}_n\}.$$

From (2.11) below, $T(\partial\mathbb{D}_m, \partial\mathbb{D}_n)$ is finite with probability 1. For the passage time of this continuum model, we have a strong law of large numbers:

Lemma 2.7. *There exists a constant* $0 < \mu_0 < \infty$ *such that*

$$\lim_{j\to\infty} \frac{T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^j})}{j} = \lim_{j\to\infty} \frac{ET(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^j})}{j} = \mu_0 \ \ a.s.$$

Proof. For short, let $X_{i,j} := -T(\partial\mathbb{D}_{2^i}, \partial\mathbb{D}_{2^j}) + 1, 0 \le i < j$. Now we verify that $X_{i,j}, 0 \le i < j$ satisfy the conditions of the subadditive ergodic theorem (see [16]):

- $X_{0,j} \le X_{0,i} + X_{i,j}$.

  If $T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^j}) > 0$, then for any chain $\mathcal{C} = \ell_1 \dots \ell_l$ connecting $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_{2^j}$, clearly we have $l \ge 2$ by the definition of chain. Then it is easy to see that we can find some $2 \le k \le l$ such that $\mathcal{C}_1 = \ell_1 \dots \ell_k$ is a chain connecting $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_{2^i}$, and $\mathcal{C}_2 = \ell_{k-1} \dots \ell_l$ is a chain connecting $\partial\mathbb{D}_{2^i}$ and $\partial\mathbb{D}_{2^j}$. Therefore,

  $$T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^i}) + T(\partial\mathbb{D}_{2^i}, \partial\mathbb{D}_{2^j}) \le T(\mathcal{C}_1) + T(\mathcal{C}_2) \le T(\mathcal{C}) + 1,$$

  which implies the above inequality. If $T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^j}) = 0$, the inequality holds obviously.

- $\{X_{jk,(j+1)k}, j \ge 1\}$ is stationary ergodic sequence for each k.

  Define the scaling transformation $\tau_k : \mathbb{R}^2 \to \mathbb{R}^2, \ \ \mathbf{x} \mapsto \mathbf{x}/2^k$. Then for each configuration ω of CLE_6, $X_{jk,(j+1)k}(\omega) = X_{k,2k}(\tau_k^{j-1}\omega)$. Since CLE_6 is invariant under scalings, τ_k is measure preserving and $\{X_{jk,(j+1)k}, j \ge 1\}$ is stationary. Now we show that τ_k is also mixing, which implies $\{X_{jk,(j+1)k}, j \ge 1\}$ is ergodic. When A, B are events which depend only on the realization of the CLE_6 inside an annulus, then $\lim_{j\to\infty} P(A \cap \tau_k^{-j} B) = P(A)P(B)$ follows immediately. For arbitrary events A and B, one approximates A and B by events which depend only on the realization of CLE_6 inside the annulus $\mathbb{D}_{1/\varepsilon} \backslash \mathbb{D}_\varepsilon$, and let $\varepsilon \to 0$. Then the result follows easily.

- The distribution of $\{X_{i,i+k}, k \ge 1\}$ does not depend on i.

  CLE_6 is invariant under scalings, which implies this immediately.

- $EX_{0,1}^+ < \infty$, where $X_{0,1}^+ := \max\{X_{0,1}, 0\}$. For each j, $EX_{0,j} \ge -Cj$, where $C < \infty$.

  It is obvious that $EX_{0,1}^+ \le 1$. From (2.11) below we know that the discrete passage time $T(\partial B(n), \partial B(2^j n)) \to_d T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^j})$ as $n \to \infty$. Then by Corollary 2.3, there exist constants $C_1, C_2, C_3 > 0$, such that for all $j \ge 1$,

  $$P(T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^j}) \ge C_1 j) \le C_2 \exp(-C_3 j),$$

  which ends the proof immediately.

Then by the subadditive ergodic theorem, there exists a constant $0 < \mu_0 < \infty$ such that

$$\lim_{j\to\infty} \frac{X_{0,j}}{j} = \lim_{j\to\infty} \frac{EX_{0,j}}{j} = -\mu_0 \ \ a.s.,$$

which ends the proof. □

Lemma 2.8.

$$\lim_{n\to\infty} \frac{Ec_n}{\log n} = \frac{\mu}{2}.$$

Proof. For short, define

$$A(k,i) := B(2^{k(i+1)}) \backslash B(2^{ki}), \quad k \geq 1, i \geq 0.$$
$$T_{k,i} := T(\partial B(2^{ki}+1), \partial B(2^{k(i+1)})), \quad k \geq 1, i \geq 0.$$

Recall the definition of $T(\partial\mathbb{D}_m, \partial\mathbb{D}_n)$ before Lemma 2.7. For the passage time defined respectively for the discrete FPP and CLE_6, we claim that for any fixed $k \geq 1$, as $i \to \infty$,

$$T_{k,i} \to_d T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^k}). \tag{2.11}$$

The proof of this claim is similar as the arguments in Section 4 in [1], but it's more complicated. First we show that for each $0 < \varepsilon < 1$, there is a $\delta > 0$, such that

$$\begin{aligned} \lim_{i\to\infty} P(&\text{for any geodesic } \gamma \text{ connecting } \partial B(2^{ki}+1) \text{ and } \partial B(2^{k(i+1)}) \text{ in } A(k,i),\\ &\text{for any closed site } x \in \gamma,\ \mathrm{dist}(x, \partial B(2^{ki}+1) \cup \partial B(2^{k(i+1)})) \geq \delta 2^{ki},\\ &\text{for any two closed sites } x,y \in \gamma,\ \mathrm{dist}(x,y) \geq \delta 2^{ki}) \geq 1-\varepsilon. \end{aligned} \tag{2.12}$$

Observe that

$$T_{k,i} = \{\text{maximal number of disjoint closed circuits which surround } \mathbf{0} \text{ in } A(k,i)\}.$$

Therefore, if γ is a geodesic connecting $\partial B(2^{ki}+1)$ and $\partial B(2^{k(i+1)})$ in $A(k,i)$, then there exist $T(\gamma)$ disjoint circuits which surround **0** in $A(k,i)$ and pass through the $T(\gamma)$ closed sites in γ respectively. Using the fact that the polychromatic half-plane 3-arm exponent is 2 (in fact, one needs a more general version, see Lemma 6.8 in [22]) and the polychromatic plane 6-arm exponent is larger that 2 (see e.g. [17]), one can get (2.12) by standard arguments.

Define

$$\begin{aligned} T'_{k,i} := \inf\{|\mathcal{S}|-1 : \ &\mathcal{S} \text{ is a sequence of open clusters, such that the first cluster}\\ &\text{intersects with } \partial B(2^{ki}+1), \text{ the last cluster intersects with } \partial B(2^{k(i+1)}),\\ &\text{and two consecutive clusters are separated by only one closed site}\}, \end{aligned}$$

if there exists no such $\mathcal{S}$, we let $T'_{k,i} = 0$. From (2.12), it is easy to see that

$$\lim_{i\to\infty} P(T_{k,i} = T'_{k,i}) = 1. \tag{2.13}$$

Now let us introduce the definition of chain for the critical site percolation on $\mathbb{T}$, which is analogous to its continuum version for CLE_6. We call a sequence of (discrete) loops $\mathcal{C} = \ell_1 \ldots \ell_l$ a (discrete) **chain** which connects $\partial B(m)$ and $\partial B(n)$, if $\mathcal{C}$ satisfies the following conditions:

- $\ell_1 \cap B(m) \neq \emptyset, \ell_l \cap \{\mathbb{R}^2 \backslash B(n)\} \neq \emptyset$, $\ell_i \subset B(n)\backslash B(m), 1 < i < l$.
- For $1 \leq i \leq l-1$, if ℓ_i is counterclockwise, then ℓ_{i+1} and ℓ_i are separated by only one site.
- For $1 \leq i \leq l-1$, if ℓ_i is clockwise, then ℓ_{i+1} is the minimal counterclockwise loop surrounding ℓ_i.

See Fig. 2. For a discrete chain $\mathcal{C}$, let

$T(\mathcal{C})$:=the number of occurrences that ℓ_{i+1} and counterclockwise loop ℓ_i are separated by only one site.

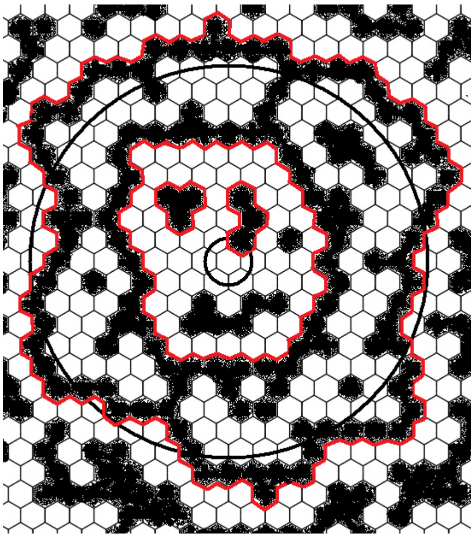

Figure 2: A discrete chain $\mathcal{C} = \ell_1\ell_2\ell_3\ell_4$ connecting $\partial B(1)$ and $\partial B(8)$. It is clear that $T(\mathcal{C}) = 2$

Define

$$T''_{k,i} := \inf\{T(\mathcal{C}) : \mathcal{C} \text{ is a chain connecting } \partial B(2^{ki}+1) \text{ and } \partial B(2^{k(i+1)})\},$$

if there exists no chain connecting $\partial B(2^{ki}+1)$ and $\partial B(2^{k(i+1)})$, let $T''_{k,i} = 0$. It is easy to get that

$$\lim_{i\to\infty} P(T'_{k,i} = T''_{k,i}) = 1. \tag{2.14}$$

By (2.12),(2.13) and (2.14), the value of $T''_{k,i}$ is determined by macroscopic loops with high probability as $i \to \infty$. It has been argued in [1], two loops touch in the scaling limit is exactly the asymptotic probability that they are separated by exactly one site on discrete lattice. Therefore, using Theorem 2.6, comparing the definitions of $T''_{k,i}$ and $T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^k})$, we have

$$T''_{k,i} \to_d T(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^k}). \tag{2.15}$$

Combining (2.13), (2.14) and (2.15), claim (2.11) follows. By Corollary 2.3, there exists a constant $C(k) > 0$, such that for all $i \geq 0$, $ET_{k,i} \leq C(k)$. This and (2.11) immediately give

$$ET_{k,i} \to ET(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^k}). \tag{2.16}$$

By the convergence of the Cesàro mean and (2.16), we have

$$\lim_{j\to\infty} \frac{\sum_{i=0}^{j} ET_{k,i}}{j} = ET(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^k}). \tag{2.17}$$

Now let us show that for each $0 < \varepsilon < 1$, there exists $k_0(\varepsilon) > 0$, such that for each $k \geq k_0$, for n sufficiently large (depending on k),

$$\frac{\sum_{i=0}^{\lfloor \log_{2^k} n \rfloor} ET_{k,i}}{Ec_n} \geq 1 - \varepsilon. \tag{2.18}$$

For $i \geq 0, k \geq 1$, denote by $N_{k,i}$ the maximum number of disjoint closed circuits which surround $\mathbf{0}$ and intersect with $\partial B(2^{ki})$. It is easy to see that

$$\sum_{i=0}^{\lfloor \log_{2^k} n \rfloor - 1} T_{k,i} \leq c_n \leq 2 + \sum_{i=0}^{\lfloor \log_{2^k} n \rfloor + 1} (T_{k,i} + N_{k,i}). \tag{2.19}$$

By RSW, FKG and BK inequality, using standard argument, we know there exists a constant $C_1 > 0$, such that for all $i \geq 0, k \geq 1$, $P(N_{k,i} \geq x) \leq \exp(-C_1 x)$. Hence there is a universal constant $C_2 > 0$ (independent of i, k), such that $EN_{k,i} < C_2$. Then by (2.19), there exists a constant $C_3 > 0$, such that for all large n,

$$\sum_{i=0}^{\lfloor \log_{2^k} n \rfloor} ET_{k,i} - C(k) \leq Ec_n \leq \sum_{i=0}^{\lfloor \log_{2^k} n \rfloor} ET_{k,i} + C_3 \log_{2^k} n, \tag{2.20}$$

where $C(k)$ is introduced before (2.16). Combining Corollary 2.4 and (2.20) gives (2.18). Then by (2.17), (2.18) and Lemma 2.7,

$$\lim_{n\to\infty} \frac{Ec_n}{\log n} = \lim_{k\to\infty} \lim_{n\to\infty} \frac{\sum_{i=0}^{\lfloor \log_{2^k} n \rfloor} ET_{k,i}}{\log n} = \lim_{k\to\infty} \frac{ET(\partial\mathbb{D}_1, \partial\mathbb{D}_{2^k})}{k \log 2} = \frac{\mu_0}{\log 2} := \frac{\mu}{2},$$

where μ_0 is from Lemma 2.7. □

3 Proof of theorem

Proof of Theorem 1.1. From Lemma 2.5 and Lemma 2.8, we have

$$\lim_{n\to\infty} \frac{c_n}{\log n} = \frac{\mu}{2} \;\; a.s. \tag{3.1}$$

Now let us use (3.1) to show (1.5). First, it is apparent that $b_{0,n} \geq c_n$. Thus if one can show that for any given $\varepsilon > 0$, as $n \to \infty$,

$$b_{0,n} - c_n \leq \varepsilon \log n \;\; a.s., \tag{3.2}$$

then (3.1) and (3.2) imply (1.5). We proceed to prove (3.2). Recall the definition of $m(p)$ before Lemma 2.1. By (2.2) and Lemma 2.2, we can choose a small constant $\delta(\varepsilon) > 0$, such that for all large q,

$$\begin{aligned} &P(b_{0,2^q} - c_{2^q} \geq (\varepsilon \log 2^q)/3) \\ &\leq P(m(\lfloor (1-\delta)q \rfloor) \geq q) \\ &\quad + P(\text{there exist } \lfloor (\varepsilon \log 2^q)/3 \rfloor \text{ disjoint closed crossing paths in } R(2^{\lfloor (1-\delta)q \rfloor}, 2^{q+1})) \\ &\leq \exp(-C_1 q) + C_2 \exp(-C_3 q). \end{aligned}$$

By (2.8),

$$P(c_{2^q} - c_{2^{q-1}} \geq (\varepsilon \log 2^q)/3) \leq \exp(-C_4 \sqrt{q}) + C_5 \exp(-C_6 q).$$

Define events: $\mathcal{A}_q := \{b_{0,2^q} - c_{2^q} \geq (\varepsilon \log 2^q)/3\} \cup \{c_{2^q} - c_{2^{q-1}} \geq (\varepsilon \log 2^q)/3\}$. Then the inequalities above and Borel-Cantelli lemma implies that

$$\text{a.s. } \mathcal{A}_q \text{ happens only finitely many times as } q \to \infty. \tag{3.3}$$

Both $\{b_{0,n}, n \geq 1\}$ and $\{c_n, n \geq 1\}$ are increasing sequences, which implies that for $2^{q-1} < n \leq 2^q$

$$b_{0,n} - c_n \leq b_{0,2^q} - c_{2^q} + c_{2^q} - c_{2^{q-1}}.$$

Combining this and (3.3) gives (3.2).

(2.84) in [12] essentially tells us that as $n \to \infty$,

$$\frac{1}{\sqrt{\log n}} [T(\mathbf{0}, (n,0)) - T(\mathbf{0}, \partial B(n/2)) - T((n,0), \partial B((n,0), n/2))] \to 0 \;\text{ in probability.}$$

Combining this and Lemma 2.8 gives (1.4).

Now let us explain why the convergence in (1.4) does not occur almost surely. To show this, with probability 1, we find a subsequence that converges to $\frac{\mu}{2}$ in the following. For $i \geq 1$, define event

$$\mathcal{B}_i := \{\text{there exists an open circuit surrounding } \mathbf{0} \text{ in } A(2^i, 2^{i+1}), \text{ with an open path connecting it to } \partial B(2^{i+1})\}.$$

By RSW and FKG, there is a universal constant $C > 0$, such that $P(\mathcal{B}_i) > C$. Then with probability 1 we can find an infinite sequence $\{i_j, j \geq 1\}$ such that A_{i_j} happens. Conditioned on A_{i_j}, there exists a $2^{i_j} < n(i_j) \leq 2^{i_j+1}$, such that $a_{0,n(i_j)} = c_{2^{i_j+1}}$. Then by (3.1) we have

$$\lim_{j\to\infty} \frac{a_{0,n(i_j)}}{\log n(i_j)} = \frac{\mu}{2} \quad a.s.$$

This completes the proof. □

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