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Erratum: The impact of selection in the Λ-Wright-Fisher model

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Abstract

This is an Erratum for paper 72 of volume 18 of Electron. Commun. Probab. (2013). The proof of statement 2) in Theorem 1.1 of this paper relies on Lemma 2.5 of which claims the transience of a certain Markov chain. While the statement of this lemma is correct, its proof contains an improper argument, which is fixed in the present note.

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The proof of statement 2) in Theorem 1.1 of [1] relies on Lemma 2.5 of [1] which claims the transience of a certain Markov chain. While the statement of this lemma is correct, its proof contains an improper argument, which is fixed in the present note.

Consider $\alpha \geq 0$ and Λ a finite measure on $[0,1]$. (As in the situation of Lemma 2.5 we assume that Λ has no mass in $\{0\}$.) Let $(R_t, t \geq 0)$ be a continuous-time Markov chain taking values in $\mathbb{N} := \{1, 2, ...\}$, whose generator is the operator $\mathcal{L}$ defined as follows: For every $g : \mathbb{N} \to \mathbb{R}$,

$$\mathcal{L}g(n) := \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}[g(n-k+1) - g(n)] + \alpha n[g(n+1) - g(n)]$$

with

$$\lambda_{n,k} := \int_0^1 x^k (1-x)^{n-k} x^{-2} \Lambda(dx).$$

In particular, $(R_t, t \geq 0)$ is an irreducible Markov chain. As in [1], put

$$\alpha^\star := -\int_0^1 \log(1-x) x^{-2} \Lambda(dx).$$

We assume that $\alpha^\star < \infty$.

Lemma 0.1 (Lemma 2.5 of [1]). *If $\alpha > \alpha^\star$ then $R_t \underset{t\to\infty}{\longrightarrow} \infty$ almost surely.*

Proof of Lemma 0.1. If, for some fixed $n_0 \in \mathbb{N}$, the function g is bounded and such that $\mathcal{L}g(n) < 0$ for all $n > n_0$, the process $(g(R_{t\wedge T^{n_0}}), t \geq 0)$, when starting from $n > n_0$, is a supermartingale; with $T^{n_0} := \inf\{t > 0, R_t < n_0\}$. Applying the martingale convergence

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theorem yields that $\mathbb{P}_n(T^{n_0} < \infty) < 1$. Therefore the process $(R_t, t \geq 0)$ is not recurrent, and by irreducibility is transient. We show that the function $g(n) := \frac{1}{\log(n+1)}$ fulfills these conditions. One has

$$\mathcal{L}g(n) = \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \left[\frac{1}{\log(n-k+2)} - \frac{1}{\log(n+1)}\right] + \alpha n \left[\frac{1}{\log(n+2)} - \frac{1}{\log(n+1)}\right].$$

On the one hand, one can easily check that

$$\alpha n \left[\frac{1}{\log(n+2)} - \frac{1}{\log(n+1)}\right] = \alpha n \frac{\log\left(\frac{n+1}{n+2}\right)}{\log(n+2)\log(n+1)} = -\alpha \frac{1}{\log(n+2)\log(n+1)}(1+o(1)).$$

On the other hand, denote by $B_n(x)$ a random variable with a binomial law (n, x). We have

$$\begin{aligned}
\mathcal{L}^0 g(n) &:= \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \left[\frac{1}{\log(n-k+2)} - \frac{1}{\log(n+1)}\right] \\
&= \int_0^1 \frac{\Lambda(dx)}{x^2} \mathbb{E}\left[\frac{\log\left(\frac{n+1}{n-B_n(x)+2}\right)}{\log(n+1)\log(n-B_n(x)+2)}\right] \\
&= \frac{1}{\log(n+1)} \int_0^1 \frac{\Lambda(dx)}{x^2} \mathbb{E}\left[\frac{-\log\left(1 - \frac{B_n(x)-1}{n+1}\right)}{\log(n+2) + \log\left(1 - \frac{B_n(x)}{n+2}\right)}\right].
\end{aligned}$$

The last equality holds true since for all $2 \leq k \leq n$, $\log\left(\frac{n+1}{n-k+2}\right) = -\log\left(1 - \frac{k-1}{n+1}\right)$ and $\log(n-k+2) = \log(n+2) + \log\left(1 - \frac{k}{n+2}\right)$. Moreover

$$|(n+1)x - (B_n(x) - 1)| \leq |nx - B_n(x)| + |x + 1|$$

and by Chebyshev's inequality, we have

$$\begin{aligned}
\mathbb{P}\left[\left|x - \frac{B_n(x)-1}{n+1}\right| > (n+1)^{-1/3}\right] &\leq \frac{\mathrm{Var}(B_n(x))}{\left((n+1)^{2/3} - (1+x)\right)^2} \\
&\leq \frac{nx(1-x)}{\left((n+1)^{2/3} - 2\right)^2}.
\end{aligned}$$

Notice that $n\left((n+1)^{2/3} - 2\right)^{-2} \underset{n\to\infty}{\sim} n^{-1/3}$ and $\int_0^1 \frac{\Lambda(dx)}{x^2} x(1-x) < \infty$, since $\alpha^\star < \infty$. Therefore, from the last expression of $\mathcal{L}^0 g(n)$ above, we have

$$\mathcal{L}^0 g(n) = \frac{1}{\log(n+1)\log(n+2)}(\alpha^\star + o(1)),$$

and thus, since $\alpha > \alpha^\star$

$$\mathcal{L}g(n) = \frac{1}{\log(n+1)\log(n+2)}\left(\alpha^\star - \alpha + o(1)\right) < 0, \text{ for } n \text{ large enough.}$$

□

We mention that R. Griffiths extended statement 2) of Theorem 1-1 in [1] to the case $\alpha = \alpha^\star$ by a different method (see [2]).

References

[1] C. Foucart. The impact of selection in the Λ-Wright-Fisher model. *Electron. Commun. Probab.*, 18:no. 72, 1–10, 2013. MR-3101637

[2] R. Griffiths. The Λ-Fleming-Viot process and a connection with Wright-Fisher diffusion. arXiv:1207.1007.

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