Recurrence for the frog model with drift on $\mathbb{Z}^d$

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Abstract

In this paper we present a recurrence criterion for the frog model on $\mathbb{Z}^d$ with an i.i.d. initial configuration of sleeping frogs and such that the underlying random walk has a drift to the right.

1 Introduction

The frog model is a certain model of interacting random walks on a graph. Imagine a graph $G = (V, E)$ with a distinguished vertex $x_0 \in V$, called the *origin*. At time 0, there is exactly one active frog at x_0 and on each vertex $x \in V \setminus \{x_0\}$ there is a number $\eta_x \in \mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$ of sleeping frogs. The frog at x_0 now starts a nearest-neighbour random walk on the graph G. If it hits a vertex x with $\eta_x > 0$ sleeping frogs, they all become active at once and start performing nearest-neighbour random walks, independently of each other and of the original frog. More generally, each time an active frogs hits a vertex $x \in V$ with $\eta_x > 0$ sleeping frogs, they all become active at once and start nearest-neighbour random walks, independently of each other and of all other frogs. In this description, the transition function of the *underlying random walk* is supposed to be the same for all frogs. The frog model is called *recurrent*, if the probability that the origin x_0 is visited infinitely often equals 1, otherwise the model is called *transient*. The frog model with $V = \mathbb{Z}^d$, E the set of nearest-neighbour edges on $\mathbb{Z}^d$, $x_0 := 0$, $\eta_x = 1$ for each $x \in \mathbb{Z}^d \setminus \{0\}$ and the underlying random walk being simple random walk (SRW) on $\mathbb{Z}^d$ was studied by Telcs and Wormald [6]. They showed in particular that the frog model is recurrent for each dimension d. This result was refined by Popov [4], who considered frogs in a random environment. More precisely, he considered the situation, where there is, for each $x \in \mathbb{Z}^d \setminus \{0\}$, originally one sleeping frog at x with probability $p(x)$ and no frog with probability $1 - p(x)$, independently of all other vertices, and found the exact rate of decay for the function $p(x)$ to distinguish transience from recurrence. Another modification of the model is to consider the frog model with death, allowing activated particles to disappear after a random, e.g. geometric, lifetime. Such a model, also with a random initial configuration of sleeping frogs, was analyzed by [1] who proved phase transition results for both survival and recurrence of the particle system using

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a slightly different definition of recurrence. Note that the frog model on $\mathbb{Z}^d$ (without death and) with SRW is trivially recurrent for $d = 1, 2$, due to Pólya's theorem. Thus, in [3] Gantert and Schmidt considered the frog model on $\mathbb{Z}$ with the underlying random walk having a drift to the right. They considered both fixed and i.i.d. random initial configurations $(\eta_x)_{x \in \mathbb{Z} \setminus \{0\}}$ of sleeping frogs and derived precise criteria to separate transience from recurrence. In the case of an i.i.d. initial configuration of sleeping frogs they also proved a $0 - 1$ law, which says that the probability of infinitely many returns to 0 equals 1, if $E[\log^+(\eta_1)] = \infty$, and equals 0, otherwise, independently of the concrete value of the drift. The purpose of the present note is to prove that the frog model on $\mathbb{Z}^d$, $d \geq 2$, with an i.i.d. initial configuration of sleeping frogs is recurrent, whenever the distribution giving the number of sleeping frogs per site is heavy-tailed enough. The paper is structured as follows: In Section 2 we give a precise description of the model we consider and state our main theorem, Theorem 2.1. In Section 3 we give the proof of Theorem 2.1 and finally, in Section 4 we give proofs of two auxiliary lemmas, which we need in Section 3 in order to prove Theorem 2.1.

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2 Setting and main theorem

As mentioned above, we consider recurrence of the frog model on $\mathbb{Z}^d$ with an i.i.d. initial configuration and such that the underlying random walk has a drift to the right. We denote by $\mathcal{S}$ the set of all possible initial configurations of sleeping frogs, i.e.

$$\mathcal{S} := \left\{ \eta = (\eta_x)_{x \in \mathbb{Z}^d \setminus \{0\}} \in \mathbb{Z}_+^{\mathbb{Z}^d \setminus \{0\}} \right\}.$$

Further, we denote by p the transition function of the underlying nearest-neighbour random walk. Thus, letting $\mathcal{E} := \{\pm e_j : 1 \leq j \leq d\}$, where e_j denotes the j-th standard basis vector in $\mathbb{R}^d$, $j = 1, \ldots, d$, we assume that $p : \mathbb{Z}^d \to [0, \infty)$ is a function such that

$$\sum_{e \in \mathcal{E}} p(e) = 1$$

and $p(x) = 0$ for all $x \in \mathbb{Z}^d \setminus \mathcal{E}$. In order to make the random walk irreducible, we will further assume that $0 < p(e) < 1$ holds for all $e \in \mathcal{E}$. Additionally, we will abuse notation to write $p(x, y) := p(y - x)$ also for the corresponding transition matrix. Since we assume that the underlying random walk has a drift to the right, we suppose that there is an $a \in (0, 1)$ such that

$$m := \sum_{e \in \mathcal{E}} p(e)e = ae_1. \tag{2.1}$$

Since the transition function p will be kept fixed throughout, we omit it from the notation. For a fixed $\eta \in \mathcal{S}$ we denote by P_η a probability measure on a suitable measurable space $(\Omega, \mathcal{F})$, which describes the evolution of the frog model with initial configuration η and underlying random walk given by the transition function p as described in the introduction. We refrain from giving a mathematical construction of the frog model with respect to η but refer the interested reader to [5]. Now, let μ be a probability distribution on $(\mathbb{Z}_+, \mathcal{P}(\mathbb{Z}_+))$ and let $\mathbb{P}_\mu$ be the corresponding product measure on $(\mathbb{Z}_+^{\mathbb{Z}^d \setminus \{0\}}, \mathcal{P}(\mathbb{Z}_+)^{\otimes \mathbb{Z}^d \setminus \{0\}})$, i.e. $\mathbb{P}_\mu = \mu^{\otimes \mathbb{Z}^d \setminus \{0\}}$. The corresponding expectation operator will be denoted by $\mathbb{E}_\mu$. Finally, we denote by P the $\mathbb{P}_\mu$-mixture of the measures P_η, i.e.

$$P(A) = \int_{\mathbb{Z}_+^{\mathbb{Z}^d \setminus \{0\}}} P_\eta(A) \mathbb{P}_\mu(d\eta), \quad A \in \mathcal{F}. \tag{2.2}$$

Thus, the measure P describes the evolution of the frog model with respect to a random i.i.d. initial configuration η. From (2.2) we can make the following easy but important observation:

An event $A \in \mathcal{F}$ holds P-a.s. if and only if it holds P_η-a.s. for $\mathbb{P}_\mu$-a.a. $\eta \in \mathcal{S}$.

With this notation at hand, we are ready to state the main result of this note:

Theorem 2.1. *If, additionally to the above assumptions, the distribution μ is such that $\mathbb{E}_\mu\big[\log^+(\eta_x)^{\frac{d+1}{2}}\big] = \sum_{j=2}^\infty \log(j)^{\frac{d+1}{2}} \mu(j) = \infty$, then the frog model with drift to the right and i.i.d. initial configuration $\eta \sim \mathbb{P}_\mu$ is recurrent, i.e.*

$$P\big(0 \text{ is visited infinitely often }\big) = 1.$$

Remark 2.2. (a) If $d = 1$, Theorem 2.1 reduces to one of the results by Gantert and Schmidt [3] that the frog model is recurrent, if $\mathbb{E}_\mu[\log^+(\eta_1)] = +\infty$.

(b) Thanks to discussion with Serguei Popov we believe that for the frog model on $\mathbb{Z}^d$, $d \geq 2$, with an i.i.d. initial configuration of sleeping frogs, in general, the question of transience and recurrence depends on the concrete value of the drift, unless the distribution of η is heavy-tailed enough, as in the situation of Theorem 2.1. Establishing a phase transition result for recurrence and transience for distributions of η with lighter tails is part of a follow-up project.

3 Proof of Theorem 2.1

First, we need to fix some more notation. Fix an integer $\alpha > 1$, which is further specified later on and for $n \in \mathbb{N} = \{1, 2, \ldots\}$ let

$$F_n := \left\{x \in \mathbb{Z}^d : \frac{3}{2}\alpha^{2n} \leq x_1 < \alpha^{2n+2} \text{ and } |x_j| \leq \alpha^n \text{ for } j = 2, \ldots, d\right\}. \tag{3.1}$$

Furthermore, for $x, y \in \mathbb{Z}^d$ we denote by $f(x, y)$ the probability that the underlying random walk ever hits y, if it starts at x. Thus, if we denote this random walk by $(X_n)_{n \geq 0}$, then $f(x, y) = P(\exists n \geq 0 : X_n = y | X_0 = x)$. If we choose, according to our assumptions, $\varepsilon > 0$ such that $\varepsilon \leq p(\pm e) \leq 1 - \varepsilon$ holds for each $e \in \mathcal{E}$, then we have the following lower bound for the probabilities $f(x, y)$:

$$f(x, y) \geq \varepsilon^{d|y-x|} \quad \text{for all } x, y \in \mathbb{Z}^d, \tag{3.2}$$

where we denote by $|x| := \max_{1 \leq j \leq d} |x_j|$ the maximum norm of a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. This follows from the fact that one can get from x to y in at most $d|y - x|$ steps. If y lies to the right of x, then one can do better. More precisely, we have the following bound.

Lemma 3.1. *For each finite constant $\gamma > 0$, there exists a constant $c_1 = c_1(\gamma, p) > 0$ such that for all $x, y \in \mathbb{Z}^d$ with $y_1 > x_1$ and $|y_j - x_j| \leq \gamma\sqrt{y_1 - x_1}$, $j = 2, \ldots, d$, we have*

$$f(x, y) \geq \frac{c_1}{(y_1 - x_1)^{\frac{d-1}{2}}}.$$

A sketch of the proof of Lemma 3.1 is given in Section 4. The following lemma about the behaviour of maxima of nonnegative i.i.d. random variables is one of the cornerstones of the proof of Theorem 2.1. Throughout, we denote by $|A|$ the cardinality of the set A.

Lemma 3.2. *Let $r > 0$ be a finite constant, J be a countably infinite index set and let $(Y_j)_{j \in J}$ be a sequence of nonnegative i.i.d. random variables such that $E[\log^+(Y_j)^r] = \infty$. Furthermore, let $(L_i)_{i \in \mathbb{N}}$ be a sequence of pairwise disjoint subsets of J such that*

$l_i := |L_i| \geq c_2 \beta^{c_3 i}$ *holds for each* $i \in \mathbb{N}$, *where* $c_2, c_3 > 0$ *and* $\beta > 1$ *are constants (Here,* β *needs not necessarily be an integer). For* $i \in \mathbb{N}$ *define*

$$M_i := \max_{j \in L_i} Y_j. \tag{3.3}$$

Then, for each finite constant $c > 0$ *it holds that*

$$P\left(M_i \geq \exp\left(c\beta^{\frac{c_3 i}{r}}\right) \text{ for infinitely many } i \in \mathbb{N}\right) = 1. \tag{3.4}$$

The proof of Lemma 3.2 is given in Section 4.

Now we can proceed to the proof of Theorem 2.1, which uses a technique from [4]. Choose the positive integer α such that

$$\alpha \geq \max\left(3, \frac{1}{c_1}\right), \tag{3.5}$$

where c_1 is the constant from Lemma 3.1. Further, we define

$$V_n := \{x \in \mathbb{Z}^d \,:\, |x| \leq \alpha^{2n}\}, \quad n \in \mathbb{N}. \tag{3.6}$$

Let us repeat the following important observation from [4]: For recurrence of the frog model, everything that matters is the *trajectories* of the activated frogs. The actual moment that a certain frog gets activated is unimportant. Thus, if we know that a certain frog starting from vertex $x \in \mathbb{Z}^d$ will sooner or later be at vertex y, we will say that the frogs at vertex y are activated by a frog from x, even if it is not the first frog to visit vertex y. We will call a vertex $x \in \mathbb{Z}^d$ *active* if at least one active frog ever visits x.

Fix $k \in \mathbb{N}$ with $k \geq 2$ and define the event

$$A_k := \big\{\text{at a certain moment and at some vertex } x_k \in V_k \setminus V_{k-1} \text{ at least}$$
$$\alpha^{(d+1)(k-1)} \text{ frogs get activated by the initial frog starting from the origin}\big\}.$$

In the following, we will implicitly be conditioning on the event A_k. Note that the event A_k only depends on the randomness coming from the path of the initial frog and from the values of the η_x, where $x \in V_k$. Define

$$B_0 := \{x_k\} \quad D_0 := \emptyset. \tag{3.7}$$

We will try to construct inductively sets $D_i \subseteq F_{k+i-1}$, $i \in \mathbb{N}$, such that with

$$B_i = F_{k+i-1} \setminus D_i$$

the following hold: We have

$$|D_i| = \alpha^{(d+1)(i+k-1)} \quad \text{and} \quad |B_i| \geq \alpha^{(d+1)(i+k-1)} \tag{3.8}$$

and all the sites in D_i are visited by frogs starting from B_{i-1}, $i \in \mathbb{N}$. Furthermore, denoting for each $i \in \mathbb{N}$ and $y \in F_{k+i-1}$ by ζ_y the indicator of the following event

$$\{\text{at least one active frog starting from } B_{i-1} \text{ eventually visits } y\},$$

we require that

$$\sum_{y \in B_i} \zeta_y \eta_y \geq \alpha^{(d+1)(i+k-1)} \tag{3.9}$$

holds for each $i \in \mathbb{N}$. Note that by the definition of the sets F_n in (3.1) we have

$$|F_n| = \alpha^{2n}\left(\alpha^2 - \frac{3}{2}\right)\left(2\alpha^n + 1\right)^{d-1} \tag{3.10}$$

and hence, since $\alpha^2 \geq 4$, we get

$$|F_n| \geq \frac{5}{2} 2^{d-1} \alpha^{n(d+1)} \tag{3.11}$$

and

$$|F_n| \leq 3^{d-1} \alpha^2 \alpha^{n(d+1)} \leq \alpha^{d+1} \alpha^{n(d+1)} \,. \tag{3.12}$$

Note that by (3.11) for all $i \in \mathbb{N}$

$$|F_{k+i-1}| - 2\alpha^{(d+1)(i+k-1)} \geq \alpha^{(d+1)(i+k-1)} \left(\frac{5}{2} 2^{d-1} - 2 \right) > 0 \,.$$

Thus, in principle, there are enough vertices in F_{k+i-1} to form disjoint sets B_i and D_i as required. The next thing to do is prove that, in fact, with high enough probability enough vertices in F_{k+i-1} are visited by frogs starting from B_{i-1} and also that the number of activated frogs is large enough for (3.9) to occur. Suppose that for $0 \leq j \leq i$ the sets B_j and D_j have already been succesfully constructed. We will soon be more precise about what this exactly means. For $i \in \mathbb{Z}_+$ we define events $G_{i,1}$, $G_{i,2}$ and G_i as follows: Let

$$G_{i,1} := G_{i,1}^{(k)} := \left\{ \sum_{y \in F_{k+i}} \zeta_y \geq 2\alpha^{(d+1)(i+k)} \right\} \,. \tag{3.13}$$

If $G_{i,1}$ happens than we can construct the set D_{i+1} by choosing exactly $\alpha^{(d+1)(i+k)}$ vertices from F_{k+i} that are visited by frogs starting from B_i according to (3.13) and let $B_{i+1} := F_{k+i} \setminus D_{i+1}$. Then, we define

$$G_{i,2} := G_{i,2}^{(k)} := \left\{ \sum_{y \in B_{i+1}} \zeta_y \eta_y \geq \alpha^{(d+1)(i+k)} \right\} \quad \text{and} \quad G_i := G_i^{(k)} := G_{i,1} \cap G_{i,2} \,. \tag{3.14}$$

We will call the ith inductive step *succesful* if G_i happens (given that $A_k, G_0, \ldots, G_{i-1}$ happen). As just explained, in this case it is possible to form subsets B_{i+1}, D_{i+1} of F_{k+i} with all the desired properties. In what follows we will implicitly be conditioning on the event $A_k \cap G_0 \cap \ldots \cap G_{i-1}$ but will suppress this from the formulas for ease of notation. Also, for the computations which follow the following remark from [4] will be crucial: Suppose that there are disjoint subsets $A, B \subseteq \mathbb{Z}^d$ and we know that for each $x \in A$ there is a frog starting from a vertex $y \in B$ which activates the frogs at vertex x. Then, all the frogs starting from A are independent, since we only allow for interaction when an active frog is waking up a sleeping frog.

Note that for all $i \in \mathbb{Z}_+$ and all $x \in F_{k+i-1}, y \in F_{k+i}$ we have

$$\frac{1}{2} \alpha^{2(k+i)} \leq (y_1 - x_1) \leq \alpha^{2(k+i+1)} \,. \tag{3.15}$$

Lemma 3.3. *Under the above assumptions and conditionally on the event $A_k \cap G_0 \cap \ldots \cap G_{i-1}$, we have for all $i \in \mathbb{Z}_+$ and all $y, z \in F_{k+i}$:*

$$E[\zeta_y] \geq 1 - \exp(-2) \tag{3.16}$$

$$\mathrm{Var}(\zeta_y) \leq 1 \tag{3.17}$$

$$\mathrm{Cov}(\zeta_y, \zeta_z) \leq \exp\left(-\alpha^{k+2(i-1)}\right) \leq \exp\left(-i\alpha^{k-2}\right) \tag{3.18}$$

Proof of Lemma 3.3. By the above remark we have

$$E[\zeta_y] = P(\zeta_y = 1) = 1 - P(\zeta_y = 0) = 1 - \prod_{x \in B_i} \left(1 - f(x,y)\right)^{\eta_x} \,. \tag{3.19}$$

Now, from Lemma 3.1, (3.15) and the fact that (3.9) holds since we are conditioning on G_{i-1}, we obtain

$$
\begin{aligned}
\prod_{x \in B_i} \left(1 - f(x,y)\right)^{\eta_x} &\leq \prod_{x \in B_i} \left(1 - \frac{c_1}{(y_1 - x_1)^{\frac{d-1}{2}}}\right)^{\eta_x} \\
&\leq \left(1 - c_1 \alpha^{-\frac{2(k+i+1)(d-1)}{2}}\right)^{\alpha^{(d+1)(k+i-1)}} \\
&= \left(1 - c_1 \alpha^{-(k+i+1)(d-1)}\right)^{\alpha^{(d+1)(k+i-1)}}.
\end{aligned}
\tag{3.20}
$$

By the inequality

$$
(1-x)^y \leq \exp(-xy)
\tag{3.21}
$$

valid for all $x \in (0,1)$ and $y > 0$, we have

$$
\begin{aligned}
\left(1 - c_1 \alpha^{-(k+i+1)(d-1)}\right)^{\alpha^{(d+1)(k+i-1)}} &\leq \exp\left(-c_1 \frac{\alpha^{(d+1)(k+i-1)}}{\alpha^{(d-1)(k+i-1)}}\right) \\
&\leq \exp\left(-c_1 \alpha^{2(k+i-1)}\right).
\end{aligned}
\tag{3.22}
$$

Now, using $k \geq 2$, $i \geq 0$ and $\alpha \geq 1/c_1$ we conclude from (3.19), (3.20) and (3.22) that

$$
E[\zeta_y] \geq 1 - \exp(-2),
$$

proving (3.16). Since $0 \leq \zeta_y \leq 1$ (3.17) is trivially true. To prove (3.18), note that

$$
\begin{aligned}
\mathrm{Cov}(\zeta_y, \zeta_z) = \mathrm{Cov}(1 - \zeta_y, 1 - \zeta_z) &= P(\zeta_y = \zeta_z = 0) - P(\zeta_y = 0)P(\zeta_z = 0) \\
&\leq P(\zeta_y = 0) \leq \exp\left(-c_1 \alpha^{2(k+i-1)}\right)
\end{aligned}
\tag{3.23}
$$

from (3.22). Using $\alpha^k \geq \alpha \geq 1/c_1$ and $\alpha^{2i} \geq i$ we obtain (3.18).

$\square$

The next lemma gives an upper bound on the probability that the event $G_{i,1}$ does not happen (conditionally on the event $A_k \cap G_0 \cap \ldots \cap G_{i-1}$).

Lemma 3.4. *There is a finite constant $c_4 = c_4(\alpha, d) > 0$, which is independent of k, such that for all $i \in \mathbb{N}$*

$$
P(G_{i,1}^c) = P\left(\sum_{y \in F_{k+i}} \zeta_y < 2\alpha^{(d+1)(i+k)}\right) \leq c_4\left(\alpha^{-(k+i)(d+1)} + \exp\left(-i\alpha^{k-2}\right)\right)
$$

and

$$
P(G_{0,1}^c) = P\left(\sum_{y \in F_k} \zeta_y < 2\alpha^{(d+1)k}\right) \leq c_4\left(\alpha^{-k(d+1)} + \exp\left(-\alpha^{k-2}\right)\right).
\tag{3.24}
$$

Proof of Lemma 3.4. By inequalities (3.11) and (3.16) we have

$$
\sum_{y \in F_{k+i}} E[\zeta_y] \geq |F_{k+i}|(1 - \exp(-2)) \geq \frac{5}{2} 2^{d-1} \alpha^{(k+i)(d+1)}(1 - \exp(-2)).
\tag{3.25}
$$

Thus, using the simple inequality $P(X \leq a) \leq P(X \leq b)$ if $a < b$ we obtain

$$
P\left(\sum_{y \in F_{k+i}} \zeta_y < 2\alpha^{(d+1)(i+k)}\right)
$$

$$
= P\left(\sum_{y \in F_{k+i}} \left(\zeta_y - E[\zeta_y]\right) < 2\alpha^{(d+1)(i+k)} - \sum_{y \in F_{k+i}} E[\zeta_y]\right)
$$

$$
\leq P\left(\sum_{y \in F_{k+i}} \left(\zeta_y - E[\zeta_y]\right) < -\alpha^{(d+1)(i+k)}\left(\frac{5}{2}2^{d-1}(1 - \exp(-2)) - 2\right)\right) \qquad (3.26)
$$

Now note that we have

$$
\frac{5}{2}2^{d-1}(1 - \exp(-2)) - 2 \geq \frac{5}{2}(1 - \exp(-2)) - 2 =: c > 0 \qquad (3.27)
$$

for all $d \geq 1$. Note that c does not depend on k. Hence, by (3.27), Chebyshev's inequality, inequalities (3.12), (3.17) and the second inequality in (3.18) we have for each $i \geq 1$.

$$
P(G_{i,1}^c) \leq c^{-2}\alpha^{-2(d+1)(i+k)}\left(\sum_{y \in F_{k+i}} \mathrm{Var}(\zeta_y) + \sum_{\substack{y,z \in F_{k+i}:\\ y \neq z}} \mathrm{Cov}(\zeta_y, \zeta_z)\right)
$$

$$
\leq c^{-2}\alpha^{-2(d+1)(i+k)}\left(\alpha^d\alpha^{(k+i)(d+1)} + \alpha^{2d}\alpha^{2(k+i)(d+1)}\exp\left(-i\alpha^{k-2}\right)\right)
$$

$$
\leq c_4\left(\alpha^{-(k+i)(d+1)} + \exp\left(-i\alpha^{k-2}\right)\right), \qquad (3.28)
$$

where $c_4 = c^{-2}\alpha^{2d}$ is also independent of k. For $i = 0$ we obtain the desired upper bound (3.24) by using the first inequality in (3.18) instead of the second one.

$\square$

Next, we aim at bounding below the conditional probability of $G_{i,2}$ given that $G_{i,1}$ happens. Note that if $G_{i,1}$ happens, the set B_{i+1} is well-defined and also we have

$$
P(G_{i,2}|G_{i,1}) \geq P\left(\sum_{j=1}^{a_i} Y_j \geq a_i\right), \qquad (3.29)
$$

where $Y_1, Y_2, \ldots$ are i.i.d. with the same distribution μ as the η_x and we write $a_i := \alpha^{(d+1)(k+i)}$, $i \in \mathbb{N}$, for short. This follows directly from independence and (3.8). Since the Y_j are nonnegative and have infinite mean, we know from Cramér's theorem (see Theorem 2.2.3 and the following Remark (c)in [2]) that with the notation $S_n := \sum_{j=1}^n Y_j$, $n \in \mathbb{N}$, we have

$$
P(S_n \leq n) \leq 2\exp\left(-nb\right), n \in \mathbb{N}, \qquad (3.30)
$$

where $b = I(1) > 0$ is the value at 1 of the Legendre-Fenchel transform $I(x)$ of the cumulant generating function of Y_1. That $I(1) > 0$ also follows from the fact that Y_1 is nonnegative and has infinite mean. From (3.29) and (3.30) we conclude that for each $i \geq 0$

$$
P(G_{i,2}|G_{i,1}) \geq P(S_{a_i} \geq a_i) \geq 1 - P(S_{a_i} \leq a_i) \geq 1 - 2\exp\left(-ba_i\right), \qquad (3.31)
$$

where we let $b := I(1) > 0$. Now, using

$$
P(G_i^c) = 1 - P(G_i) = 1 - P(G_{i,2}|G_{i,1})P(G_{i,1}) = 1 - P(G_{i,2}|G_{i,1})\left(1 - P(G_{i,1}^c)\right)
$$

$$
\leq 1 - P(G_{i,2}|G_{i,1}) + P(G_{i,1}^c)
$$

and $a_i \geq i\alpha^k$, from Lemma 3.4 and (3.31) we immediately infer the following lemma.

Lemma 3.5. *With the constant $c_4 = c_4(\alpha, d)$ from Lemma 3.4 we have*

$$P(G_i^c) \leq c_4\Big(\alpha^{-(k+i)(d+1)} + \exp\big(-i\alpha^{k-2}\big)\Big) + 2\exp\big(-ib\alpha^k\big)\,, i \in \mathbb{N}\,, \tag{3.32}$$

and

$$P(G_0^c) \leq c_4\Big(\alpha^{-k(d+1)} + \exp\big(-\alpha^{k-2}\big)\Big) + 2\exp\big(-b\alpha^k\big)\,. \tag{3.33}$$

Now, for $x \geq 0$, define the function

$$g(x) := c_4\left(\frac{\alpha^{-x(d+1)}}{1 - \alpha^{-(d+1)}} + \frac{\exp\big(-\alpha^{x-2}\big)}{1 - \exp\big(-\alpha^{x-2}\big)} + \exp\big(-\alpha^{x-2}\big)\right) \tag{3.34}$$

$$+ 2\left(\exp\big(-b\alpha^x\big) + \frac{\exp\big(-b\alpha^x\big)}{1 - \exp\big(-b\alpha^x\big)}\right) \tag{3.35}$$

and note that

$$\lim_{x \to \infty} g(x) = 0\,. \tag{3.36}$$

From Lemma 3.4 and the multiplication rule for conditional probabilites, we obtain that under our initial assumption that the event A_k happens we have

$$P\Big(\bigcap_{i=0}^{\infty} G_i\Big) = \lim_{m \to \infty} P\Big(\bigcap_{i=0}^{m} G_i\Big) = \lim_{m \to \infty} \prod_{i=0}^{m}\big(1 - P\big(G_i^c | G_0 \cap \ldots \cap G_{i-1}\big)\big)$$

$$\geq \lim_{m \to \infty}\big(1 - \sum_{i=0}^{m} P\big(G_i^c | G_0 \cap \ldots \cap G_{i-1}\big)\big)$$

$$= 1 - \sum_{i=0}^{\infty} P\big(G_i^c | G_0 \cap \ldots \cap G_{i-1}\big) \geq 1 - g(k)\,, \tag{3.37}$$

where we have used the simple inequality

$$\prod_{i=0}^{m}(1 - p_i) \geq 1 - \sum_{i=0}^{m} p_i$$

valid for numbers $p_0, \ldots, p_m \in [0, 1]$.

Proposition 3.6. *Fix $k \in \mathbb{N}$. Assume for the frog model that the i.i.d. random variables η_x, $x \in \mathbb{Z}^d \setminus \{0\}$ satisfy $\mathbb{E}_\mu[\log^+(\eta_x)^{\frac{d+1}{2}}] = \infty$. Then, if the event A_k happens and, thus, B_0 can be constructed as in (3.7), we have*

$$P\Big(0 \text{ is visited infinitely often} \mid \bigcap_{i=0}^{\infty} G_i\Big) = 1\,.$$

Proof of Proposition 3.6. First note that, if $k \geq 1$ is fixed, the sets D_i, $i \in \mathbb{N}$, satisfy $D_i \subseteq F_{k+i-1}$ and, hence, we have $D_i \cap V_k = \emptyset$ and also $D_i \cap \bigcup_{j \in \mathbb{Z}_+} B_j = \emptyset$ for each $i \in \mathbb{N}$. The event A_k does not depend on the values of the random variables η_x for $x \notin V_k$. Furthermore, the event $\bigcap_{j \in \mathbb{Z}_+} G_j$ only depends on the η_x such that $x \in A_k \cup \bigcup_{j \in \mathbb{Z}_+} B_j$. Thus, after conditioning on A_k and on $\bigcap_{j \in \mathbb{Z}_+} G_j$, by independence, we still have the i.i.d. property for the η_x, where $x \in \bigcup_{i \in \mathbb{Z}_+} D_i$. This will allow us to apply Lemma 3.2 below. Note that for each fixed configuration η_x, $x \in \bigcup_{i \in \mathbb{Z}_+} D_i$, by (3.2) we have

$$\sum_{i=1}^{\infty} \sum_{x \in D_i} \eta_x f(x, 0) \geq \sum_{i=1}^{\infty} \sum_{x \in D_i} \eta_x \varepsilon^{d|x|} \geq \sum_{i=1}^{\infty} \varepsilon^{d\alpha^{2k+2i}} \sum_{x \in D_i} \eta_x$$

$$\geq \sum_{i=1}^{\infty} \delta^{\alpha^{2i}} M_i\,, \tag{3.38}$$

where $\delta := \varepsilon^{d\alpha^{2k}} \in (0,1)$ and $M_i := \max_{x \in D_i} \eta_x$, $i \in \mathbb{N}$. For $i \in \mathbb{N}$ let
$l_i := |D_i| = \alpha^{(k-1)(d+1)}\alpha^{i(d+1)}$. Then, by using Lemma 3.2 with $L_i := D_i$, $c := -\log \delta$,
$c_2 = \alpha^{(k-1)(d+1)}$, $c_3 = d+1$, $r = \frac{d+1}{2}$ and $\beta = \alpha$ we obtain that $\mathbb{P}_\mu$-a.s.

$$M_i \geq \exp\left(c\alpha^{2i}\right) \text{ for infinitely many } i \in \mathbb{N}. \tag{3.39}$$

Hence, $\mathbb{P}_\mu$-a.s., there is a strictly increasing sequence $(i_m)_{m \in \mathbb{N}}$ of positive integers such that for all $m \in \mathbb{N}$

$$M_{i_m} \geq \exp\left(c\alpha^{2i_m}\right). \tag{3.40}$$

Thus, from (3.38) and (3.40) we have $\mathbb{P}_\mu$-a.s.

$$\sum_{i=1}^{\infty} \sum_{x \in D_i} \eta_x f(x, 0) \geq \sum_{m=1}^{\infty} \delta^{\alpha^{2i_m}} M_{i_m} \geq \sum_{m=1}^{\infty} \delta^{\alpha^{2i_m}} \exp\left(c\alpha^{2i_m}\right)$$

$$= \sum_{m=1}^{\infty} 1 = \infty. \tag{3.41}$$

By construction, for each $i \in \mathbb{N}$, the frogs in D_i get activated by frogs starting from B_{i-1}. Hence, by the remark before Lemma 3.3, all frogs in $\bigcup_{i=1}^{\infty} D_i$ are independent. Hence, from (3.41) and the second Borel-Cantelli lemma we conclude that $\mathbb{P}_\mu$-a.s.

$$P_\eta\left(0 \text{ is visited infinitely often } \big| \bigcap_{i=0}^{\infty} G_i\right) = 1.$$

Thus, also

$$P\left(0 \text{ is visited infinitely often } \big| \bigcap_{i=0}^{\infty} G_i\right) = 1,$$

as claimed.

$\square$

Now, note that from (3.37) and Proposition 3.6 we have

$$P\left(0 \text{ is visited infinitely often }\right) \geq P\left(0 \text{ is visited infinitely often } \big| \bigcap_{i=0}^{\infty} G_i\right) P\left(\bigcap_{i=0}^{\infty} G_i\right)$$

$$\geq 1 - g(k). \tag{3.42}$$

Since $\lim_{k \to \infty} g(k) = 0$ by (3.42) the proof of Theorem 2.1 will be completed, if we can show that P-a.s. the event A_k happens for arbitrarily large $k \in \mathbb{N}$. This is guaranteed by the following lemma.

Lemma 3.7. *We have*

$$P\left(\limsup_{k \to \infty} A_k\right) = 1.$$

Proof of Lemma 3.7. Denote by π the path of the initial frog starting from the origin. By the properties of the underlying random walk, clearly, π contains infinitely many different vertices. We are going to use Lemma 3.2 with $J = \pi$, $Y_x = \eta_x$, $x \in \pi$, and $r = (d+1)/2$. The pairwise disjoint sets L_i, $i \in \mathbb{N}$, are constructed inductively as follows: Let L_1 contain the first $\alpha^2 - 1$ pairwise different vertices in $\pi \setminus \{0\}$. Clearly, $L_1 \subseteq V_1$. If L_{i-1} for $i \geq 2$ has already been constructed, let L_i contain exactly the next $\alpha^{2i} - \alpha^{2i-2}$ vertices in π, which are not contained in V_{i-1}. Then, $L_i \subseteq V_i \setminus V_{i-1}$. Note that the sets L_i satisfy $l_i := |L_i| \geq c_2\alpha^{2i}$, where $c_2 = 1 - \alpha^{-2}$. Hence, from Lemma 3.2 (with $c_3 = 2$, $c = 1$, $\beta = \alpha$ and $r = (d+1)/2$) we conclude that $\mathbb{P}_\mu$-a.s.

$$M_i = \max_{x \in L_i} \eta_x \geq \exp\left(\alpha^{\frac{4i}{d+1}}\right) \text{ infinitely often.} \tag{3.43}$$

In particular, $\mathbb{P}_\mu$-a.s. for each $k_0 \in \mathbb{N}$ there exists a $k \geq k_0$ such that

$$M_k \geq \alpha^{(k-1)(d+1)},$$

implying that P-a.s. the event A_k happens for arbitrarily large values of k.

$\square$

4 Proofs of auxiliary lemmas

This section is devoted to the proofs of Lemmas 3.2 and 3.1. In order to prove Lemma 3.2 we need some facts about the behaviour of the maxima of nonnegative i.i.d. random variables, some of which rely on the following simple lemma on real sequences:

Lemma 4.1. *Let $u : [0, \infty) \to [0, \infty)$ be an increasing and invertible function and let $(y_n)_{n \in \mathbb{N}}$ be a sequence of numbers in the interval $[a, \infty)$. For $n \in \mathbb{N}$ let $m_n := \max_{1 \leq j \leq n} y_j$. Then, the following two conditions are equivalent:*

(i) $m_n \geq u^{-1}(n)$ *for infinitely many* $n \in \mathbb{N}$

(ii) $y_n \geq u^{-1}(n)$ *for infinitely many* $n \in \mathbb{N}$

Proof of Lemma 4.1. Of course, (ii) trivially implies (i). So let us prove the converse. Let

$$n_0 := \inf\{n \in \mathbb{N} \,:\, m_n \geq u^{-1}(n)\}.$$

By (i) n_0 is finite and $m_{n_0} = y_{n_0}$. Hence, there is an $n \in \mathbb{N}$ such that $y_n \geq u^{-1}(n)$. It thus suffices to show that for each $n_1 \in \mathbb{N}$ with $y_{n_1} \geq u^{-1}(n_1)$ there is a further $n_2 > n_1$ such that $y_{n_2} \geq u^{-1}(n_2)$. Since u^{-1} is unbounded, there is a $k \in \mathbb{N}$ such that $u^{-1}(k) > y_{n_1}$. By (i) there is an $n > k$ such that

$$m_n \geq u^{-1}(n) > u^{-1}(k) > y_{n_1},$$

since u^{-1} is also increasing. Now, choose $n_2 \in \{k+1, \ldots, n\}$ minimal such that $m_{n_2} \geq u^{-1}(n)$. Then, $m_{n_2-1} < u^{-1}(n)$ and

$$u^{-1}(n_2) \leq u^{-1}(n) \leq m_{n_2} = \max(m_{n_1-1}, y_{n_2}) = y_{n_2},$$

since $m_{n_1-1} < u^{-1}(n)$.

$\square$

For a sequence $(Y_j)_{j \in \mathbb{N}}$ of nonnegative random variables and $n \in \mathbb{N}$ we define

$$M'_n := \max_{1 \leq j \leq n} Y_j. \tag{4.1}$$

Lemma 4.2. *Let $(Y_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of nonnegative random variables and let $u : [0, \infty) \to [0, \infty)$ be an increasing and invertible function.*

(a) *If $E[u(Y_1)] < \infty$, then $P(M'_n < u^{-1}(n)$ eventually $) = 1$.*

(b) *If $E[u(Y_1)] = \infty$, then $P(M'_n \geq u^{-1}(n)$ infinitely often $) = 1$.*

Proof. We first prove (a). Since the Y_n are identically distributed and also u^{-1} is increasing, we have

$$\sum_{n=1}^\infty P(Y_n \geq u^{-1}(n)) = \sum_{n=1}^\infty P(Y_1 \geq u^{-1}(n)) \leq \int_0^\infty P(Y_1 \geq u^{-1}(x))dx$$

$$= \int_0^\infty P(u(Y_1) \geq x)dx = E[u(Y_1)] < \infty.$$

From the first Borel-Cantelli lemma we conclude that $P(Y_n \geq u^{-1}(n) \text{ infinitely often }) = 0$ and from Lemma 4.1 we obtain $P(M'_n \geq u^{-1}(n) \text{ infinitely often }) = 0$, which is equivalent to the assertion.

Now, we turn to the proof of (b). By assumption we have

$$\sum_{n=0}^{\infty} P(Y_n \geq u^{-1}(n)) = \sum_{n=0}^{\infty} P(Y_1 \geq u^{-1}(n)) \geq \int_0^{\infty} P(Y_1 \geq u^{-1}(x)) dx$$
$$= E[u(Y_1)] = \infty.$$

By independence, the second Borel-Cantelli lemma implies that

$$P(M'_n \geq u^{-1}(n) \text{ infinitely often }) \geq P(Y_n \geq u^{-1}(n) \text{ infinitely often }) = 1.$$

$\square$

Corollary 4.3. *Let $(Y_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of nonnegative random variables and let $r > 0$.*

(a) *If $E[\log^+(Y_1)^r] < \infty$, then for all constants $c, L > 0$*

$$P\Big(\max_{1 \leq i \leq \lfloor Ln^r \rfloor} Y_i < \exp\big(cL^{1/r}n\big) \text{ eventually } \Big) = 1.$$

(b) *If $E[\log^+(Y_1)^r] = \infty$, then for every constant $c > 0$*

$$P\Big(M'_n \geq \exp\big(cn^{1/r}\big) \text{ infinitely often } \Big) = 1.$$

(c) *If $E[\log^+(Y_1)^r] = \infty$, then for every constant $c > 0$ and every non-decreasing sequence $(s_i)_{i \in \mathbb{N}}$ of positive integers such that $\lim_{i \to \infty} s_i = \infty$ and $\inf_{i \geq 2} \frac{s_{i-1}}{s_i} > 0$*

$$P\Big(M'_{s_i} \geq \exp\big(cs_i^{1/r}\big) \text{ for infinitely many } i \Big) = 1.$$

Proof. (a) follows from Lemma 4.2 (a) by choosing $u(x) = (\log^+(x)/c)^r$ and noting that $M'_n < \exp\big(cn^{1/r}\big)$ eventually implies $\max_{1 \leq i \leq \lfloor Ln^r \rfloor} Y_i < \exp\big(cL^{1/r}n\big)$ eventually. Similarly, (b) follows from Lemma 4.2 (b). To prove (c) choose a set G with $P(G) = 1$ according to (b) such that for all $\omega \in G$ there exists a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ (depending on ω) with

$$M'_{n_k}(\omega) \geq \exp\big(\tilde{c}n_k^{1/r}\big) \text{for all } k \in \mathbb{N},$$

where $\tilde{c} := c(\inf_{i \geq 2} s_{i-1}/s_i)^{-1/r} < \infty$ by the assumptions on the sequence $(s_i)_{i \in \mathbb{N}}$. Then, for each $\omega \in G$ and for infinitely many values of $i \in \mathbb{N}$ there is a $k = k_i$ such that $s_{i-1} < n_k \leq s_i$. The claim now follows from the chain of inequalities

$$M'_{s_i}(\omega) \geq M_{n_k}(\omega) \geq \exp\big(\tilde{c}n_k^{1/r}\big) \geq \exp\big(s_i^{1/r}\tilde{c}\big(\tfrac{s_{i-1}}{s_i}\big)^{1/r}\big) \geq \exp\big(cs_i^{1/r}\big).$$

$\square$

Proof of Lemma 3.2. For $i \in \mathbb{N}$ define

$$M_i^\star := \max_{j \in \bigcup_{k \leq i} L_i} Y_j = \max_{k \leq i} M_k. \qquad (4.2)$$

Note that by disjointness of the sets L_i we have for the cardinality of $\bigcup_{k \leq i} L_i$:

$$\Big| \bigcup_{k \leq i} L_i \Big| = \sum_{k=1}^i l_k \geq \sum_{k=1}^i c_2 \beta^{c_3 k} = c_2 \beta^{c_3} \frac{\beta^{c_3 i} - 1}{\beta^{c_3} - 1} \geq \lceil \tilde{c}\beta^{c_3 i} \rceil =: s_i, \qquad (4.3)$$

where $\tilde{c} > 0$ is a constant depending only on c_2, c_3 and β. Hence, for each $i \in \mathbb{N}$, $M_i^\star$ is stochastically larger than M'_{s_i} from Corollary 4.3 (c) and the integer sequence $(s_i)_{i \in \mathbb{N}}$ satisfies the above assumptions. In particular, we have

$$P\Big(M_i^\star \geq \exp\big(c'{s_i}^{1/r}\big) \text{ for infinitely many } i\Big) = 1 \qquad (4.4)$$

for each finite constant $c' > 0$. This immediately implies that

$$P\Big(M_i^\star \geq \exp\big(c\beta^{\frac{c_3 i}{r}}\big) \text{ for infinitely many } i\Big) = 1 \qquad (4.5)$$

for each finite constant $c > 0$. Now using

$$M_i^\star = \max_{1 \leq k \leq i} M_k$$

the claim follows from Lemma 4.1 applied to the function $u(x) = r\frac{\log\log x - \log c}{c_3 \log \beta}$.

$\square$

Sketch of the proof of Lemma 3.1. First note that the probability $f(x, y)$ is also the probability that the continuous time random walk (CTRW)

$$(X_t)_{t>0} = \big(X_t^{(1)}, \ldots, X_t^{(d)}\big)_{t>0}$$

corresponding to p ever visits y if it is starting at x. The benefit of working in continuous time here is that for CTRW the coordinates are independent, which is not true for discrete time random walks. Because of (2.1), letting $\tau := \inf\{t > 0 : X_t^{(1)} = y_1\}$, we know that $P_x(\tau < \infty) = 1$. Furthermore,

$$\begin{aligned}
f(x, y) = P_x\big(\exists\, t > 0 : X_t = y\big) &\geq P_x\big(X_\tau = y\big) \\
&= \int_0^\infty P_x\big(X_\tau = y \mid \tau = t\big) P_x\big(\tau \in dt\big) \\
&= \int_0^\infty P_x\big((X_t^{(2)}, \ldots, X_t^{(d)}) = (y_2, \ldots, y_d)\big) P_x\big(\tau \in dt\big) \\
&\geq \int_{\gamma_1(y_1 - x_1)}^{\gamma_2(y_1 - x_1)} P_x\big((X_t^{(2)}, \ldots, X_t^{(d)}) = (y_2, \ldots, y_d)\big) P_x\big(\tau \in dt\big), \qquad (4.6)
\end{aligned}$$

where $0 < \gamma_1 < \gamma_2 < \infty$ are chosen such that $P_x\big(\gamma_1(y_1 - x_1) \leq \tau \leq \gamma_2(y_1 - x_1)\big) \geq 1/2$. Now, since $|(y_2 - x_2, \ldots, y_d - x_d)| \leq \gamma\sqrt{y_1 - x_1}$, by the local CLT for continuous time random walks there is a universal constant $c > 0$ such that

$$P_x\big((X_t^{(2)}, \ldots, X_t^{(d)}) = (y_2, \ldots, y_d)\big) \geq \frac{c}{t^{\frac{d-1}{2}}} \qquad (4.7)$$

for all $t \geq \gamma_1(y_1 - x_1)$. Thus, from (4.6) and (4.7) we get

$$\begin{aligned}
f(x, y) &\geq \frac{c}{\big(\gamma_2(y_1 - x_1)\big)^{\frac{d-1}{2}}} \int_{\gamma_1(y_1 - x_1)}^{\gamma_2(y_1 - x_1)} P_x\big(\tau \in dt\big) \\
&\geq \frac{c}{2\big(\gamma_2(y_1 - x_1)\big)^{\frac{d-1}{2}}},
\end{aligned}$$

yielding the claim with $c_1 := \frac{1}{2}c\gamma_2^{-\frac{d-1}{2}}$.

$\square$

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