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SOME REMARKS ON THE HEAT FLOW FOR FUNCTIONS AND FORMS

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Abstract

This note is concerned with the differentiation of heat semigroups on Riemannian manifolds. In particular, the relation $dP_tf = P_tdf$ is investigated for the semigroup generated by the Laplacian with Dirichlet boundary conditions. By means of elementary martingale arguments it is shown that well-known properties which hold on complete Riemannian manifolds fail if the manifold is only BM-complete. In general, even if M is flat and f smooth of compact support, $\|dP_tf\|_\infty$ cannot be estimated on compact time intervals in terms of f or df.

1 Introduction

Let (M, g) be a Riemannian manifold and Δ its Laplacian. Consider the minimal heat semigroup associated to $\frac{1}{2}\Delta$ on functions given by

$$(P_tf)(x) = \mathbb{E}\big[\big(f \circ X_t(x)\big)\, 1_{\{t<\zeta(x)\}}\big] \tag{1.1}$$

where $X_{\bullet}(x)$ is Brownian motion on M starting at x, with (maximal) lifetime $\zeta(x)$. Denote by $W_{0,\bullet}\colon T_xM \to T_{X_\bullet(x)}M$ the linear transport on M along $X_\bullet(x)$ determined by the following pathwise covariant equation:

$$\begin{cases} \frac{D}{dr}\, W_{0,r}\, v = \frac{1}{2}\,\mathrm{Ric}\big(W_{0,r}\, v,\, \cdot\,\big)^{\#} \\ \quad W_{0,0}\, v = v. \end{cases} \tag{1.2}$$

By definition, $\frac{D}{dr} = /\!/_{0,r}\, \frac{d}{dr}\, /\!/_{0,r}^{-1}$ where $/\!/_{0,\bullet}$ denotes parallel transport along $X_\bullet(x)$. For 1-forms $\alpha \in \Gamma(T^*M)$ let

$$(P_t^{(1)}\alpha)v = \mathbb{E}\big[\alpha_{X_t(x)}\, W_{0,t}\, v\, 1_{\{t<\zeta(x)\}}\big], \quad v \in T_xM. \tag{1.3}$$

It is a well-known consequence of the spectral theorem that on a complete Riemannian manifold M

$$d\,P_t f = P_t^{(1)} df \tag{1.4}$$

holds for all $f \in C_c^\infty(M)$ (compactly supported C^∞ functions on M) if, for instance,

$$\mathbb{E}\big[\|W_{0,t}\|\,1_{\{X_t(x)\in K\}}\,1_{\{t<\zeta(x)\}}\big] < \infty \tag{1.5}$$

for any $x \in M$ and any compact subset $K \subset M$. Indeed, (1.4) holds true for the semigroups associated to the self-adjoint extensions of the Laplacian on functions, resp. 1-forms. These semigroups defined by the spectral theorem can be identified with the stochastic versions (1.1) and (1.3) as soon as (1.3) is well-defined. The identification can be done, for instance, with straightforward martingale arguments by exhausting the manifold through a sequence of regular domains.

Note that from the defining equation (1.2) one gets

$$\|W_{0,t}\| \le \exp\left\{-\frac{1}{2}\int_0^t \underline{\mathrm{Ric}}(X_s(x))\,ds\right\}$$

where $\underline{\mathrm{Ric}}(x)$ is the smallest eigenvalue of the Ricci tensor Ric_x at x. Thus (1.5) reads as a condition imposing lower bounds on the Ricci curvature of M.

The Brownian motions $X_{\textbf{.}}(x)$ may be constructed as solutions of a globally defined (non-intrinsic) Stratonovich SDE on M of the form

$$dX = A(X)\circ dZ + A_0(X)\,dt \tag{1.6}$$

with $A \in \Gamma(\mathbb{R}^r \otimes TM)$, $A_0 \in \Gamma(TM)$ and Z an $\mathbb{R}^r$-valued Brownian motion on some filtered probability space satisfying the usual completeness conditions. For $x \in M$, let

$$\mathcal{F}_t(x) := \mathcal{F}_t^{X(x)} \equiv \sigma\big\{X_s(x) : 0 \le s \le t\big\} \tag{1.7}$$

be the filtration generated by $X(x)$ starting at x. Then, by [4], A and A_0 in the SDE (1.6) can be chosen in such a way that

$$W_{0,t}\,v\,1_{\{X_t(x)\in K\}} = /\!/_{0,t}\,\mathbb{E}^{\mathcal{F}_t(x)}\big[/\!/_{0,t}^{-1}(T_xX_t)\,v\,1_{\{X_t(x)\in K\}}\big]. \tag{1.8}$$

Suppose that, instead of (1.5), we have

$$\mathbb{E}\big[\|T_xX_t\|\,1_{\{X_t(x)\in K\}}\,1_{\{t<\zeta(x)\}}\big] < \infty \tag{1.9}$$

for any $x \in M$ and any compact subset $K \subset M$. Then

$$(P_t^{(1)}df)v = \mathbb{E}\big[(df)_{X_t(x)}\,T_xX_t\,v\,1_{\{t<\zeta(x)\}}\big], \quad v \in T_xM. \tag{1.10}$$

Thus, supposing for simplicity that (M,g) is BM-complete, i.e., $\zeta(x) \equiv \infty$ a.s. for all $x \in M$, relation (1.4) comes down to a matter of differentiation under the integral.

This point of view rises the question whether completeness of M is an essential ingredient for (1.4) to hold. However, we show that (1.4) may fail on metrically incomplete manifolds, even if the manifold is flat and BM-complete. Even then, $\limsup_{t\to 0+} \|dP_tf\|_\infty$ may be infinite for compactly supported $f \in C^\infty(M)$.

2 Differentiation of semigroups

We follow the methods of [7]. In the sequel we write occasionally $T_x f$ instead of df_x for the differential of a function f to avoid mix-up with stochastic differentials. Finally, we denote by $B(M)$ the bounded measurable functions on M and by $bC^1(M)$ the bounded C^1-functions on M with bounded derivative.

Lemma 2.1 *Let (M,g) be a Riemannian manifold and $f \in B(M)$. Fix $t>0$, $x \in M$, and $v \in T_xM$. Then*

$$N_s := T_{X_s(x)}(P_{t-s}f)\, T_xX_s\, v\,, \qquad 0 \le s < t \wedge \zeta(x),$$
$$\bar N_s := T_{X_s(x)}(P_{t-s}f)\, W_{0,s}v\,, \qquad 0 \le s < t \wedge \zeta(x),$$

are local martingales (with respect to the underlying filtration).

Proof To see the first claim, note that $(P_{t-\bullet}f)\big(X_\bullet(x)\big)$ is a local martingale depending on x in a differentiable way. Thus, the derivative with respect to x is again a local martingale, see [1]. The second claim is reduced to the first one by conditioning with respect to $\mathcal{F}_\bullet(x)$ to filter out redundant noise. The second part may also be checked directly using the Weitzenböck formula

$$d\Delta f \equiv \Delta^{(1)}df = \Delta^{\mathrm{hor}}df - \mathrm{Ric}(df^\#, \,\boldsymbol{\cdot}\,) \tag{2.1}$$

where $\Delta^{(1)}$ is the Laplacian on 1-forms and $\Delta^{\mathrm{hor}}df$ the horizontal Laplacian on $\mathrm{O}(M)$ acting on df when considered as equivariant function on $\mathrm{O}(M)$. Indeed, by lifting things up to the orthonormal frame bundle $\mathrm{O}(M)$ over M, we can write

$$\bar N_s = F(s,U_s)\cdot U_s^{-1}W_{0,s}\,v$$

where U is a horizontal lift of the BM $X_\bullet(x)$ to $\mathrm{O}(M)$ (i.e., a horizontal BM on $\mathrm{O}(M)$ with generator $\frac12\Delta^{\mathrm{hor}}$) and

$$F\colon [0,t]\times \mathrm{O}(M) \to \mathbb{R}^d, \quad F_i(s,u) := (dP_{t-s}f)_{\pi(u)}(ue_i)\,, \quad i=1,\dots,d=\dim M.$$

Then $d\bar N_s \overset{\mathrm{m}}{=} 0$ (equality modulo differentials of local martingales) follows by means of Itô's formula. □

Notation For the Brownian motion $X_\bullet(x)$ on M, let

$$B = \int_0^\bullet //_{0,r}^{-1} \circ dX_r(x)$$

denote the anti-development of $X_\bullet(x)$ taking values in T_xM. By definition, B is a BM on T_xM satisfying

$$A\big(X(x)\big)\,dZ = //_{0,\bullet}\,dB.$$

Lemma 2.2 *Let (M,g) be a Riemannian manifold, $f \in B(M)$, $x \in M$ and $t > 0$. Let $\Theta_{0,\bullet}\colon T_xM \to T_{X_\bullet(x)}M$ be linear maps such that*

$$T_{X_s(x)}(P_{t-s}f)\,\Theta_{0,s}\,v\,,\quad 0 \le s < t \wedge \zeta(x),$$

is a continuous local martingale. Then

$$T_{X_s(x)}(P_{t-s}f)\,\Theta_{0,s}\,h_s - \int_0^s (T_{X_r(x)}P_{t-r}f)\,\Theta_{0,r}\,dh_r,\quad 0 \le s < t \wedge \zeta(x), \tag{2.2}$$

is again a continuous local martingale for any adapted T_xM-valued process h of locally bounded variation. In particular,

$$T_{X_s(x)}(P_{t-s}f)\,\Theta_{0,s}\,h_s - (P_{t-s}f)\big(X_s(x)\big)\int_0^s \big\langle \Theta_{0,r}\,\dot h_r, //_{0,r}\,dB_r\big\rangle,\quad 0 \le s < t \wedge \zeta(x),$$

is a local martingale for any adapted process h with paths in the Cameron-Martin space $\mathbb{H}([0,t],T_xM)$, i.e., $h_\bullet(\omega) \in \mathbb{H}([0,t],T_xM)$ for almost all ω.

Proof Indeed, by Itô's lemma,

$$\begin{aligned} d\big(T_{X_s(x)}(P_{t-s}f)\,\Theta_{0,s}\,h_s\big) &= \big(T_{X_s(x)}(P_{t-s}f)\,\Theta_{0,s}\big)\,dh_s + d\big(T_{X_s(x)}(P_{t-s}f)\,\Theta_{0,s}\big)\cdot h_s \\ &\overset{\mathrm{m}}{=} \big(T_{X_s(x)}(P_{t-s}f)\,\Theta_{0,s}\big)\,dh_s \end{aligned}$$

where $\overset{\mathrm{m}}{=}$ stands for equality modulo local martingales. The second part can be seen using the formula

$$(P_{t-s}f)\big(X_s(x)\big) = \int_0^s T_{X_r(x)}(P_{t-r}f)\,//_{0,r}\,dB_r\,.$$

This proves the Lemma. □

Lemma 2.2 leads to explicit formulae for dP_tf by means of appropriate choices for h which make the local martingales in Lemma 2.2 to uniformly integrable martingales. This can be done as in [7].

Theorem 2.3 [7] *Let $f\colon M \to \mathbb{R}$ be bounded measurable, $x \in M$ and $v \in T_xM$. Then, for any bounded adapted process h with paths in $\mathbb{H}(\mathbb{R}_+,T_xM)$ such that $\big(\int_0^{\tau_D\wedge t} |\dot h_s|^2\,ds\big)^{1/2} \in L^1$, and the property that $h_0 = v$, $h_s = 0$ for all $s \ge \tau_D \wedge t$, the following formula holds:*

$$d(P_tf)_xv = -\mathbb{E}\left[f\big(X_t(x)\big)\,1_{\{t<\zeta(x)\}}\int_0^{\tau_D\wedge t}\big\langle W_{0,s}(\dot h_s), //_{0,s}\,dB_s\big\rangle\right] \tag{2.3}$$

where τ_D is the first exit time of $X(x)$ from some relatively compact open neighbourhood D of x.

Theorem 2.4 *Let (M,g) be a* BM*-complete Riemannian manifold such that* Ric $\geq \alpha$ *for some constant* α.
(i) *For $f \in bC^1(M)$ the relation $dP_sf = P_s^{(1)}df$ holds for $0 \leq s \leq t$ if and only if*

$$\sup_{0\leq s\leq t} \|dP_sf\|_\infty < \infty. \tag{2.4}$$

(ii) *Let $f \in C^1(M)$ be bounded such that* (2.4) *is satisfied. Then, for $t > 0$,*

$$\|dP_tf\|_\infty \leq \Big(\Big(\frac{1-e^{-\alpha t}}{\alpha}\Big)^{1/2}\frac{1}{t}\,\|f\|_\infty\Big) \wedge \Big(e^{-\alpha\, t/2}\,\|df\|_\infty\Big) \tag{2.5}$$

with the convention $(1-e^{-\alpha t})/\alpha = t$ for $\alpha = 0$.

Proof (i) Of course, $dP_sf = P_s^{(1)}df$ implies (2.4) in case df is bounded. On the other hand, let $f \in C^1(M)$ such that (2.4) holds. Condition (2.4) ensures the local martingale

$$\bar N_s = (dP_{t-s}f)_{X_s(x)}\, W_{0,s}\, v, \quad v \in T_xM,$$

of Lemma 2.1 to be a martingale for $0 \leq s \leq t$, which gives by taking expectations

$$(dP_tf)_xv = \mathbb{E}\big[(df)_{X_t(x)}\, W_{0,t}v\big] = P_t^{(1)}df(v).$$

(ii) As in (i), condition (2.4) for $f \in C^1(M)$ implies $(dP_tf)_xv = \mathbb{E}\big[(df)_{X_t(x)}\, W_{0,t}v\big]$ which shows $|d(P_tf)_x| \leq e^{-\alpha\, t/2}\,\|df\|_\infty$. On the other hand, by Lemma 2.2,

$$T_{X_\bullet(x)}(P_{t-\bullet}f)\, W_{0,\bullet}\, h_\bullet - (P_{t-\bullet}f)\big(X_\bullet(x)\big)\int_0^\bullet \big\langle W_{0,r}\,\dot h_r, //_{0,r}\, dB_r\big\rangle \tag{2.6}$$

is a local martingale for any adapted process h with $h_\bullet(\omega) \in \mathbb{H}([0,t], T_xM)$. If we take $h_s := (1-s/t)v$ where $v \in T_xM$, then by means of assumption (2.4) and the bound on the Ricci curvature, (2.6) is seen to be a uniformly integrable martingale, hence

$$d(P_tf)_xv = -\frac{1}{t}\,\mathbb{E}\Big[f\circ X_t(x)\int_0^t \big\langle W_{0,r}\, v, //_{0,r}\, dB_r\big\rangle\Big].$$

Thus

$$\begin{aligned} |d(P_tf)_x| &\leq \frac{1}{t}\|f\|_\infty\left(\mathbb{E}\int_0^t \|W_{0,r}\|^2\, dr\right)^{1/2} \\ &\leq \frac{1}{t}\|f\|_\infty\left(\int_0^t e^{-\alpha r}\, dr\right)^{1/2} \leq \frac{1}{t}\Big(\frac{1-e^{-\alpha t}}{\alpha}\Big)^{1/2}\|f\|_\infty \end{aligned}$$

which shows part (ii). □

Remark 2.5 [8] Let M be an arbitrary Riemannian manifold and $D \subset M$ an open set with compact closure and nonempty smooth boundary. Let $f \in B(M)$. Then, for $x \in D$ and $t > 0$,

$$|d(P_tf)_x| \leq c\,\|f\|_\infty$$

with a constant c depending on t, $\dim M$, $\operatorname{dist}(x, \partial D)$ and the lower bound of Ric on D. This follows from Theorem 2.3 with an explicit choice for h. See [8] for the details.

Remark 2.6 In the abstract framework of the Γ_2-theory of Bakry and Emery (e.g. [2]) lower bounds on the Ricci curvature $\mathrm{Ric} \geq \alpha$ (i.e. $\Gamma_2 \geq \alpha\Gamma$) may be expressed equivalently in terms of the semigroup as

$$|dP_t f|^2 \leq e^{-\alpha t}\, P_t |df|^2, \quad t \geq 0,$$

for f in a sufficiently large algebra of bounded functions on M. However, in general, the setting does not include the Laplacian on metrically incomplete manifolds. On such spaces, we may have $\limsup_{t\to 0+} \|dP_t f\|_\infty = \infty$ for $f \in C_c^\infty(M)$, as can be seen from the examples below.

3 An example

Let $\mathbb{R}^2\backslash\{0\}$ be the plane with origin removed. For $n \geq 2$, let M_n be an n-fold covering of $\mathbb{R}^2\backslash\{0\}$ equipped with the flat Riemannian metric. See [6] for a detailed analysis of the heat kernel on such BM-complete spaces. In terms of polar coordinates $x = (r, \vartheta)$ on M_n with $0 < r < \infty$, $0 \leq \vartheta < 2n\pi$,

$$h(x) = \cos(\vartheta/n)\, J_{1/n}(r) \tag{3.1}$$

is a bounded eigenfunction of Δ on M_n (with eigenvalue -1); here $J_{1/n}$ denotes the Bessel function of order $1/n$. Note that $J_{1/n}(r) = O(r^{1/n})$ as $r \searrow 0$, consequently dh is unbounded on M_n. The martingale property of

$$m_t = e^{t/2}\,(h \circ X_t(x)), \quad t \geq 0,$$

implies $P_t h = e^{-t/2} h$ which means that $dP_t h$ is unbounded on M_n as well.

Example 3.1 On M_n the relation $dP_t f = P_t^{(1)} df$ fails in general for compactly supported $f \in C^\infty(M_n)$. If this happens, then by Theorem 2.4 (i),

$$\sup_{0\leq s\leq t} \|dP_s f\|_\infty = \infty \tag{3.2}$$

for $f \in C^\infty(M_n)$ of compact support.

Proof Otherwise (3.2) holds true for all compactly supported $f \in C^\infty(M_n)$. Fix $t > 0$. Then by Theorem 2.4 (ii)

$$\|dP_t f\|_\infty \leq \frac{1}{\sqrt{t}}\, \|f\|_\infty \tag{3.3}$$

for any compactly supported $f \in C^\infty(M_n)$. On the other hand, we may choose a sequence (f_ℓ) of nonnegative, compactly supported elements in $C^\infty(M_n)$ such that $f_\ell \nearrow h^c := h + c$ with h given by (3.1) and c a constant such that $h + c \geq 0$. But then (see Chavel [3] p. 187 Lemma 3; note that this is a local argument which can be applied on any open relatively compact subset of M)

$$P_t f_\ell \nearrow P_t h^c \quad \text{and} \quad dP_t f_\ell \to dP_t h^c \quad \text{as } \ell \to \infty.$$

By (3.3) we would have

$$\|dP_t h^c\|_\infty \leq \frac{1}{\sqrt{t}}\, \|h^c\|_\infty\,,$$

in contradiction to $\|dP_t h^c\|_\infty = e^{-t/2}\, \|dh\|_\infty = \infty$. □

Remark 3.2 In [5] it is shown that if a stochastic dynamical system of the type (1.6) is strongly 1-complete, and if for each compact set K there is a $\delta > 0$ such that

$$\sup_{x\in K} \mathbb{E}\,\|T_xX_s\|^{1+\delta} < \infty,$$

then $d\,P_tf = P_t^{(1)}df$ holds true for functions $f \in bC^1(M)$. Example 3.1 shows that the strong 1-completeness is necessary and can not be replaced by completeness.

On M_n consider the heat equation for 1-forms

$$\begin{cases} \frac{\partial}{\partial t}\,\alpha = \frac{1}{2}\Delta^{(1)}\alpha \\ \alpha|_{t=0} = df \end{cases} \tag{3.4}$$

where $f \in C^\infty(M_n)$. Take $f \in C^\infty(M_n)$ of compact support with $dP_tf \neq P_t^{(1)}df$. Then

$$\alpha_t^1 := P_t^{(1)}df \quad \text{and} \quad \alpha_t^2 := dP_tf$$

define two different smooth solutions to (3.4). Note that $\|\alpha_t^i\| \in L^2$, $i = 1, 2$.

Corollary 3.3 *On the n-fold cover M_n of the punctured plane ($n \geq 2$) there are infinitely many nontrivial classical solutions to*

$$\begin{cases} \frac{\partial}{\partial t}\,\alpha = \frac{1}{2}\Delta^{(1)}\alpha \\ \alpha|_{t=0} = 0 \end{cases}$$

of the form $\alpha_t = P_t^{(1)}df - dP_tf$ with $f \in C^\infty(M_n)$ of compact support.

References

[1] M. Arnaudon and A. Thalmaier, Stability of stochastic differential equations in manifolds. *Séminaire de Probabilités,* XXXII, 188–214. Lecture Notes in Math. **1686**. Berlin: Springer, 1998.

[2] D. Bakry and M. Ledoux, Levy-Gromov's isoperimetric inequality for an infinite dimensional diffusion generator. *Invent. math.* **123** (1996), 259–281.

[3] I. Chavel, *Eigenvalues in Riemannian geometry.* New York: Academic Press, 1984.

[4] K. D. Elworthy and M. Yor, Conditional expectations for derivatives of certain stochastic flows. *Séminaire de Probabilités*, XXVII, 159–172. Lecture Notes in Math. **1557**. Berlin: Springer, 1993.

[5] X.-M. Li, Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds. *Probab. Theory Relat. Fields* **100** (1994), 485–511.

[6] R. S. Strichartz, Harmonic analysis on constant curvature surfaces with point singularities. *J. Funct. Anal.* **91** (1990), 37–116.

[7] A. Thalmaier, On the differentiation of heat semigroups and Poisson integrals. *Stochastics and Stochastics Reports* **61** (1997), 297–321.

[8] A. Thalmaier and F.-Y. Wang, Gradient estimates for harmonic functions on regular domains in Riemannian manifolds. *J. Funct. Anal.* **155** (1998), 109–124.