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Attractivity of solutions of Riemann–Liouville fractional differential equations

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Abstract. Some new weakly singular integral inequalities are established by a new method, which generalize some results of this type in some previous papers. By these new integral inequalities, we present the attractivity of solutions for Riemann-Liouville fractional differential equations. Finally, several examples are given to illustrate our main results.

Keywords: weakly singular integral inequalities, Riemann–Liouville fractional derivative, fractional differential equations, attractivity.

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1 Introduction

The study of fractional differential equations has been of great interest in the past three decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications in various sciences. In particular, the existence, uniqueness and stability results of fractional differential equations have been studied by many papers and books. In recent years, many researchers have began to investigate the attractivity of solutions of fractional differential equations. For example, Furati and Tatar [4] investigated the asymptotic behavior for solutions of a weighted Cauchy-type nonlinear fractional problem. Kassim, Furati and Tatar [8] studied the asymptotic behavior of solutions for a class of nonlinear fractional differential equations involving two Riemann–Liouville fractional derivatives of different orders. Zhou et al. [13] studied the attractivity of solutions for fractional evolution equations with Riemann–Liouville fractional derivative. Gallegos and Duarte-Mermoud [5] studied the asymptotic behavior of solutions to Riemann–Liouville fractional systems. Tuan et al. [11] presented some results for existence of global solutions and attractivity for multidimensional fractional differential equations involving Riemann–Liouville derivative. Cong, Tuan and Trinh [2] presented some distinct asymptotic properties of solutions to Caputo fractional differential equations.

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In this paper, we first study the following weakly singular integral inequality

$$u(t) \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u^{\mu}(s) ds, \qquad t \in (0, +\infty), \tag{1.1}$$

where $a, b > 0$, $\alpha > 0$, $\delta \geq 0$, $0 < \beta < 1$ and $0 < \mu \leq 1$. We know that weakly singular integral inequalities are well-known tools in the study of the fractional differential equations. The pioneering work of weakly singular integral inequalities was investigated by Henry [7]. In 1981, Henry [7, p. 190] studied the following weakly singular integral inequality

$$u(t) \leq at^{\alpha-1} + b \int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds, \qquad t \in (0, +\infty), \tag{1.2}$$

where α, β, γ are positive with $\beta + \gamma > 1$ and $\alpha + \gamma > 1$. Webb [12] also studied the following weakly singular Gronwall inequality

$$u(t) \leq at^{-\alpha} + b + c \int_0^t (t-s)^{-\beta} s^{-\gamma} u(s) ds, \qquad \text{for a.e. } t \in (0, T], \tag{1.3}$$

where $0 < \alpha, \beta, \gamma < 1$ with $\alpha + \gamma < 1$ and $\beta + \gamma < 1$. Recently, Zhu [14] considered the following inequality

$$u(t) \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u(s) ds, \qquad t \in (0, +\infty), \tag{1.4}$$

where $\alpha > \delta \geq 0$ and $0 < \beta < 1$. Zhu [15] also considered the following weakly singular integral inequality

$$u(t) \leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u^{\mu}(s) ds, \qquad t \in (0, +\infty), \tag{1.5}$$

where $1 > \alpha \geq \delta \geq 0$, $0 < \mu < 1$ and $0 < \beta < 1$. Some results of this type are also proved by Denton and Vatsala [3], Haraux [6], Kong and Ding [9].

Applying weakly singular integral inequality (1.1), we begin to investigate the attractivity of solutions of fractional differential equation

$$\begin{cases} D_{0^+}^{\beta} x(t) = f(t, x(t)), \\ \lim_{t \to 0^+} t^{1-\beta} x(t) = x_0, \end{cases} \tag{1.6}$$

where $\beta \in (0, 1)$ and $t \in (0, +\infty)$. As far as I know, there have been few papers to study the attractivity of fractional differential equation (1.6) by weakly singular integral inequalities. The conclusion and the method of the proof in this paper seem to be new.

The outline of this paper is as follows. In Section 2, we introduce some notations, definitions and theorems needed in our proofs. In Section 3, we obtain some new results concerning weakly singular integral inequalities. In the last Section, we give some sufficient conditions on the attractivity of solutions of fractional differential equation (1.6). Finally, some examples are given to illustrate our main results.

2 Preliminaries

In this section, we introduce some notations, definitions and theorems which will be needed later.

Let $\alpha \in (0, 1)$, we denote $C_{\alpha}(0, +\infty) = \{x(t) : x(t) \in C(0, +\infty) \text{ and } t^{\alpha} x(t) \in C[0, +\infty)\}$. $L^p_{Loc}[0, +\infty)$ $(p \geq 1)$ is the space of all real valued functions which are Lebesgue integrable over every bounded subinterval of $[0, +\infty)$.

Definition 2.1. [10, p. 33] Let $\beta \in (0,1)$, The operator $I^{\beta}_{0^+}$, defined on $L^1[0,T]$ by

$$I^{\beta}_{0^+}\varphi(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{\varphi(s)}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t \in [0,T]$$

is called the Riemann–Liouville fractional integral operator of order β.

Definition 2.2. [10, p. 35] Let $\beta \in (0,1)$, The operator $D^{\beta}_{0^+}$, defined by

$$D^{\beta}_{0^+}\varphi(t) = \frac{d}{dt}I^{1-\beta}_{0^+}\varphi(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{\varphi(s)}{(t-s)^{\beta}}ds, \qquad \text{a.e. } t \in [0,T],$$

where $I^{1-\beta}_{0^+}\varphi(t)$ is an absolutely continuous function, is called the Riemann–Liouville fractional differential operator of order β.

Definition 2.3. The solution $x(t) \in C_{1-\beta}(0,+\infty)$ of fractional differential equation (1.6) is said to be attractive if $\lim_{t\to+\infty} x(t) = 0$.

Using the Hölder inequality, Zhu [15] obtained the following inequality.

Lemma 2.4. *Let $0 < \beta < 1$. Suppose that $s^{1-\beta}\rho(s) \in L^p[0,1]$, where $p > \frac{1}{\beta}$. Then*

$$\left|\int_0^t (\frac{t}{t-s})^{1-\beta}\rho(s)ds\right| \le \frac{2^{\frac{1}{q}}t^{\beta-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^t s^{p(1-\beta)}|\rho(s)|^p ds\right)^{\frac{1}{p}} \tag{2.1}$$

for $t \in [0,1]$, where $q = \frac{p}{p-1}$.

Recently, Zhu [15, Corollary 4.5] obtained the following result which is very useful for the study of the main purpose of this paper.

Theorem 2.5. *Let $0 < \beta < 1$ and $0 < \mu \le 1$. Suppose $f : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist nonnegative functions $l(t)$ and $k(t)$ such that*

$$|f(t,x)| \le l(t)|x|^{\mu} + k(t)$$

for all $(t,x) \in (0,+\infty)\times\mathbb{R}$, where $t^{(1-\mu)(1-\beta)}l(t) \in C(0,+\infty)\bigcap L^p_{Loc}[0,+\infty)$ and $t^{1-\beta}k(t) \in C(0,+\infty)\bigcap L^p_{Loc}[0,+\infty)$, $p > \frac{1}{\beta}$. Then the fractional differential equation (1.6) has at least one global solution in $C_{1-\beta}(0,+\infty)$.

3 Weakly singular integral inequalities

In this section, we are now to prove some results concerning weakly singular integral inequalities, which can be used to study the attractivity of solutions for fractional differential equation (1.6). We first study the weakly singular integral inequality (1.1) for the case $\mu = 1$.

Theorem 3.1. *Let $a, b > 0$, $\alpha > 0$, $\delta \ge 0$ and $0 < \beta < 1$. Let $l(t)$ be a nonnegative, continuous function on $(0,+\infty)$ and $t^{\alpha_1}l(t) \in L^p_{Loc}[0,+\infty)$, where $\alpha_1 = \min\{1-\alpha-\beta, -\delta\}$ and $p > \frac{1}{\beta}$. Let $t^{\alpha}u(t)$ be a continuous, nonnegative function on $[0,+\infty)$ with*

$$u(t) \le at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1}l(s)u(s)ds, \qquad t \in (0,\infty). \tag{3.1}$$

Then

$$u(t) \leq at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t) \exp\left(\int_0^t \frac{L(s)}{p} ds\right), \qquad t \in (0, +\infty), \tag{3.2}$$

where $A(t) = \int_0^t 2^{p-1} a^p s^{p(1-\alpha-\beta)} l^p(s) ds$, $L(t) = \frac{4^{p-1} b^p t^{p(\beta-\delta)-1} l^p(t)}{(q\beta-q+1)^{\frac{p}{q}}}$ *and* $q = \frac{p}{p-1}$.

Proof. Applying Lemma 2.4, we have

$$\begin{aligned} u(t) &\leq at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u(s) ds \\ &= at^{-\alpha} + bt^{\beta-\delta-1} \int_0^t (\frac{t}{t-s})^{1-\beta} l(s) u(s) ds \\ &\leq at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^p(s) ds\right)^{\frac{1}{p}}. \end{aligned} \tag{3.3}$$

From (3.3), we obtain

$$t^{1-\beta} l(t) u(t) \leq at^{1-\alpha-\beta} l(t) + \frac{2^{\frac{1}{q}} bt^{\beta-\delta-\frac{1}{p}} l(t)}{(q\beta - q + 1)^{\frac{1}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^p(s) ds\right)^{\frac{1}{p}}. \tag{3.4}$$

Since $t^{\alpha_1} l(t) \in L^p_{Loc}[0, +\infty)$, then $t^{p(1-\alpha-\beta)} l^p(t) \in L^1_{Loc}[0, +\infty)$ and $t^{p(\beta-\delta)-1} l^p(t) \in L^1_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$. Therefore we get

$$\begin{aligned} \int_0^t s^{p(1-\beta)} l^p(s) u^p(s) ds &\leq \int_0^t \left[as^{1-\alpha-\beta} l(s) + \frac{2^{\frac{1}{q}} bs^{\beta-\delta-\frac{1}{p}} l(s)}{(q\beta - q + 1)^{\frac{1}{q}}} \left(\int_0^s \tau^{p(1-\beta)} l^p(\tau) u^p(\tau) d\tau\right)^{\frac{1}{p}}\right]^p ds \\ &\leq \int_0^t 2^{p-1} a^p s^{p(1-\alpha-\beta)} l^p(s) ds \\ &+ \int_0^t \frac{4^{p-1} b^p s^{p(\beta-\delta)-1} l^p(s)}{(q\beta - q + 1)^{\frac{p}{q}}} \int_0^s \tau^{p(1-\beta)} l^p(\tau) u^p(\tau) d\tau ds. \end{aligned} \tag{3.5}$$

Let $W(t) = \int_0^t s^{p(1-\beta)} l^p(s) u^p(s) ds$, then we get

$$W(t) \leq A(t) + \int_0^t L(s) W(s) ds. \tag{3.6}$$

In (3.6), we know that $A(t)$ is a nondecreasing function on $[0, +\infty)$ and using the Gronwall integral inequality [1, Corollary 1.2], we obtain

$$W(t) \leq A(t) \exp\left(\int_0^t L(s) ds\right). \tag{3.7}$$

From (3.3) and (3.7), we get

$$u(t) \leq at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t) \exp\left(\int_0^t \frac{L(s)}{p} ds\right). \tag{3.8}$$

Thus, we complete the proof. □

As a consequence of Theorem 3.1, we can immediately obtain the following result for the case $\alpha = 1-\beta$ and $\delta = 0$.

Theorem 3.2. *Let $a, b > 0$ and $0 < \beta < 1$. Let $l(t)$ be a nonnegative and continuous function on $(0, +\infty)$ with $l(t) \in L^p_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$, and $t^{1-\beta}u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$u(t) \le at^{\beta-1} + b\int_0^t (t-s)^{\beta-1} l(s)u(s)ds, \qquad t \in (0,\infty). \tag{3.9}$$

Then

$$u(t) \le at^{\beta-1} + \frac{2^{\frac{1}{q}} bt^{2\beta-1-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t)\exp\left(\int_0^t \frac{L(s)}{p}ds\right), \qquad t \in (0,+\infty), \tag{3.10}$$

where $A(t) = \int_0^t 2^{p-1}a^p l^p(s)ds$, $L(t) = \frac{4^{p-1}b^p t^{p\beta-1} l^p(t)}{(q\beta-q+1)^{\frac{p}{q}}}$ and $q = \frac{p}{p-1}$.

Example 3.3. Suppose that $t^{\frac{1}{4}}u(t)$ is a continuous, nonnegative function on $[0, +\infty)$ and $u(t)$ satisfies the following inequality

$$u(t) \le t^{-\frac{1}{4}} + t^{-\frac{1}{3}}\int_0^t (t-s)^{-\frac{1}{3}} \frac{u(s)}{1+s}ds, \qquad t \in (0,+\infty). \tag{3.11}$$

By Theorem 3.1, let $p = 2$, then we get

$$u(t) \le t^{-\frac{1}{4}} + 6^{\frac{1}{2}}t^{-\frac{1}{2}}\left(\int_0^t \frac{2s^{\frac{1}{6}}}{(1+s)^2}ds\right)^{\frac{1}{2}} \exp\left(\int_0^t \frac{6s^{\frac{-1}{3}}}{(1+s)^2}ds\right), \qquad t \in (0,+\infty). \tag{3.12}$$

We know

$$\int_0^t \frac{s^{\frac{1}{6}}}{(1+s)^2}ds \le \int_0^{+\infty} \frac{s^{\frac{1}{6}}}{(1+s)^2}ds = \mathrm{B}(7/6, 5/6) = \frac{\pi}{3}$$

and

$$\int_0^t \frac{s^{\frac{-1}{3}}}{(1+s)^2}ds \le \int_0^{+\infty} \frac{s^{\frac{-1}{3}}}{(1+s)^2}ds = \mathrm{B}(2/3, 4/3) = \frac{2\sqrt{3}\pi}{9},$$

where $\mathrm{B}(p,q) = \int_0^1 (1-s)^{p-1}s^{q-1}ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ $(p, q > 0)$ is the Beta function, and $\Gamma(p) = \int_0^{+\infty} s^{p-1}\exp(-s)ds$ $(p > 0)$ is the Gamma function.

Then we obtain

$$u(t) \le t^{-\frac{1}{4}} + 2\sqrt{\pi}\exp\left(\frac{4\sqrt{3}\pi}{3}\right)t^{-\frac{1}{2}}, \qquad t \in (0,+\infty), \tag{3.13}$$

and $u(t) \to 0$ as $t \to +\infty$.

Now, we investigate the weakly singular integral inequality (1.1) when $0 < \mu < 1$.

Theorem 3.4. *Let $a, b > 0$, $\alpha > 0$, $\delta \ge 0$, $0 < \beta < 1$ and $0 < \mu < 1$. Let $l(t)$ be a nonnegative, continuous function on $(0, +\infty)$ with $t^{\alpha_2}l(t) \in L^p_{Loc}[0, +\infty)$, where $\alpha_2 = \min\{1-\alpha\mu-\beta, (\beta-\delta-1)\mu+1-\beta\}$ and $p > \frac{1}{\beta}$. Let $t^{\alpha}u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$u(t) \le at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1} l(s)u^{\mu}(s)ds, \qquad t \in (0,\infty). \tag{3.14}$$

Then

$$u(t) \leq at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s)ds \right)^{\frac{1}{p(1-\mu)}}, \qquad t \in (0, +\infty), \tag{3.15}$$

where $A(t) = \int_0^t 2^{p-1} a^{p\mu} s^{p(1-\alpha\mu-\beta)} l^p(s)ds$, $L(t) = \frac{4^{p-1} b^{p\mu} t^{(2p\beta-p\delta-p-1)\mu+p-p\beta} l^p(t)}{(q\beta-q+1)^{\frac{p\mu}{q}}}$ *and* $q = \frac{p}{p-1}$.

Proof. From the inequality (3.14), using the same procedure as in the proof of the inequality (3.3), we have

$$u(t) \leq at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s)ds \right)^{\frac{1}{p}}. \tag{3.16}$$

From (3.16), we know

$$u^\mu(t) \leq a^\mu t^{-\alpha\mu} + \frac{2^{\frac{1}{q}} b^\mu t^{(2\beta-\delta-1-\frac{1}{p})\mu}}{(q\beta - q + 1)^{\frac{\mu}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s)ds \right)^{\frac{\mu}{p}} \tag{3.17}$$

and

$$t^{1-\beta} l(t) u^\mu(t) \leq a^\mu t^{-\alpha\mu+1-\beta} l(t) + \frac{2^{\frac{1}{q}} b^\mu t^{(2\beta-\delta-1-\frac{1}{p})\mu+1-\beta} l(t)}{(q\beta - q + 1)^{\frac{\mu}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s)ds \right)^{\frac{\mu}{p}}. \tag{3.18}$$

Since $t^{\alpha_2} l(t) \in L^p_{Loc}[0, +\infty)$, then $t^{p(1-\alpha\mu-\beta)} l^p(t) \in L^1_{Loc}[0, +\infty)$ and $t^{(2p\beta-p\delta-p-1)\mu+p-p\beta} l^p(t) \in L^1_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$. Then we obtain

$$\begin{aligned} &\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s)ds \\ &\quad \leq \int_0^t 2^{p-1} a^{p\mu} s^{p(1-\alpha\mu-\beta)} l^p(s)ds \\ &\quad\quad + \int_0^t \frac{4^{p-1} b^{p\mu} s^{(2p\beta-p\delta-p-1)\mu+p-p\beta} l^p(s)}{(q\beta - q + 1)^{\frac{p\mu}{q}}} \left(\int_0^s \tau^{p(1-\beta)} l^p(\tau) u^{p\mu}(\tau)d\tau \right)^\mu ds. \end{aligned} \tag{3.19}$$

Let $W(t) = \int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s)ds$, then we get

$$W(t) \leq A(t) + \int_0^t L(s) W^\mu(s)ds. \tag{3.20}$$

Using the Bihari integral inequality [1, Corollary 5.3], we obtain

$$W(t) \leq \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s)ds \right)^{\frac{1}{1-\mu}}. \tag{3.21}$$

From (3.16) and (3.21), we get

$$u(t) \leq at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s)ds \right)^{\frac{1}{p(1-\mu)}}. \tag{3.22}$$

Thus, we complete the proof. □

As a consequence of Theorem 3.4, we can obtain the following result when $\alpha = 1 - \beta$ and $\delta = 0$.

Theorem 3.5. *Let $a > 0$, $b > 0$, $0 < \beta < 1$ and $0 < \mu < 1$. Let $l(t)$ be a nonnegative and continuous function on $(0, +\infty)$ with $t^{(1-\mu)(1-\beta)} l(t) \in L^p_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$, and $t^{1-\beta} u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$u(t) \leq at^{\beta-1} + b \int_0^t (t-s)^{\beta-1} l(s) u^{\mu}(s) ds, \qquad t \in (0, \infty). \tag{3.23}$$

Then

$$u(t) \leq at^{\beta-1} + \frac{2^{\frac{1}{q}} b t^{2\beta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s) ds \right)^{\frac{1}{p(1-\mu)}}, \qquad t \in (0, +\infty), \tag{3.24}$$

where $A(t) = \int_0^t 2^{p-1} a^{p\mu} s^{p(1-\mu)(1-\beta)} l^p(s) ds$, $L(t) = \frac{4^{p-1} b^{p\mu} t^{(2p\beta - p - 1)\mu + p - p\beta} l^p(t)}{(q\beta - q + 1)^{\frac{p\mu}{q}}}$ and $q = \frac{p}{p-1}$.

Example 3.6. Suppose that $t^{\frac{1}{3}} u(t)$ is a continuous, nonnegative function on $[0, +\infty)$ and $u(t)$ satisfies the inequality

$$u(t) \leq t^{-\frac{1}{3}} + t^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{3}} s^{\frac{-1}{2}} u^{\frac{1}{3}}(s) ds, \qquad t \in (0, +\infty). \tag{3.25}$$

Let $p = 2$, using Theorem 3.4, we get

$$\begin{aligned} u(t) &\leq t^{\frac{-1}{3}} + 6^{\frac{1}{2}} t^{\frac{-2}{3}} \left((\frac{9}{2})^{\frac{2}{3}} t^{\frac{8}{27}} + 12 \cdot 3^{\frac{1}{3}} t^{\frac{2}{9}} \right)^{\frac{3}{4}} \\ &\leq t^{\frac{-1}{3}} + \sqrt{27} t^{\frac{-4}{9}} + 12\sqrt{3} t^{\frac{-1}{2}}, \qquad t \in (0, +\infty) \end{aligned} \tag{3.26}$$

and $u(t) \to 0$ as $t \to +\infty$.

In [15, Theorem 3.4], Zhu studied the weakly singular integral inequality (3.14) when $t^{(1-\mu)\alpha - \delta} l(t) \in L^p_{Loc}[0, +\infty)$, where $1 > \alpha \geq \delta \geq 0$ and $p > \max\{\frac{1}{\beta}, \frac{1}{1-\alpha+\delta}\}$. In fact, the conclusion is also correct when $1 > \alpha > 0$ and $1 > \delta \geq 0$. In the inequality (3.25), since $t^{\frac{-7}{9}} \notin L^p[0, +\infty)$ when $p > \frac{3}{2}$, then Theorem 3.4 in [15] cannot be used to solve the inequality (3.25).

Remark 3.7. In Theorem 3.4, since $0 < \mu < 1$, then $\alpha_1 < \alpha_2$. If $l(t)$ is a nonnegative and continuous function on $(0, +\infty)$ satisfying $t^{\alpha_1} l(t) \in L^p_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$, then we can get $t^{\alpha_2} l(t) \in L^p_{Loc}[0, +\infty)$. Therefore, the hypothesis of function $l(t)$ in Theorem 3.4 is weaker than that imposed in Theorem 3.1.

Zhu [14, Theorem 3.4] obtained some results for the inequality (3.1) when $\alpha > \delta \geq 0$. Zhu [15, Theorem 3.3] studied the inequality (3.1) when $1 > \alpha \geq \delta \geq 0$. In Theorem 3.1, we study the inequality (3.1) when $\alpha > 0$ and $\delta \geq 0$. Therefore, our result generalizes some results in [14, 15].

Denton and Vatsala [3, Theorem 2.8] studied the inequality (3.1) for the special case $\alpha = 1 - \beta$ and $\delta = 0$. Henry [7, Exercise 3, p. 190] discussed the inequality (3.1) for the case $\delta = 0$ and $l(t) = t^{\gamma - 1}$. Some similar results of the inequality (3.1) were proved in Haraux [6, Lemma 10, p. 112], Kong and Ding [9, Theorem 2.7], Webb [13, Theorem 3.9] and Zhu [14, Theorem 3.6]. As far as I know, there have been few papers to study the inequality (1.1), and the methods of proof in Theorem 3.1 and Theorem 3.4 seem to be new.

4 Attractivity of fractional differential equations

In this section, we present the main results of this paper. We first study the attractivity of solutions of fractional differential equation (1.6) when $|f(t,x)| \leq l(t)|x|$.

Theorem 4.1. *Let $0 < \beta < 1$ and $\lambda > \beta$. Let $l(t)$ be a nonnegative function with $l(t) \in C(0,+\infty) \bigcap L^p_{Loc}[0,+\infty)$, where $p > 1$ and $\beta > \frac{1}{p} > 2\beta - 1$, and there exists a nonnegative constant K such that*

$$t^\lambda l(t) \leq K \tag{4.1}$$

for all $t \in [1,+\infty)$. Suppose $f : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and

$$|f(t,x)| \leq l(t)|x|$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. Then the solution of fractional differential equation (1.6) is attractive.

Proof. Using Theorem 2.5, we know that the fractional differential equation (1.6) has at least one global solution $x(t) \in C_{1-\beta}(0,+\infty)$ and $x(t)$ also satisfies the following Volterra integral equation

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds, \qquad t \in (0,+\infty). \tag{4.2}$$

Then we have

$$|x(t)| \leq |x_0| t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)|x(s)|ds, \qquad t \in (0,+\infty). \tag{4.3}$$

Then by Theorem 3.2, we obtain

$$|x(t)| \leq |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}} t^{2\beta-1-\frac{1}{p}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t) \exp\left(\int_0^t \frac{L(s)}{p} ds \right), \qquad t \in (0,+\infty), \tag{4.4}$$

where $A(t) = \int_0^t 2^{p-1}|x_0|^p l^p(s)ds$, $L(t) = \frac{4^{p-1} t^{p\beta-1} l^p(t)}{\Gamma^p(\beta)(q\beta-q+1)^{\frac{p}{q}}}$ and $q = \frac{p}{p-1}$.

From (4.1) and $\lambda > \beta > \frac{1}{p}$, we know

$$l^p(t) \leq K^p t^{-p\lambda}, \qquad t \in [1,+\infty)$$

and $\int_1^{+\infty} K^p s^{-p\lambda} ds$ is convergent. Then we obtain that $\int_1^{+\infty} 2^{p-1}|x_0|^p l^p(s)ds$ is also convergent and there exists a nonnegative constant M_1 such that $A(t) \leq M_1$ for all $t \in (0,+\infty)$. Since $\lambda > \beta$ and

$$t^{p\beta-1} l^p(t) \leq K^p t^{p\beta-p\lambda-1}, \qquad t \in [1,+\infty),$$

we know that $\int_1^{+\infty} K^p s^{p\beta-p\lambda-1} ds$ is convergent. Then we obtain that $\int_1^{+\infty} s^{p\beta-1} l^p(s)ds$ and $\int_1^{+\infty} L(s)ds$ are also convergent, and there exists a nonnegative constant M_2 such that $\int_0^t \frac{L(s)}{p} ds \leq M_2$ for all $t \in (0,+\infty)$.

Therefore, from (4.4) and $\beta > \frac{1}{p} > 2\beta - 1$, we get

$$|x(t)| \leq |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}} t^{2\beta-1-\frac{1}{p}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} M_1^{\frac{1}{p}} \exp(M_2), \qquad t \in (0,+\infty), \tag{4.5}$$

and

$$\lim_{t\to+\infty} |x(t)| = 0. \tag{4.6}$$

Thus, we complete the proof. □

We now discuss the case when $|f(t,x)| \leq l(t)|x|^{\mu}$ for all $(t,x) \in (0,+\infty) \times \mathbb{R}$, where $0 < \mu < 1$.

Theorem 4.2. *Let $0 < \mu < 1$, $0 < \beta < 1$ and $\lambda > \beta$. Let $l(t)$ be a nonnegative function with $t^{(1-\mu)(1-\beta)}l(t) \in C(0,+\infty) \bigcap L^{p}_{Loc}[0,+\infty)$, where $p > 1$ with $\beta > \frac{1}{p} > 2\beta - 1$, and there exists a nonnegative constant K such that*

$$t^{\lambda} l(t) \leq K \tag{4.7}$$

for all $t \in [1,+\infty)$. Suppose $f : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$|f(t,x)| \leq l(t)|x|^{\mu}$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. Then the solution of fractional differential equation (1.6) is attractive.

Proof. Using the same procedure as in the proof of Theorem 4.1, we know that the global solution $x(t) \in C_{1-\beta}(0,+\infty)$ of equation (1.6) satisfies the following Volterra integral equation

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds, \qquad t \in (0,+\infty), \tag{4.8}$$

and

$$|x(t)| \leq |x_0| t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)|x(s)|^{\mu} ds, \qquad t \in (0,+\infty). \tag{4.9}$$

Then by Theorem 3.5, for $t \in (0,+\infty)$, we obtain

$$\begin{aligned} |x(t)| &\leq |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}} t^{2\beta-1-\frac{1}{p}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu)\int_0^t L(s)ds \right)^{\frac{1}{p(1-\mu)}} \\ &= |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} \left(\left(\frac{A(t)}{t^{p+1-2p\beta}} \right)^{1-\mu} + \frac{(1-\mu)\int_0^t L(s)ds}{t^{(p+1-2p\beta)(1-\mu)}} \right)^{\frac{1}{p(1-\mu)}}, \end{aligned} \tag{4.10}$$

where $A(t) = \int_0^t 2^{p-1}|x_0|^{p\mu} s^{p(1-\mu)(1-\beta)} l^p(s)ds$, $L(t) = \frac{4^{p-1} t^{(2p\beta-p-1)\mu+p-p\beta} l^p(t)}{\Gamma^{p\mu}(\beta)(q\beta-q+1)^{\frac{p\mu}{q}}}$ and $q = \frac{p}{p-1}$.

Since $1 > \beta > \frac{1}{p} > 2\beta - 1$ and $\lambda > \beta$, using L'Hôspital's rule, we get

$$\begin{aligned} \lim_{t\to+\infty} \frac{\int_0^t s^{p(1-\mu)(1-\beta)} l^p(s)ds}{t^{p+1-2p\beta}} &= \lim_{t\to+\infty} \frac{t^{p(1-\mu)(1-\beta)} l^p(t)}{(p+1-2p\beta)t^{p-2p\beta}} \\ &\leq \lim_{t\to+\infty} \frac{K^p t^{p(1-\mu)(1-\beta)-p\lambda}}{(p+1-2p\beta)t^{p-2p\beta}} \\ &= \lim_{t\to+\infty} \frac{K^p t^{p\mu(\beta-1)+p(\beta-\lambda)}}{(p+1-2p\beta)} \\ &= 0. \end{aligned} \tag{4.11}$$

In (4.11), if $\int_0^t s^{p(1-\mu)(1-\beta)} l^p(s)ds$ is a bounded function for $t \in [0,+\infty)$, we can also obtain this conclusion.

Since $\lambda > \beta$, using L'Hôspital's rule, we obtain

$$\begin{aligned}\lim_{t\to+\infty}\frac{\int_0^t s^{(2p\beta-p-1)\mu+p-p\beta}l^p(s)ds}{t^{(p+1-2p\beta)(1-\mu)}} &= \lim_{t\to+\infty}\frac{t^{(2p\beta-p-1)\mu+p-p\beta}l^p(t)}{(p+1-2p\beta)(1-\mu)t^{(p+1-2p\beta)(1-\mu)-1}} \\ &\le \lim_{t\to+\infty}\frac{K^p t^{(2p\beta-p-1)\mu+p-p(\beta+\lambda)}}{(p+1-2p\beta)(1-\mu)t^{(p+1-2p\beta)(1-\mu)-1}} \\ &= \lim_{t\to+\infty}\frac{K^p t^{p(\beta-\lambda)}}{(p+1-2p\beta)(1-\mu)} \\ &= 0.\end{aligned} \tag{4.12}$$

In (4.12), if $\int_0^t s^{(2p\beta-p-1)\mu+p-p\beta}l^p(s)ds$ is a bounded function for $t \in [0,+\infty)$, we can also obtain this conclusion.

In (4.10), using (4.11) and (4.12), we obtain

$$\lim_{t\to+\infty}|x(t)| = 0. \tag{4.13}$$

Thus, we complete the proof. □

Example 4.3. Consider the following Riemann–Liouville fractional differential equation

$$\begin{cases} D_{0^+}^{\frac{2}{3}}x(t) = \frac{x(t)}{\sqrt{t}(1+\sqrt{t})}, \\ \lim_{t\to 0^+} t^{\frac{1}{3}}x(t) = 1. \end{cases} \tag{4.14}$$

Let $\lambda = 1$ and $\frac{3}{2} < p < 2$, using Theorem 4.1 and the inequality (4.4), we know that the solution $x(t) \in C_{\frac{1}{3}}(0,+\infty)$ of the equation (4.14) is attractive, and

$$x(t) \le t^{\frac{-1}{3}} + Mt^{\frac{p-3}{3p}}, \tag{4.15}$$

where $M = M(p)$ is a nonnegative constant and $\lim_{p\to\frac{3}{2}^+} M(p) = +\infty$.

Example 4.4. Consider the following Riemann–Liouville fractional differential equation

$$\begin{cases} D_{0^+}^{\frac{1}{2}}x(t) = t^{\frac{-2}{3}}x^{\frac{1}{2}}(t), \\ \lim_{t\to 0^+} t^{\frac{1}{2}}x(t) = 1. \end{cases} \tag{4.16}$$

Let $\lambda = \frac{2}{3}$ and $2 < p < \frac{12}{5}$, using Theorem 4.2 and the inequality (4.10), we get that the solution $x(t) \in C_{\frac{1}{2}}(0,+\infty)$ of the equation (4.16) is attractive, and

$$x(t) \le t^{\frac{-1}{2}} + M_1t^{\frac{-1}{3}} + M_2t^{\frac{-5}{12}}, \tag{4.17}$$

where $M_1 = M_1(p)$ and $M_2 = M_2(p)$ are nonnegative constants.

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