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NEW RESULTS FOR TIME REVERSED SYMPLECTIC DYNAMIC SYSTEMS AND QUADRATIC FUNCTIONALS

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Abstract. In this paper, we examine time scale symplectic (or Hamiltonian) systems and the associated quadratic functionals which contain a forward shift in the time variable. Such systems and functionals have a close connection to Jacobi systems for calculus of variations and optimal control problems on time scales. Our results, among which we consider the Reid roundabout theorem, generalize the corresponding classical theory for time reversed discrete symplectic systems, as well as they complete the recently developed theory of time scale symplectic systems.

1. Introduction

Time scale symplectic (or Hamiltonian) systems constitute a basis for the study of Jacobi equations arising in the optimal control theory [4, 13, 16] as well as for generalizations of Sturm–Liouville differential and difference equations [1, 14]. In such optimization problems, the second variation is a quadratic functional whose definiteness indicates the potential optimality of a given candidate, along which the functional is evaluated. Traditionally, two types of optimal control problems are studied in the literature, see e.g. [11,16]. The first problem has the form

$$\text{minimize} \quad \mathcal{F}(x,u) := K(x(a),x(b)) + \int_a^b L(t,x^\sigma(t),u(t))\,\Delta t \tag{C}$$

subject to $x \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$, $u \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$, and

$$x^\Delta(t) = f(t,x^\sigma(t),u(t)), \quad t \in [a,\rho(b)]_{\mathbb{T}},$$
$$\varphi(x(a),x(b)) = 0,$$

while the second one is

$$\text{minimize} \quad \underline{\mathcal{F}}(x,u) := \underline{K}(x(a),x(b)) + \int_a^b \underline{L}(t,x(t),u(t))\,\Delta t \tag{$\underline{\mathrm{C}}$}$$

subject to $x \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$, $u \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$, and

$$x^\Delta(t) = \underline{f}(t,x(t),u(t)), \quad t \in [a,\rho(b)]_{\mathbb{T}},$$
$$\underline{\varphi}(x(a),x(b)) = 0.$$

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These two problems are formulated in the framework of dynamic equations on time scales. We refer to [4, 5] for basic notation and terminology in this recent theory.

Problems (C) and ($\underline{\text{C}}$) differ in the presence or absence of the forward shift $\sigma(t)$ in the state variable $x^\sigma(t) := x(\sigma(t))$ or $x(t)$, respectively, appearing both in the objective functional and the equation of motion. Although the problem (C) has been much more preferred in the literature, in [11, Section 3] it was proven that it is equivalent to problem ($\underline{\text{C}}$) via a transformation involving the implicit function theorem. In addition, in [16] it was shown that the second order optimality conditions for both problems (C) and ($\underline{\text{C}}$) lead to a linear time scale dynamic system

$$x^\Delta = \mathbb{A}(t)\, x + \mathbb{B}(t)\, u, \quad u^\Delta = \mathbb{C}(t)\, x + \mathbb{D}(t)\, u, \quad t \in [a, \rho(b)]_{\mathbb{T}}. \tag{$\mathbb{S}$}$$

System ($\mathbb{S}$) is called a *time scale symplectic system* (or a Hamiltonian system), since its coefficients satisfy the identity

$$\mathbb{S}^T(t)\, J + \mathcal{J}\, \mathbb{S}(t) + \mu(t)\, \mathbb{S}^T(t)\, \mathcal{J}\, \mathbb{S}(t) = 0 \quad \text{for all } t \in [a, \rho(b)]_{\mathbb{T}}, \tag{1.1}$$

where

$$\mathbb{S}(t) := \begin{pmatrix} \mathbb{A}(t) & \mathbb{B}(t) \\ \mathbb{C}(t) & \mathbb{D}(t) \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.2}$$

Condition (1.1) yields that the fundamental matrix of system ($\mathbb{S}$) is symplectic, which is the defining property for the continuous time linear Hamiltonian systems and the discrete time symplectic systems. It is known that system ($\mathbb{S}$) has the equivalent form

$$x^\Delta = -\mathbb{D}^T(t)\, x^\sigma + \mathbb{B}^T(t)\, u^\sigma, \quad u^\Delta = \mathbb{C}^T(t)\, x^\sigma - \mathbb{A}^T(t)\, u^\sigma, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{1.3}$$

see [6, 8]. In the discrete time theory, system (1.3) is called a *time-reversed symplectic system*, see the pioneer work [3].

In this paper we consider a general time-reversed symplectic dynamic system

$$x^\Delta = \underline{\mathbb{A}}(t)\, x^\sigma + \underline{\mathbb{B}}(t)\, u^\sigma, \quad u^\Delta = \underline{\mathbb{C}}(t)\, x^\sigma + \underline{\mathbb{D}}(t)\, u^\sigma, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{$\underline{\mathbb{S}}$}$$

whose coefficients satisfy the identity

$$\underline{\mathbb{S}}^T(t)\, \mathcal{J} + \mathcal{J}\, \underline{\mathbb{S}}(t) - \mu(t)\, \underline{\mathbb{S}}^T(t)\, \mathcal{J}\, \underline{\mathbb{S}}(t) = 0 \quad \text{for all } t \in [a, \rho(b)]_{\mathbb{T}}, \tag{1.4}$$

where the matrix $\mathcal{J}$ is defined in (1.2) and

$$\underline{\mathbb{S}}(t) := \begin{pmatrix} \underline{\mathbb{A}}(t) & \underline{\mathbb{B}}(t) \\ \underline{\mathbb{C}}(t) & \underline{\mathbb{D}}(t) \end{pmatrix}. \tag{1.5}$$

In the analogy with the discrete time theory we show that, rather than system ($\mathbb{S}$), it is the system ($\underline{\mathbb{S}}$) in terms of which the natural second order optimality conditions for problem ($\underline{\text{C}}$) should be formulated. This is nicely demonstrated by a series of new results which we derive for the quadratic forms (suppressing the argument t)

$$\left.\begin{aligned} \mathbb{Q}(\eta, q) := \eta^T \mathbb{C}^T (I + \mu\mathbb{A})\, \eta + 2\, \mu\, \eta^T \mathbb{C}^T \mathbb{B}\, q + q^T (I + \mu\mathbb{D}^T)\, \mathbb{B}\, q, \\ \eta \in \mathrm{C}^1_{\mathrm{prd}}[a, b]_{\mathbb{T}},\ q \in \mathrm{C}_{\mathrm{prd}}[a, \rho(b)]_{\mathbb{T}},\ \text{with} \\ \eta^\Delta(t) = \mathbb{A}(t)\, \eta(t) + \mathbb{B}(t)\, q(t),\ t \in [a, \rho(b)]_{\mathbb{T}}, \end{aligned}\right\} \tag{1.6}$$

and

$$\left.\begin{aligned} &\underline{\mathbb{Q}}(\eta,q) := \eta^T\underline{\mathbb{C}}^T(I-\mu\underline{\mathbb{A}})\,\eta - 2\,\mu\,\eta^T\underline{\mathbb{C}}^T\underline{\mathbb{B}}\,q + q^T(I-\mu\underline{\mathbb{D}}^T)\,\underline{\mathbb{B}}\,q, \\ &\qquad \eta\in \mathrm{C}^1_{\mathrm{prd}}[a,b]_{\mathbb{T}},\ q\in \mathrm{C}_{\mathrm{prd}}[a,\rho(b)]_{\mathbb{T}},\ \text{with} \\ &\qquad \eta^\Delta(t) = \underline{\mathbb{A}}(t)\,\eta^\sigma(t) + \underline{\mathbb{B}}(t)\,q^\sigma(t),\ t\in[a,\rho(b)]_{\mathbb{T}} \end{aligned}\right\} \tag{1.7}$$

corresponding to systems ($\mathbb{S}$) and ($\underline{\mathbb{S}}$), respectively. This way we complete and clarify the relationship between various quadratic functionals considered in [16, Section 4]. We establish a generalization of the discrete Reid roundabout theorem in [3, Theorem 1] to the time-reversed symplectic system ($\underline{\mathbb{S}}$) on arbitrary time scales. Such a result can be regarded as an analog of the corresponding Reid roundabout theorems for nabla time scale symplectic systems in [12, Theorem 8.1] and system ($\mathbb{S}$) in [9, Theorem 6.1]. As a consequence we also prove the exact relation between the time scale quadratic functionals involving $\mathbb{Q}$ and $\underline{\mathbb{Q}}$.

2. Jacobi, Hamiltonian, and symplectic systems for problem ($\underline{\mathrm{C}}$)

In this section we motivate the time scale symplectic system ($\underline{\mathbb{S}}$) and the quadratic form $\underline{\mathbb{Q}}$ through their origin in the variational theory over time scales. In [16, Section 4] and [13, Section 4.2], it is shown that the Jacobi system for the nonlinear optimal control problem ($\underline{\mathrm{C}}$) has the form

$$\left.\begin{aligned} \eta^\Delta = \underline{\mathcal{A}}(t)\,\eta + \underline{\mathcal{B}}(t)\,v, \quad q^\Delta = -\underline{\mathcal{A}}^T(t)\,q^\sigma + \underline{P}(t)\,\eta + \underline{Q}(t)\,v, \\ -\underline{\mathcal{B}}^T(t)\,q^\sigma + \underline{Q}^T(t)\,\eta + \underline{R}(t)\,v = 0, \end{aligned}\right\}\quad t\in[a,\rho(b)]_{\mathbb{T}}. \tag{$\underline{\mathrm{J}}$}$$

Here the coefficient matrices are determined by the data of problem ($\underline{\mathrm{C}}$). We recall that η and v are the variations of the state x and control u from problem ($\underline{\mathrm{C}}$) and that q is the momentum variable. Throughout the paper we are given the dimensions $m,n\in\mathbb{N}$ with $m\le n$.

Notation 2.1 (Jacobi system ($\underline{\mathrm{J}}$))**.** The $\mathrm{C}_{\mathrm{prd}}$ matrix functions $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{P}$, $\underline{Q}$, $\underline{R}$ on $[a,\rho(b)]_{\mathbb{T}}$, the $\mathrm{C}^1_{\mathrm{prd}}$ vectors η, q on $[a,b]_{\mathbb{T}}$, and the $\mathrm{C}_{\mathrm{prd}}$ vector v on $[a,\rho(b)]_{\mathbb{T}}$ have the following dimensions: $\underline{\mathcal{A}},\underline{P}\in\mathbb{R}^{n\times n}$, $\underline{\mathcal{B}},\underline{Q}\in\mathbb{R}^{n\times m}$, $\underline{R}\in\mathbb{R}^{m\times m}$, and $\eta,q\in\mathbb{R}^n$, $v\in\mathbb{R}^m$. In addition, $\underline{P}$ and $\underline{R}$ are symmetric, $I+\mu\underline{\mathcal{A}}$ is invertible, and we define the $\mathrm{C}_{\mathrm{prd}}$ matrices $\tilde{\underline{\mathcal{A}}}\in\mathbb{R}^{n\times n}$ and $\underline{S}\in\mathbb{R}^{m\times m}$ on $[a,\rho(b)]_{\mathbb{T}}$ by

$$\tilde{\underline{\mathcal{A}}} := (I+\mu\underline{\mathcal{A}})^{-1}, \quad \underline{S} := \underline{R} - \mu\underline{\mathcal{B}}^T\tilde{\underline{\mathcal{A}}}^T\underline{Q}.$$

Moreover, if the matrix $\underline{R}$ is invertible, we define the matrix $\underline{T}\in\mathbb{R}^{n\times n}$ by

$$\underline{T} := I - \mu\tilde{\underline{\mathcal{A}}}\underline{\mathcal{B}}\underline{R}^{-1}\underline{Q}^T.$$

With system ($\underline{\mathrm{J}}$) we consider the quadratic form

$$\left.\begin{aligned} &\underline{\omega}(\eta,v) := \eta^T\underline{P}\,\eta + 2\,\eta^T\underline{Q}\,v + v^T\underline{R}\,v, \\ &\qquad \eta\in \mathrm{C}^1_{\mathrm{prd}}[a,b]_{\mathbb{T}},\ v\in \mathrm{C}_{\mathrm{prd}}[a,\rho(b)]_{\mathbb{T}},\ \text{with} \\ &\qquad \eta^\Delta(t) = \underline{\mathcal{A}}(t)\,\eta(t) + \underline{\mathcal{B}}(t)\,v(t),\ t\in[a,\rho(b)]_{\mathbb{T}}. \end{aligned}\right\} \tag{2.1}$$

When the matrix $\underline{R}$ is invertible, Jacobi system ($\underline{\mathrm{J}}$) can be written as the linear Hamiltonian system

$$\eta^\Delta = \underline{A}(t)\,\eta + \underline{B}(t)\,q^\sigma, \quad q^\Delta = \underline{C}(t)\,\eta - \underline{A}^T(t)\,q^\sigma, \quad t \in [a,\rho(b)]_{\mathbb{T}}. \tag{$\underline{\mathrm{H}}$}$$

The coefficients in system ($\underline{\mathrm{H}}$) have the following properties.

Notation 2.2 (Hamiltonian system ($\underline{\mathrm{H}}$))**.** The $\mathrm{C}_{\mathrm{prd}}$ matrix functions $\underline{A}$, $\underline{B}$, $\underline{C}$ on $[a,\rho(b)]_{\mathbb{T}}$ and the $\mathrm{C}^1_{\mathrm{prd}}$ vectors η, q on $[a,b]_{\mathbb{T}}$ have the following dimensions: $\underline{A},\underline{B},\underline{C} \in \mathbb{R}^{n\times n}$ and $\eta, q \in \mathbb{R}^n$. In addition, $\underline{B}$ and $\underline{C}$ are symmetric, $I + \mu\underline{A}$ is invertible, and we define the $\mathrm{C}_{\mathrm{prd}}$ matrix $\underline{\tilde{A}} \in \mathbb{R}^{n\times n}$ on $[a,\rho(b)]_{\mathbb{T}}$ by

$$\underline{\tilde{A}} := (I + \mu\underline{A})^{-1}.$$

In the definitions of $\underline{T}$ and $\underline{\tilde{A}}$ in Notations 2.1 and 2.2 we slightly differ from [16, Section 4], where these two matrices are transposed. The current definitions appear to be more convenient. The quadratic form associated with system ($\underline{\mathrm{H}}$) has the form

$$\left.\begin{aligned} &\underline{\Omega}(\eta,q) := \eta^T\underline{C}\,\eta + q^T\underline{B}\,q, \\ &\qquad\qquad \eta \in \mathrm{C}^1_{\mathrm{prd}}[a,b]_{\mathbb{T}},\ q \in \mathrm{C}_{\mathrm{prd}}[a,\rho(b)]_{\mathbb{T}},\ \text{with} \\ &\qquad\qquad \eta^\Delta(t) = \underline{A}(t)\,\eta(t) + \underline{B}(t)\,q^\sigma(t),\ t \in [a,\rho(b)]_{\mathbb{T}}. \end{aligned}\right\} \tag{2.2}$$

Note that the forward shift is now in q^σ instead of the traditional shift in η^σ in [16, Section 4].

Notation 2.3 (Symplectic system ($\underline{\mathbb{S}}$))**.** The $\mathrm{C}_{\mathrm{prd}}$ matrix functions $\underline{\mathbb{A}}$, $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\underline{\mathbb{D}}$ on $[a,\rho(b)]_{\mathbb{T}}$ and the $\mathrm{C}^1_{\mathrm{prd}}$ vectors η, q on $[a,b]_{\mathbb{T}}$ have the following dimensions: $\underline{\mathbb{A}},\underline{\mathbb{B}},\underline{\mathbb{C}},\underline{\mathbb{D}} \in \mathbb{R}^{n\times n}$ and $\eta, q \in \mathbb{R}^n$. In addition, the matrix $\underline{\mathbb{S}} \in \mathbb{R}^{2n\times 2n}$ defined in (1.5) satisfies identity (1.4).

The quadratic form $\underline{\mathbb{Q}}$ associated with system ($\underline{\mathbb{S}}$) is defined in (1.7). Note that identity (1.4) is equivalent with

$$\underline{\mathbb{S}}(t)\,\mathcal{J} + \mathcal{J}\,\underline{\mathbb{S}}^T(t) - \mu(t)\,\underline{\mathbb{S}}(t)\,\mathcal{J}\,\underline{\mathbb{S}}^T(t) = 0 \quad \text{for all } t \in [a,\rho(b)]_{\mathbb{T}}, \tag{2.3}$$

since at the right-dense points $t \in [a,b)_{\mathbb{T}}$ equations (1.4) and (2.3) coincide, while at the right-scattered points $t \in [a,\rho(b)]_{\mathbb{T}}$ they are equivalent with $I - \mu(t)\,\underline{\mathbb{S}}(t)$ and $I - \mu(t)\,\underline{\mathbb{S}}^T(t)$ being symplectic. Identities (1.4) and (2.3) then reduce, respectively, to

$$\underline{\mathbb{A}}^T + \underline{\mathbb{D}} - \mu\,(\underline{\mathbb{A}}^T\,\underline{\mathbb{D}} - \underline{\mathbb{C}}^T\,\underline{\mathbb{B}}) = 0, \quad (I - \mu\underline{\mathbb{A}}^T)\,\underline{\mathbb{C}} \ \text{ and } \ (I - \mu\underline{\mathbb{D}}^T)\,\underline{\mathbb{B}} \ \text{ symmetric}, \tag{2.4}$$

$$\underline{\mathbb{A}}^T + \underline{\mathbb{D}} - \mu\,(\underline{\mathbb{D}}\,\underline{\mathbb{A}}^T - \underline{\mathbb{C}}\,\underline{\mathbb{B}}^T) = 0, \quad (I - \mu\underline{\mathbb{D}})\,\underline{\mathbb{C}}^T \ \text{ and } \ (I - \mu\underline{\mathbb{A}})\,\underline{\mathbb{B}}^T \ \text{ symmetric}. \tag{2.5}$$

We now review the results connecting the systems ($\underline{\mathrm{J}}$), ($\underline{\mathrm{H}}$), and ($\underline{\mathbb{S}}$) and their quadratic forms. These results are from [16] and [13], except of the result in Propositions 2.5, 2.8, and 2.10 which are new. In particular, as displayed below in the three mentioned propositions, it is the natural connection of the quadratic form $\underline{\Omega}$ corresponding to Hamiltonian system ($\underline{\mathrm{H}}$) with the quadratic forms $\underline{\omega}$ and $\underline{\mathbb{Q}}$ that shows the importance of these objects. For example, in [2, Section 5] it is shown that higher order Sturm–Liouville dynamic equations lead to the Hamiltonian system ($\underline{\mathrm{H}}$). Therefore, the variational methods based on the corresponding quadratic forms known for the Hamiltonian and symplectic systems ($\underline{\mathrm{H}}$) and ($\underline{\mathbb{S}}$) can now be utilized also for some types of higher order Sturm–Liouville equations on time scales.

Proposition 2.4 (Jacobi ($\underline{\mathrm{J}}$) to Hamiltonian ($\underline{\mathrm{H}}$))**.** *Assume that $\underline{\mathcal{A}}, \underline{\mathcal{B}}, \underline{P}, \underline{Q}, \underline{R}, \underline{T}$ satisfy the conditions in Notation 2.1 with $\underline{R}$ and $\underline{T}$ invertible. Then the Jacobi system* ($\underline{\mathrm{J}}$) *is the Hamiltonian system* ($\underline{\mathrm{H}}$)*, whose coefficients*

$$\underline{A} = \underline{\mathcal{A}} - \underline{\mathcal{B}}\underline{R}^{-1}\underline{Q}^T, \quad \underline{B} = \underline{\mathcal{B}}\underline{R}^{-1}\underline{\mathcal{B}}^T, \quad \underline{C} = \underline{P} - \underline{Q}\underline{R}^{-1}\underline{Q}^T \tag{2.6}$$

with $\tilde{\underline{A}} = \underline{T}^{-1}\tilde{\underline{\mathcal{A}}}$ satisfy the conditions in Notation 2.2.

Proof. See [16, Proposition 4.4] or [13, Formula (45)]. ■

The pairs of functions (η, v) and (η, q) in definitions (2.1), (2.2), (1.6), and (1.7) of the quadratic forms $\underline{\omega}$, $\underline{\Omega}$, $\mathbb{Q}$, and $\underline{\mathbb{Q}}$ are called respectively $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible, $(\underline{A}, \underline{B})$-admissible, $(\mathbb{A}, \mathbb{B})$-admissible, and $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible. The results in Propositions 2.5, 2.8, and 2.10 below represent the correct parallel versions to [16, Propositions 2.10, 2.11, 3.7].

Proposition 2.5 (Quadratic forms for ($\underline{\mathrm{J}}$) and ($\underline{\mathrm{H}}$))**.** *Assume that $\underline{\mathcal{A}}, \underline{\mathcal{B}}, \underline{P}, \underline{Q}, \underline{R}$ satisfy the conditions in Notation 2.1 with $\underline{R}$ invertible and let $\underline{A}, \underline{B}, \underline{C}$ be given by* (2.6)*. If (η, q) is $(\underline{A}, \underline{B})$-admissible, then the pair (η, v) with $v := \underline{R}^{-1}(\underline{\mathcal{B}}^T q^\sigma - \underline{Q}^T \eta)$ is $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible, and in this case $\underline{\omega}(\eta, v) = \underline{\Omega}(\eta, q^\sigma)$.*

Proof. The result follows by direct calculations. ■

Proposition 2.6 (Hamiltonian ($\underline{\mathrm{H}}$) to symplectic ($\underline{\mathbb{S}}$))**.** *Assume that $\underline{A}, \underline{B}, \underline{C}$ satisfy the conditions in Notation 2.2. Then the Hamiltonian system* ($\underline{\mathrm{H}}$) *is the symplectic system* ($\underline{\mathbb{S}}$)*, whose coefficients*

$$\underline{\mathbb{A}} = \tilde{\underline{A}}\underline{A}, \quad \underline{\mathbb{B}} = \tilde{\underline{A}}\underline{B}, \quad \underline{\mathbb{C}} = \underline{C}\tilde{\underline{A}}, \quad \underline{\mathbb{D}} = -\mu\,\underline{C\tilde{A}B} - \underline{A}^T \tag{2.7}$$

with $I - \mu\underline{\mathbb{A}} = \tilde{\underline{A}}$ satisfy the conditions in Notation 2.3.

Proof. The result follows by direct calculations with the aid of properties (2.4)–(2.5). ■

Proposition 2.7 (Symplectic ($\underline{\mathbb{S}}$) to Hamiltonian ($\underline{\mathrm{H}}$))**.** *Assume that $\underline{\mathbb{A}}, \underline{\mathbb{B}}, \underline{\mathbb{C}}, \underline{\mathbb{D}}$ satisfy the conditions in Notation 2.3. Then the the symplectic system* ($\underline{\mathbb{S}}$) *is the Hamiltonian system* ($\underline{\mathrm{H}}$)*, whose coefficients*

$$\underline{A} = (I - \mu\underline{\mathbb{A}})^{-1}\underline{\mathbb{A}}, \quad \underline{B} = (I - \mu\underline{\mathbb{A}})^{-1}\,\underline{\mathbb{B}}, \quad \underline{C} = \underline{\mathbb{C}}\,(I - \mu\underline{\mathbb{A}})^{-1} \tag{2.8}$$

with $\tilde{\underline{A}} = I - \mu\underline{\mathbb{A}}$ satisfy the conditions in Notation 2.2.

Proof. The result follows by direct calculations, in which we use the properties of the coefficients of system ($\underline{\mathbb{S}}$) displayed in (2.5). ■

Proposition 2.8 (Quadratic forms for ($\underline{\mathrm{H}}$) and ($\underline{\mathbb{S}}$))**.** *Assume that*

(i) *either $\underline{A}, \underline{B}, \underline{C}$ satisfy the conditions in Notation 2.2 and $\underline{\mathbb{A}}, \underline{\mathbb{B}}, \underline{\mathbb{C}}, \underline{\mathbb{D}}$ are given by* (2.7)*,*

(ii) *or $\underline{\mathbb{A}}, \underline{\mathbb{B}}, \underline{\mathbb{C}}, \underline{\mathbb{D}}$ satisfy the conditions in Notation 2.3 with $I - \mu\underline{\mathbb{A}}$ invertible and $\underline{A}, \underline{B}, \underline{C}$ are given by* (2.8)*.*

Then a pair (η, q) is $(\underline{A}, \underline{B})$-admissible if and only if it is $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible, and in this case $\underline{\Omega}(\eta, q^\sigma) = \underline{\mathbb{Q}}(\eta^\sigma, q^\sigma)$.

Proof. The result follows by direct calculations. ■

Combining Propositions 2.4 and 2.6 yields the transition from Jacobi system ($\underline{\mathrm{J}}$) to symplectic system ($\underline{\mathbb{S}}$). This, however, requires the invertibility of the matrices $\underline{R}$ and $\underline{T}$. As the result of [16, Lemma 4.9] shows, the invertibility of the matrix $\underline{S}$ alone is a weaker condition than the invertibility of $\underline{R}$ and $\underline{T}$. This way, we may transform system ($\underline{\mathrm{J}}$) into system ($\underline{\mathbb{S}}$) directly by bypassing the Hamiltonian system ($\underline{\mathrm{H}}$).

Proposition 2.9 (Jacobi ($\underline{\mathrm{J}}$) to symplectic ($\underline{\mathbb{S}}$))**.** *Assume that $\underline{\mathcal{A}}, \underline{\mathcal{B}}, \underline{P}, \underline{Q}, \underline{R}, \underline{S}$ satisfy the conditions in Notation 2.1 with $\underline{S}$ invertible. Then the Jacobi system ($\underline{\mathrm{J}}$) is the symplectic system ($\underline{\mathbb{S}}$), whose coefficients*

$$\left.\begin{aligned} \underline{\mathbb{A}} &= \underline{\tilde{\mathcal{A}}}\,(\underline{\mathcal{A}} - \underline{\mathcal{B}}\underline{S}^{T-1}\underline{Q}^T\underline{\tilde{\mathcal{A}}}), & \underline{\mathbb{B}} &= \underline{\tilde{\mathcal{A}}\mathcal{B}}\underline{S}^{T-1}\underline{\mathcal{B}}^T, \\ \underline{\mathbb{C}} &= \underline{P\tilde{\mathcal{A}}} - (\underline{Q} - \mu\, \underline{P\tilde{\mathcal{A}}\mathcal{B}})\, \underline{S}^{T-1}\underline{Q}^T\underline{\tilde{\mathcal{A}}}, & \underline{\mathbb{D}} &= (\underline{Q} - \mu\, \underline{P\tilde{\mathcal{A}}\mathcal{B}})\, \underline{S}^{T-1}\underline{\mathcal{B}}^T - \underline{\mathcal{A}}^T \end{aligned}\right\} \tag{2.9}$$

with $I - \mu\underline{\mathbb{A}} = \underline{\tilde{\mathcal{A}}} + \mu\underline{\tilde{\mathcal{A}}\mathcal{B}S}^{T-1}\underline{Q}^T\underline{\tilde{\mathcal{A}}}$ satisfy the conditions in Notation 2.3. Thus, the resulting symplectic system ($\underline{\mathbb{S}}$) is the Hamiltonian system ($\underline{\mathrm{H}}$) if and only if the matrix $\underline{R}$ is invertible.

Proof. The result follows from [16, Theorem 4.8], when the symplectic system obtained from [16, Theorem 4.8] is written in the time-reversed form ($\underline{\mathbb{S}}$), see the relation between the systems ($\mathbb{S}$) and (1.3) in Section 1. The last assertion is a consequence of [16, Lemma 4.9]. ∎

Proposition 2.10 (Quadratic forms for ($\underline{\mathrm{J}}$) and ($\underline{\mathbb{S}}$))**.** *Assume that $\underline{\mathcal{A}}, \underline{\mathcal{B}}, \underline{P}, \underline{Q}, \underline{R}$ satisfy the conditions in Notation 2.1 with $\underline{S}$ invertible and let $\underline{\mathbb{A}}, \underline{\mathbb{B}}, \underline{\mathbb{C}}, \underline{\mathbb{D}}$ be given by* (2.9)*. If (η, q) is $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible, then the pair (η, v) with*

$$v := \underline{S}^{T-1}(\underline{\mathcal{B}}^T q^\sigma - \underline{Q}^T\underline{\tilde{\mathcal{A}}}\,\eta^\sigma) \tag{2.10}$$

is $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible, and in this case $\underline{\omega}(\eta, v) = \underline{\mathbb{Q}}(\eta^\sigma, q^\sigma)$.

Proof. The result follows by direct calculations. ∎

From the above result we can easily deduce the relationship between the quadratic functionals associated with systems ($\underline{\mathrm{J}}$) and ($\underline{\mathbb{S}}$). Let $M \in \mathbb{R}^{r\times 2n}$ and $\Gamma \in \mathbb{R}^{2n\times 2n}$ be given matrices with $r \le 2n$, which define the boundary conditions of η and the cost of the endpoints of η. For $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible pairs (η, v) we define the quadratic functional

$$\underline{\mathcal{F}}(\eta, v) := \begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix}^T \Gamma \begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix} + \int_a^b \underline{\omega}(\eta, v)(t)\,\Delta t,$$

and for $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible pairs (η, q) the quadratic functional

$$\underline{\mathbb{F}}(\eta, q) := \begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix}^T \Gamma \begin{pmatrix}\eta(a)\\ \eta(b)\end{pmatrix} + \int_a^b \underline{\mathbb{Q}}(\eta^\sigma, q^\sigma)(t)\,\Delta t. \tag{2.11}$$

The quadratic functional $\underline{\mathcal{F}}$ is the second variation of the optimal control problem ($\underline{\mathrm{C}}$), see [11, Theorem 9.7]. The result below implies through the latter reference that necessary conditions for the nonnegativity of the functional $\underline{\mathbb{F}}$ provide at the same time necessary optimality conditions for the optimal control problem ($\underline{\mathrm{C}}$).

Corollary 2.11. *Assume that $\underline{\mathcal{A}}, \underline{\mathcal{B}}, \underline{P}, \underline{Q}, \underline{R}$ satisfy the conditions in Notation 2.1 with $\underline{S}$ invertible and let $\underline{\mathbb{A}}, \underline{\mathbb{B}}, \underline{\mathbb{C}}, \underline{\mathbb{D}}$ be given by (2.9). If the functional $\underline{\mathcal{F}}$ is nonnegative, then the functional $\underline{\mathbb{F}}$ is nonnegative as well. That is, if $\underline{\mathcal{F}}(\eta, v) \geq 0$ for every $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible pair (η, v) with*

$$M \begin{pmatrix} \eta(a) \\ \eta(b) \end{pmatrix} = 0, \tag{2.12}$$

then $\underline{\mathbb{F}}(\eta, q) \geq 0$ for every $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible pair (η, q) satisfying (2.12).

Proof. If (η, q) is $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible and satisfies (2.12), then Proposition 2.10 yields that (η, v) with v given by (2.10) is $(\underline{\mathcal{A}}, \underline{\mathcal{B}})$-admissible and $\underline{\mathbb{F}}(\eta, q) = \underline{\mathcal{F}}(\eta, v) \geq 0$, by our assumption. ∎

3. Reid roundabout theorem for symplectic system ($\underline{\mathbb{S}}$)

In this section we derive the so-called Reid roundabout theorem for the time-reversed symplectic system ($\underline{\mathbb{S}}$). In addition, we establish the exact connection between the quadratic functionals involving the forms $\underline{\mathbb{Q}}$ and $\mathbb{Q}$ associated with systems ($\underline{\mathbb{S}}$) and ($\mathbb{S}$), respectively.

Identity (1.4) implies that the Wronskian matrix of any two solutions of system ($\underline{\mathbb{S}}$) is constant on $[a, b]_{\mathbb{T}}$, that is, if $Z = (X, U)$ and $\tilde{Z} = (\tilde{X}, \tilde{U})$ are two $2n \times k$ solutions of ($\underline{\mathbb{S}}$), then

$$Z^T(t)\, \mathcal{J} \tilde{Z}(t) = X^T(t)\, \tilde{U}(t) - U^T(t)\, \tilde{X}(t) \equiv C \quad \text{on } [a, b]_{\mathbb{T}},$$

where $C \in \mathbb{R}^{k \times k}$ is a constant matrix. A solution $Z = (X, U)$ of ($\underline{\mathbb{S}}$) is called a conjoined basis if $Z(t) \in \mathbb{R}^{2n \times n}$ and $Z^T(t)\, \mathcal{J} Z(t) \equiv 0$ on $[a, b]_{\mathbb{T}}$.

With system ($\underline{\mathbb{S}}$) we consider the Riccati matrix dynamic equation

$$\underline{R}[W](t) = 0, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \qquad \underline{R}[W] := W^{\Delta} - \underline{\mathbb{C}} - \underline{\mathbb{D}}\, W^{\sigma} + W(\underline{\mathbb{A}} + \underline{\mathbb{B}}\, W^{\sigma}), \tag{$\underline{\mathbb{R}}$}$$

and the quadratic functional $\underline{\mathbb{F}}$ over $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible pairs (η, q) introduced in (2.11). We say that the functional $\underline{\mathbb{F}}$ is positive, if $\underline{\mathbb{F}}(\eta, q) > 0$ for every $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible (η, q) satisfying (2.12) and $\eta \not\equiv 0$. In particular, we are interested in the zero endpoints case for which $M = I$ and $\Gamma = 0$, i.e., endpoints constraint (2.12) has the form $\eta(a) = 0 = \eta(b)$. In this case, the functional $\underline{\mathbb{F}}$ will be denoted by $\underline{\mathbb{F}}_0$, that is,

$$\underline{\mathbb{F}}_0(\eta, q) := \int_a^b \underline{\mathbb{Q}}(\eta^{\sigma}, q^{\sigma})(t)\, \Delta t. \tag{3.1}$$

Next we present the first result of this section, compare with [13, Theorem 3.1] and [12, Theorem 8.1], in which we characterize the positivity of the functional $\underline{\mathbb{F}}_0$ over the zero endpoints.

Theorem 3.1 (Reid roundabout theorem for system ($\underline{\mathbb{S}}$))**.** *The following are equivalent.*

(i) *The functional $\underline{\mathbb{F}}_0$ is positive.*

(ii) *There exists a conjoined basis (X, U) of ($\underline{\mathbb{S}}$) with $X(t)$ is invertible on $[a, b]_{\mathbb{T}}$ and*

$$X^{\sigma}(t)\, X^{-1}(t)\, \underline{\mathbb{B}}(t) \geq 0 \quad \textit{for all } t \in [a, \rho(b)]_{\mathbb{T}}.$$

(iii) *There exists a symmetric solution $W(t)$ on $[a, b]_{\mathbb{T}}$ of the Riccati equation ($\underline{\mathbb{R}}$) with*

$$\underline{\mathbb{P}}(t) := \underline{\mathbb{B}}(t) - \mu(t)\, [\underline{\mathbb{D}}^T(t) - \underline{\mathbb{B}}^T(t)\, W(t)]\, \underline{\mathbb{B}}(t) \geq 0 \quad \textit{for all } t \in [a, \rho(b)]_{\mathbb{T}}. \tag{3.2}$$

In the literature there are several other characterizations of the positivity of a quadratic functional. For the functional $\underline{\mathbb{F}}_0$, one can prove such conditions in terms of the principal solution $(\hat{\underline{X}}, \hat{\underline{U}})$ of $(\underline{\mathbb{S}})$ at $t = b$ with nondecreasing kernel of $\hat{\underline{X}}(t)$ on $[a, b]_{\mathbb{T}}$, certain perturbed quadratic functionals, and the implicit Riccati equations, see [7,8,10,12]. Also, the result of Theorem 3.1 can be derived for more general endpoint constraints by the standard time scales methods e.g. in [7,9,12]. Observe that in the discrete case the result of Theorem 3.1 provides yet another two equivalent conditions to the list in [3, Theorem 1]. With respect to this reference we note that our functional $\underline{\mathbb{F}}_0$ has the opposite sign than the corresponding time-reversed functional in [3, Theorem 1(vi)].

We emphasize that the transformation of the coefficients of system $(\underline{\mathbb{S}})$ into a standard symplectic system $(\mathbb{S})$ displayed in (3.6) below does not yield the functional $\underline{\mathbb{F}}_0$, but rather a different functional with η and q in its argument instead of the required η^σ and q^σ. Therefore, a direct proof of Theorem 3.1 is essentially needed. First we present an important tool.

Lemma 3.2 (Picone identity)**.** *Assume that $W(t)$ on $[a, b]_{\mathbb{T}}$ is a symmetric solution of the Riccati matrix equation $(\underline{\mathbb{R}})$. Then for any $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissible $\underline{z} = (\eta, q)$ we have*

$$\underline{\mathbb{Q}}(\underline{z}^\sigma)(t) = [\eta^T(t)\, W(t)\, \eta(t)]^\Delta + [w^\sigma(t)]^T\, \underline{\mathbb{P}}(t)\, w^\sigma(t) \quad \text{on } [a, \rho(b)]_{\mathbb{T}}, \tag{3.3}$$

where $w := q - W\eta$ on $[a, \rho(b)]_{\mathbb{T}}$ and $\underline{\mathbb{P}}$ is defined in (3.2)*. In addition, we have*

$$\eta^\sigma - \mu \underline{\mathbb{P}}\, w^\sigma = [I - \mu\, (\underline{\mathbb{D}}^T - \underline{\mathbb{B}}^T W)]\, \eta \quad \text{on } [a, \rho(b)]_{\mathbb{T}}. \tag{3.4}$$

Proof. The $(\underline{\mathbb{A}}, \underline{\mathbb{B}})$-admissibility of $\underline{z}$ implies that $\eta = (I - \mu \underline{\mathbb{A}})\, \eta^\sigma - \mu \underline{\mathbb{B}}\, q^\sigma$. Formula (3.3) then follows by direct calculations from the time scales product rule for $(\eta^T W \eta)^\Delta = (\eta^\Delta)^T W \eta + (\eta^\sigma)^T (W^\Delta \eta^\sigma + W \eta^\Delta)$ with the aid of the Riccati equation $(\underline{\mathbb{R}})$, the symmetry of W, and identity (2.4)(i). Formula (3.4) is also proven by direct calculations, in which the identity

$$[I - \mu\, (\underline{\mathbb{D}}^T - \underline{\mathbb{B}}^T W)]\, [I - \mu\, (\underline{\mathbb{A}} + \underline{\mathbb{B}}\, W^\sigma)] = I - \mu^2\, \underline{\mathbb{B}}^T \underline{R}[W] \quad \text{on } [a, \rho(b)]_{\mathbb{T}} \tag{3.5}$$

is utilized, compare with [15, Lemma 3.6]. ■

Proof of Theorem 3.1. The implication "(i) ⇒ (ii)" is proven in a similar way as in [9, Theorem 6.1]. The implication "(ii) ⇒ (iii)" is based on the Riccati substitution $W := UX^{-1}$ on $[a, b]_{\mathbb{T}}$. Finally, the implication "(iii) ⇒ (i)" follows from the Picone identity in Lemma 3.2. ■

Next we establish the second main result of this section, which relates the positivity of the quadratic functionals corresponding to time scale symplectic systems $(\underline{\mathbb{S}})$ and $(\mathbb{S})$. Parallel to Notation 2.3 we specify the conditions for coefficients of system $(\mathbb{S})$.

Notation 3.3 (Symplectic system $(\mathbb{S})$)**.** The $\mathrm{C}_{\mathrm{prd}}$ matrix functions $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ on $[a, \rho(b)]_{\mathbb{T}}$ and the $\mathrm{C}^1_{\mathrm{prd}}$ vectors η, q on $[a, b]_{\mathbb{T}}$ have the following dimensions: $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D} \in \mathbb{R}^{n\times n}$ and $\eta, q \in \mathbb{R}^n$. In addition, the matrix $\mathbb{S} \in \mathbb{R}^{2n\times 2n}$ defined in (1.2) satisfies identity (1.1).

Similarly to the definition of the functional $\underline{\mathbb{F}}_0$ in (3.1) we put

$$\mathbb{F}_0(\eta, q) := \int_a^b \mathbb{Q}(\eta, q)(t)\, \Delta t$$

over $(\mathbb{A},\mathbb{B})$-admissible (η,q) satisfying $\eta(a)=0=\eta(b)$, where the quadratic form $\mathbb{Q}$ is defined in (1.6). The relationship between the coefficients of systems $(\mathbb{S})$ and $(\underline{\mathbb{S}})$ is given by the following formulas, see (1.3). For $t\in[a,\rho(b)]_{\mathbb{T}}$ we have

$$\mathbb{S}(t)=\mathcal{J}\,\underline{\mathbb{S}}^T(t)\,\mathcal{J}=\begin{pmatrix}-\underline{\mathbb{D}}^T(t) & \underline{\mathbb{B}}^T(t)\\ \underline{\mathbb{C}}^T(t) & -\underline{\mathbb{A}}^T(t)\end{pmatrix}, \tag{3.6}$$

$$\underline{\mathbb{S}}(t)=\mathcal{J}\,\mathbb{S}^T(t)\,\mathcal{J}=\begin{pmatrix}-\mathbb{D}^T(t) & \mathbb{B}^T(t)\\ \mathbb{C}^T(t) & -\mathbb{A}^T(t)\end{pmatrix}. \tag{3.7}$$

The following result is a generalization of the discrete time result in [3, Theorem 1(i), (vi)] to arbitrary time scales.

Theorem 3.4 (Quadratic forms for $(\underline{\mathbb{S}})$ and $(\mathbb{S})$)**.** *Assume that the coefficients in systems* $(\underline{\mathbb{S}})$ *and* $(\mathbb{S})$ *satisfy the conditions in Notations 2.3 and 3.3 and that they are related by the formulas in* (3.6)–(3.7). *Then the functional* $\underline{\mathbb{F}}_0$ *is positive if and only if the functional* $\mathbb{F}_0$ *is positive.*

Proof. By Theorem 3.1, we know that the positivity of $\underline{\mathbb{F}}_0$ is equivalent with condition (iii) in Theorem 3.1. This means, by the formulas in (3.7), that the function $W(t)$ satisfies the equation

$$W^{\Delta}=\mathbb{C}+\mathbb{D}\,W-W^{\sigma}(\mathbb{A}+\mathbb{B}\,W),\quad \text{on } [a,\rho(b)]_{\mathbb{T}}. \tag{3.8}$$

In addition, from (3.5) we know that the matrices

$$I-\mu\,(\underline{\mathbb{D}}^T-\underline{\mathbb{B}}^TW)=I+\mu\,(\mathbb{A}+\mathbb{B}\,W)\quad\text{and}\quad I-\mu\,(\underline{\mathbb{A}}+\underline{\mathbb{B}}\,W^{\sigma})=I+\mu\,(\mathbb{D}^T-\mathbb{B}^TW^{\sigma})$$

are inverses of each other. Therefore, the matrix

$$\mathbb{P}:=[I+\mu\,(\mathbb{D}^T-\mathbb{B}^TW^{\sigma})]\,\mathbb{B}=[I-\mu\,(\underline{\mathbb{A}}+\underline{\mathbb{B}}\,W^{\sigma})]\,\underline{\mathbb{P}}\,[I-\mu\,(\underline{\mathbb{A}}+\underline{\mathbb{B}}\,W^{\sigma})]^T\geq 0$$

on $[a,\rho(b)]_{\mathbb{T}}$. Hence, we showed that the function $W(t)$ satisfies the Riccati equation condition in [9, Theorem 6.1]. Since the converse from (3.8) and $\mathbb{P}\geq 0$ on $[a,\rho(b)]_{\mathbb{T}}$ to condition (iii) in Theorem 3.1 is done quite similarly, we have by [9, Theorem 6.1] the stated equivalence of the positivity of the functionals $\underline{\mathbb{F}}_0$ and $\mathbb{F}_0$. ∎

Remark 3.5. Note that the proof of Theorem 3.4 is much simpler than the corresponding proof of the discrete time statement in [3, Theorem 1(i), (vi)]. On the other hand, one direction in the proof above can be shown by the same method as in [3, Theorem 1]. For convenience, we use in the calculations the matrix $\mathcal{K}:=\left(\begin{smallmatrix}0&0\\ I&0\end{smallmatrix}\right)\in\mathbb{R}^{2n\times 2n}$. With this matrix the $(\underline{\mathbb{A}},\underline{\mathbb{B}})$-admissibility or $(\mathbb{A},\mathbb{B})$-admissibility of $z=(\eta,q)$ reads as $(\mathcal{K}z)^{\Delta}=\mathcal{K}\,\underline{\mathbb{S}}\,z^{\sigma}$ or $(\mathcal{K}z)^{\Delta}=\mathcal{K}\,\mathbb{S}\,z$, respectively. Moreover, the quadratic forms $\underline{\mathbb{Q}}$ and $\mathbb{Q}$ are

$$\underline{\mathbb{Q}}(z^{\sigma})=(z^{\sigma})^T(\underline{\mathbb{S}}^T\mathcal{K}+\mathcal{K}\,\underline{\mathbb{S}}-\mu\,\underline{\mathbb{S}}^T\mathcal{K}\,\underline{\mathbb{S}})\,z^{\sigma},\quad \mathbb{Q}(z)=z^T(\mathbb{S}^T\mathcal{K}+\mathcal{K}\,\mathbb{S}+\mu\,\mathbb{S}^T\mathcal{K}\,\mathbb{S})\,z.$$

Assume that $\mathbb{F}_0$ is positive. Let $\underline{z}=(\eta,q)$ be $(\underline{\mathbb{A}},\underline{\mathbb{B}})$-admissible with $\eta(a)=0=\eta(b)$ and $\eta\not\equiv 0$. Define $z:=(I+\mu\mathbb{S})^{-1}\,\underline{z}^{\sigma}$ on $[a,\rho(b)]_{\mathbb{T}}$, and $z(b):=0$ if b is left-scattered. Then from $(\mathcal{K}\underline{z})^{\Delta}=\mathcal{K}\,\underline{\mathbb{S}}\,\underline{z}^{\sigma}$ we get upon the multiplication by μ that $\mathcal{K}\underline{z}^{\sigma}-\mathcal{K}\underline{z}=\mu\mathcal{K}\,\underline{\mathbb{S}}\,\underline{z}^{\sigma}$. Thus, $\mathcal{K}\underline{z}=\mathcal{K}\,(I-\mu\underline{\mathbb{S}})\,\underline{z}^{\sigma}$. But since by (3.6) we have $(I+\mu\mathbb{S})^{-1}=I-\mu\underline{\mathbb{S}}$, it follows that $\mathcal{K}\underline{z}=\mathcal{K}\,(I+\mu\mathbb{S})^{-1}\,\underline{z}^{\sigma}=\mathcal{K}z$ on $[a,\rho(b)]_{\mathbb{T}}$, and hence on $[a,b]_{\mathbb{T}}$. Therefore, the first components of $\underline{z}$ and z are identical, with the result that $\mathcal{K}z \in \mathrm{C}^1_{\mathrm{prd}}$. In turn, since the identity $(I-\mu\underline{\mathbb{S}})^{-1} = -\mathcal{J}\,(I-\mu\underline{\mathbb{S}}^T)\,\mathcal{J}$ holds, we get

$$\begin{aligned}(\mathcal{K}z)^\Delta &= (\mathcal{K}\underline{z})^\Delta = \mathcal{K}\,\underline{\mathbb{S}}\,\underline{z}^\sigma = \mathcal{K}\,\underline{\mathbb{S}}\,(I+\mu\mathbb{S})\,z = \mathcal{K}\,\underline{\mathbb{S}}\,(I-\mu\underline{\mathbb{S}})^{-1}\,z \\ &= -\,\mathcal{K}\,\underline{\mathbb{S}}\,\mathcal{J}\,(I-\mu\underline{\mathbb{S}}^T)\,\mathcal{J}z = -\,\mathcal{K}(\underline{\mathbb{S}}\,\mathcal{J} - \mu\,\underline{\mathbb{S}}\,\mathcal{J}\,\underline{\mathbb{S}}^T)\,\mathcal{J}z \\ &\overset{(2.3)}{=} \mathcal{K}\,\mathcal{J}\,\underline{\mathbb{S}}^T\mathcal{J}z \overset{(3.6)}{=} \mathcal{K}\,\mathbb{S}\,z\end{aligned}$$

on $[a,\rho(b)]_{\mathbb{T}}$. This shows that z is $(\mathbb{A},\mathbb{B})$-admissible, so that our assumption implies $\mathbb{F}_0(z) > 0$. If we now prove that $\underline{\mathbb{Q}}(\underline{z}^\sigma) = \mathbb{Q}(z)$, then $\underline{\mathbb{F}}_0(\underline{z}) = \mathbb{F}_0(z) > 0$ follows. To this end we fix an arbitrary $t \in [a,\rho(b)]_{\mathbb{T}}$. If t is right-dense, then $\mu(t)=0$ and so from (2.4) and (1.1) we have $\underline{\mathbb{B}}$, $\underline{\mathbb{C}}$, $\mathbb{B}$, $\mathbb{C}$ symmetric, $\underline{\mathbb{D}} = -\underline{\mathbb{A}}^T$, and $\mathbb{D} = -\mathbb{A}^T$ at this point t. In addition, formula (3.6) yields that $\underline{\mathbb{S}} = \mathbb{S}$ at t, while from the definition of z in which now $\mu = 0$ we get $z = \underline{z}$ at t. Consequently, if t is right-dense, then

$$\mathbb{Q}(z) = z^T(\mathbb{S}^T\mathcal{K} + \mathcal{K}\,\mathbb{S})\,z = \underline{z}^T(\underline{\mathbb{S}}^T\mathcal{K} + \mathcal{K}\,\underline{\mathbb{S}})\,\underline{z} = \underline{\mathbb{Q}}(\underline{z}) = \underline{\mathbb{Q}}(\underline{z}^\sigma).$$

If t is right-scattered, then $\mu(t) > 0$, and at t we have

$$\begin{aligned}\mu\,\mathbb{Q}(z) &= z^T\,[(I+\mu\mathbb{S}^T)\,\mathcal{K}\,(I+\mu\mathbb{S}) - \mathcal{K}\,]\,z \\ &= z^T(I+\mu\mathbb{S}^T)\,[\mathcal{K} - (I+\mu\mathbb{S}^T)^{-1}\mathcal{K}\,(I+\mu\mathbb{S})^{-1}]\,(I+\mu\mathbb{S})\,z \\ &= (\underline{z}^\sigma)^T\,[\mathcal{K} - (I-\mu\underline{\mathbb{S}}^T)\,\mathcal{K}\,(I-\mu\underline{\mathbb{S}})]\,\underline{z}^\sigma = \mu\,\underline{\mathbb{Q}}(\underline{z}^\sigma).\end{aligned}$$

Therefore, $\mathbb{Q}(z) = \underline{\mathbb{Q}}(\underline{z}^\sigma)$ and we conclude that the functional $\underline{\mathbb{F}}_0$ is positive as well. Note that the converse statement cannot be proven by the same method on general time scales. This is due to the fact that the corresponding "definition" of the $(\underline{\mathbb{A}},\underline{\mathbb{B}})$-admissible pair $\underline{z}$ in terms of a given $(\mathbb{A},\mathbb{B})$-admissible z has the form of $\underline{z}^\sigma := (I+\mu\mathbb{S})\,z$ on $[a,\rho(b)]_{\mathbb{T}}$ with $\underline{z}(a) = 0$ if a is right-scattered, which does not provide a correct formula for $\underline{z}$, but rather a formula for $\underline{z}^\sigma$. This approach does work only on purely discrete time scales as in [3].

4. Conclusion

In this paper we demonstrated the utility of the time-reversed symplectic dynamic system ($\underline{\mathbb{S}}$) in the theory of optimal control problems on time scales. We showed that, rather than the usual system ($\mathbb{S}$), the time-reversed system ($\underline{\mathbb{S}}$) is more convenient for the optimal control problem ($\underline{\mathrm{C}}$), since it naturally preserves the structure of the associated quadratic forms. We obtained the Reid roundabout theorem for the time-reversed system ($\underline{\mathbb{S}}$) and as its consequence we also established the exact connection between the quadratic functionals associated with the time-reversed symplectic system ($\underline{\mathbb{S}}$) and the standard symplectic system ($\mathbb{S}$).

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